A STANDARD ZERO FREE REGION FOR RANKIN SELBERG L-FUNCTIONS

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Abstract. A standard zero free region is obtained for Rankin Selberg L-functions $L(s, f\times\widetilde{f})$ where f is an almost everywhere tempered Maass form on $GL(n)$ and f is not necessarily self dual. The method is based on the theory of Eisenstein series generalizing a work of Sarnak.

§1 Introduction

Let π_1, π_2 be cuspidal automorphic representations of $GL(n_i, \mathbb{A}_F)$ $(i = 1, 2)$ for some number field F whose central characters are trivial on $\mathbb{R}^+$ imbedded diagonally in the (archimedean) idèles. As pointed out in [Moreno, 1985], [Sarnak, 2004], de La Vallée Poussin's proof of the prime number theorem can be applied to any Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$ provided one of the π_i $(i = 1, 2)$ is self dual. In this case, the zero free region for $L(\sigma + it, \pi_1 \times \pi_2)$ takes the form

$$\sigma \; > \; 1 - \frac{c}{\log\big(Q_{\pi_1} Q_{\pi_2} \cdot \big(|t| + 2\big)\big)},$$

where $c > 0$ is an absolute constant depending only on n_1, n_2, F and Q_{π_1}, Q_{π_2} are the analytic conductors (see [Iwaniec-Sarnak, 1999] for the definition of analytic conductor) of π_1, π_2, respectively. A zero free region of the form

$$\sigma \; > \; 1 - \frac{c}{\Big(\log\big(Q_{\pi_1} Q_{\pi_2} \cdot \big(|t| + 2\big)\big)\Big)^B},$$

for some fixed $B > 0$ is called a standard zero free region.

Prior to this work, a standard zero free region for $L(s, \pi_1 \times \pi_2)$ was not known in the case that both π_1, π_2 are not self dual. The best zero-free region (non self dual case) known to date is due to [Brumley, 2006], and is of the form

$$\sigma \; > \; 1 - \frac{c}{\big(Q_{\pi_1} Q_{\pi_2} \cdot \big(|t| + 2\big)\big)^N}, \tag{1.1}$$

where $c > 0, N > 1$ depend only on n_1, n_2, F. The bound (1.1) was generalized to L-functions of Langlands-Shahidi type in [Gelbart-Lapid, 2006]. They invoked the method

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of [Sarnak, 2004] which made use of Eisenstein series, the Maass–Selberg relations, and a sieving argument. Sarnak's work can be viewed as an effectuation of the non-vanishing results of [Jacquet-Shalika, 1976] for the standard L-function.

In this paper we also follow [Sarnak, 2004], obtaining a much stronger result than (1.1), i.e., a standard zero free region, for a restricted class of Rankin-Selberg L-functions. Our main theorems are the following.

Theorem 1.2 *Let π be an irreducible cuspidal unramified representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$ for $n \geq 2$ which is tempered at all finite primes (except possibly a set of measure zero). Let $\widetilde{\pi}$ be the contragredient representation of π. Then a zero free region for $L\left(\sigma + it, \pi \times \widetilde{\pi}\right)$ is given by*

$$\sigma \; > \; 1 \; - \; \frac{c}{\big(\log(|t|+2)\big)^5},$$

where $c > 0$ is a fixed constant (independent of t) which depends at most on π.

Theorem 1.3 *Let π be an irreducible cuspidal unramified representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$ for $n \geq 2$ which is tempered at all finite primes (except possibly a set of Dirichlet density zero). Let f be a Maass cusp form in the space of π with dual $\widetilde{f} \in \widetilde{\pi}$. Then*

$$|L(1+it, \; f \times \widetilde{f}\,)| \; \gg \; \frac{1}{\big(\log(|t|+2)\big)^3} \tag{1.4}$$

for all $t \in \mathbb{R}$ and where the constant implied by the $\gg$-symbol is effectively computable and depends at most on f.

Theorem 1.2 follows from the mean value theorem together with theorem 1.3 and the fact that $L'(\sigma + it, f \times \widetilde{f}\,) \ll_{\pi,c} \big(\log(|t|+2)\big)^2$ for $1 - c/\big(\log(|t|+2)\big)^5 \leq \sigma \leq 1$. When $|t|$ is less than any fixed positive constant, the lower bound (1.4) was already proved in [Brumley, 2012]. To simplify the exposition we shall assume $|t| \gg 1$ for the remainder of this paper, where the constant implied by the $\gg$-symbol is sufficiently large and effectively computable.

Theorem 1.3 can be viewed as an effectuation of Shahidi's non-vanishing result [Shahidi, 1981] and an improvement of (1.1). Theorem 1.3 will have many applications since the relative trace formula for $GL(2n)$ (see [Jacquet-Lai, 1985], [Lapid, 2006]) will have a spectral contribution which involves integrating the Rankin-Selberg L-function $L(1+it, \pi \times \widetilde{\pi})^{-1}$ over $t \in \mathbb{R}$. It is crucial to have good lower bounds for $L(1+it, \pi \times \widetilde{\pi})$.

The key step in the proof of theorem 1.3 is the introduction of the integral $\mathcal{I}$ which is defined as $|L(1+2int, \; f \times \widetilde{f}\,)|^2$ multiplied by a certain transform of a smoothly truncated Eisenstein series on $GL(2n, \mathbb{A}_{\mathbb{Q}})$. The precise definition of $\mathcal{I}$ is given in 11.1. By the use of the Maass-Selberg relation it is possible to obtain an almost sharp upper bound for $\mathcal{I}$. The upper bound is given in theorem 11.2. Remarkably, it is also possible to obtain an almost sharp lower bound for $\mathcal{I}$ using bounds for Whittaker functions and a sieving argument. The lower bound is given in theorem 12.1. The combination of the upper and lower bounds for $\mathcal{I}$ immediately prove theorem 1.3.

We believe that the methods of this paper can be vastly generalized to global (not necessarily unramified) irreducible cuspidal automorphic representations of reductive groups over a number field, leading to standard zero free regions for all Rankin-Selberg L-functions. It is very likely that the assumptions of temperedness at finite primes can also be dropped.

To keep technicalities as simple as possible we work over $\mathbb{A}_{\mathbb{Q}}$ with the assumption that π is globally unramified. In this case, it is only necessary to do archimedean computations. The assumption that π is globally unramified can be removed. It is not hard to modify the proof we present if π is ramified at ∞. If, in addition, π is ramified at a finite set of primes then the entire proof will go through except that the $\gg$-constant in the lower bound for $|L(1+it,\ f\times\widetilde{f}\,)|$ will now depend on the finite set of ramified primes. This is clear because the key integrals coming up in the proof will all be Eulerian and the ramified primes will only involve a finite Euler product. So bounds for these integrals will only change by a finite factor when doing the proof in the ramified case.

§2 Basic Notation:

We shall be working with

$$G := GL(n,\mathbb{R}), \qquad \Gamma := GL(n,\mathbb{Z}), \qquad K := O(n,\mathbb{R}),$$

for $n \geq 2$. Every $z \in GL(n,\mathbb{R})\big/(K\cdot\mathbb{R}^\times)$ can be given in Iwasawa form

$$z = xy \tag{2.1}$$

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$

with $x_{i,j} \in \mathbb{R}$ $(1 \leq i < j \leq n)$ and $y_i > 0$ $(1 \leq i \leq n-1)$.

For $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ consider the I_ν-function

$$I_\nu(z) := \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j}, \tag{2.2}$$

where

$$b_{i,j} = \begin{cases} ij & \text{if } i+j \leq n, \\ (n-i)(n-j) & \text{if } i+j \geq n. \end{cases}$$

Associated to $\nu \in \mathbb{C}^{n-1}$ there are Langlands parameters $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ defined as follows. For each $1 \leq i \leq n-1$ define

$$B_i(\nu) := \sum_{j=1}^{n-1} b_{i,j}\nu_j.$$

Then the Langlands parameters associated to (n, ν) are given by

$$\alpha_i := \begin{cases} B_{n-1}(\nu) + \frac{1-n}{2}, & \text{if } i = 1, \\ B_{n-i}(\nu) - B_{n-i+1}(\nu) + \frac{2i-n-1}{2}, & \text{if } 1 < i < n, \\ -B_1(\nu) + \frac{n-1}{2}, & \text{if } i = n. \end{cases} \tag{2.3}$$

Fix a partition $n = n_1 + n_2 + \cdots + n_r$ with $1 \leq n_1, n_2, \ldots, n_r < n$. We define the standard parabolic subgroup

$$\mathcal{P} := P_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} GL(n_1) & * & \cdots & * \\ 0 & GL(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL(n_r) \end{pmatrix} \right\} \tag{2.4}$$

with nilpotent radical

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}, \tag{2.5}$$

and standard Levi

$$M^{\mathcal{P}} := \left\{ \begin{pmatrix} GL(n_1) & 0 & \cdots & 0 \\ 0 & GL(n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL(n_r) \end{pmatrix} \right\}. \tag{2.6}$$

Define

$$\mathfrak{h}^n := G/(K \cdot \mathbb{R}^\times).$$

For a function $F : G/KR^\times \to \mathbb{C}$ and $\gamma \in \Gamma$, we define the slash operator by

$$F(z) \Big|_\gamma := F(\gamma z), \qquad (z \in G/KR^\times). \tag{2.7}$$

For two functions $F_1, F_2 \in \mathcal{L}^2\big(SL(n,\mathbb{Z})\backslash\mathfrak{h}^n\big)$, we define the inner product

$$\langle F_1, F_2 \rangle := \int\limits_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n} F_1(z)\overline{F_2(z)}\, d^*z, \tag{2.8}$$

where $d^*z := d^\times x\, d^\times y = \prod\limits_{1 \leq i < j \leq n} dx_{i,j} \prod\limits_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k$ is the left invariant measure.

§3 The maximal parabolic Eisenstein series $\boldsymbol{E_{P_{2n-1,1}}}$:

Let $z \in GL(2n, \mathbb{R})\big/(K \cdot \mathbb{R}^\times)$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. Let $P_{2n-1,1}$ denote the maximal parabolic subgroup as in (2.4). We may then define as in proposition 10.7.5 of [Goldfeld 2006] the maximal parabolic Eisenstein series

$$E_{P_{2n-1,1}}(z,s) := \sum_{\gamma\, \in\, (P_{2n-1,1}(\mathbb{Z}) \cap SL(2n,\mathbb{Z}))\backslash SL(2n,\mathbb{Z})} \operatorname{Det}(\gamma z)^s,$$

and the completed maximal parabolic Eisenstein series

$$E^*_{P_{2n-1,1}}(z,s) = \pi^{-ns}\Gamma(ns)\zeta(2ns)E_{P_{2n-1,1}}(z,s) = E^*_{P_{2n-1,1}}(\,{}^t z^{-1}, 1-s)$$

where ${}^t z$ denotes the transpose of the matrix z.

For $u \in \mathbb{R}$ with $u > 0$, and $z = xy$ as in (2.1), define the theta function

$$\theta_z(u) := \sum_{(a_1, a_2, \ldots, a_{2n}) \in \mathbb{Z}^{2n}} e^{-\pi\left(b_1^2 + b_2^2 + \cdots + b_{2n}^2\right)\cdot u}, \tag{3.1}$$

where

$$\begin{aligned} b_1 &= a_1 Y_1, \\ b_2 &= (a_1 x_{1,2} + a_2)\, Y_2 \\ &\;\;\vdots \\ b_k &= (a_1 x_{1,k} + a_2 x_{2,k} + \cdots + a_{k-1} x_{k-1,k} + a_k)\, Y_k \\ &\;\;\vdots \\ b_{2n} &= (a_1 x_{1,2n} + a_2 x_{2,2n} + \cdots + a_{2n})\, Y_{2n} \end{aligned} \tag{3.2}$$

and

$$Y_k := y_1 y_2 \cdots y_{2n-k}\left[y_1^{2n-1} y_2^{2n-2} \cdots y_{2n-1}\right]^{-\frac{1}{2n}}, \qquad (\text{for } 1 \le k \le 2n).$$

In the formula above for Y_k we define $y_0 = 1$.

It then follows as in the proof of proposition 10.7.5 [Goldfeld 2006] that

(3.3)

$$E^*_{P_{2n-1,1}}(z,s) = \int_1^\infty \left[\theta_z(u) - 1\right] u^{ns}\, \frac{du}{u} \;+\; \int_1^\infty \left[\theta_{w\,{}^t z^{-1} w}(u) - 1\right] u^{n(1-s)}\, \frac{du}{u} - \frac{1}{n}\left(\frac{1}{1-s} + \frac{1}{s}\right).$$

where w is the long element in the Weyl group. We shall use (3.3) to prove the following proposition.

Proposition 3.4 *Let $z \in GL(2n,\mathbb{R})\big/(K\cdot\mathbb{R}^\times)$ as in (2.1). Let $y_0 = 1$ and assume that $y_i \ge 1$ $(1 \le i \le 2n-1)$. Then for $w \in \mathbb{C}$ with $\Re(w) = \frac{1}{2}$, we have*

$$\left|E_{P_{2n-1,1}}(z,w)\right| \ll \frac{e^{\frac{\pi}{2}n|w|}}{|w|^{\frac{n-1}{2}}} \sum_{1\le k\le 2n} \Bigg[\Big(y_1y_2^2 \;\cdots\; y_{2n-k}^{2n-k}\Big)^{\frac{1}{2}}\Big(y_{2n-k+1}^{k-1}y_{2n-k+2}^{k-2} \;\cdots\; y_{2n-1}\Big)^{\frac{1}{2}}$$

$$+\Big(y_1y_2^2 \;\cdots\; y_{2n-k}^{2n-k}\Big)^{\frac{k-1}{2n}}\Big(y_{2n-k+1}^{k-1}y_{2n-k+2}^{k-2} \;\cdots\; y_{2n-1}\Big)^{\frac{2n-k+1}{2n}}\Bigg]\cdot\log(y_1\cdots y_{2n-1}),$$

where the implied constant only depends on n.

Proof of Proposition 3.4: The factor $\frac{e^{\frac{\pi}{2}n|w|}}{|w|^{\frac{n-1}{2}}}$ comes from Stirling's asymptotic formula for $|\Gamma(nw)|^{-1}$, which arises when passing from $E^*_{P_{2n-1,1}}(z,w)$ to $E_{P_{2n-1,1}}(z,w)$.

It follows from (3.3) that for $w\in\mathbb{C}$ with $\Re(w)=\frac{1}{2}$, to bound $E_{P_{2n-1,1}}(z,w)$, it is enough to bound the integral

$$\int_1^\infty \left|\theta_z(u)-1\right| u^{n/2}\,\frac{du}{u}.$$

The other integral can be bounded similarly.

First of all, we note that by (3.1), (3.2), we have

$$\theta_z(u)-1 \;=\; \sum_{k=1}^{2n} \sum_{\substack{a_k\ne 0\\ a_i=0\;(1\le i\le k-1)}} \sum_{a_{k+1},\dots,a_{2n}\in\mathbb{Z}} e^{-\pi u\left[a_k^2Y_k^2\,+\,b_{k+1}^2\,+\,\cdots\,+\,b_{2n}^2\right]}. \tag{3.5}$$

Observe that for $t>0$ and $\alpha\in\mathbb{R}$, we may compute (using the Poisson summation formula) that

$$\sum_{n\in\mathbb{Z}} e^{-(n+\alpha)^2t} = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e^{-(x+\alpha)^2t}e^{2\pi inx}\,dx \;\ll\; 1+\frac{1}{t^{\frac{1}{2}}}. \tag{3.6}$$

It immediately follows from (3.2), (3.5), (3.6) that

$$\left|\theta_z(u)-1\right| \ll \sum_{k=1}^{2n-1}\sum_{a_k\ne 0} e^{-\pi u a_k^2Y_k^2}\prod_{i=k+1}^{2n}\left(1+\frac{1}{Y_i\cdot u^{\frac{1}{2}}}\right) + \sum_{a_{2n}\ne 0} e^{-\pi u\,a_{2n}^2Y_{2n}^2}.$$

Consequently

$$\int_1^\infty \left|\theta_z(u)-1\right| u^{n/2}\,\frac{du}{u} \;\ll\; \sum_{k=1}^{2n-1}\int_{Y_k^2}^{\infty}\sum_{a_k\ne 0} e^{-\pi u a_k^2}\prod_{i=k+1}^{2n}\left(1+\frac{1}{Y_iY_k^{-1}\cdot u^{\frac{1}{2}}}\right) Y_k^{-n}\,u^{n/2}\,\frac{du}{u}$$

$$+\int_{Y_{2n}^2}^{\infty}\sum_{a_{2n}\ne 0} e^{-\pi u a_{2n}^2}\,Y_{2n}^{-n}\,u^{n/2}\,\frac{du}{u}.$$

We now define

$$\mathcal{I}_k(z) := \begin{cases} \int\limits_{Y_k^2}^{\infty} \sum\limits_{a_k\neq 0} e^{-\pi u a_k^2} \prod\limits_{i=k+1}^{2n} \left(1+\frac{1}{Y_iY_k^{-1}\cdot u^{\frac{1}{2}}}\right) Y_k^{-n}\, u^{n/2}\, \frac{du}{u}, & \text{if } 1\le k\le 2n-1,\\ \int\limits_{Y_{2n}^2}^{\infty} \sum\limits_{a_{2n}\neq 0} e^{-\pi u a_{2n}^2}\, Y_{2n}^{-n}\, u^{n/2}\, \frac{du}{u}, & \text{if } k=2n. \end{cases}$$

so that

$$\int\limits_1^{\infty} \big|\theta_z(u)-1\big|\, u^{n/2}\, \frac{du}{u} \ll \sum_{k=1}^{2n} \mathcal{I}_k(z).$$

The bound for $\mathcal{I}_k(z)$ will depend on whether $Y_k \ge 1$ or $Y_k < 1$. Clearly

$$Y_1 \ge Y_2 \ge \cdots \ge Y_{2n}.$$

Case 1: $Y_k \ge 1$ and $1 \le k \le 2n-1$

In this case we have

$$\begin{aligned} \mathcal{I}_k(z) &\ll \int\limits_1^{\infty} \sum_{a_k\neq 0} e^{-\pi u a_k^2} \left[1+Y_{k+1}^{-1}Y_k\right] \cdots \left[1+Y_{2n}^{-1}Y_k\right] Y_k^{-n}\, u^{n/2}\, \frac{du}{u} \\ &\ll Y_{k+1}^{-1}\cdots Y_{2n}^{-1}\, Y_k^{n-k}. \end{aligned} \tag{3.6}$$

since $Y_j^{-1}Y_k \ge 1$ for $j > k$.

Case 2: $Y_k < 1$ and $1 \le k \le 2n-1$

We write

$$\begin{aligned} \mathcal{I}_k(z) &= \left(\int\limits_{Y_k^2}^{1} + \int\limits_1^{\infty}\right) \sum_{a_k\neq 0} e^{-\pi u a_k^2} \prod_{i=k+1}^{2n} \left(1+\frac{1}{Y_iY_k^{-1}\cdot u^{\frac{1}{2}}}\right) Y_k^{-n}\, u^{n/2}\, \frac{du}{u} \\ &:= I_1 + I_2. \end{aligned}$$

Note that for $u \le 1$, we have

$$\sum_{a_k\neq 0} e^{-\pi a_k^2 u} \ll u^{-\frac{1}{2}}.$$

It follows that

$$
\begin{aligned}
I_1 &= \int\limits_{Y_k^2}^{1} \sum_{a_k\neq 0} e^{-\pi u a_k^2} \prod_{i=k+1}^{2n} \left(1+\frac{1}{Y_iY_k^{-1}\cdot u^{\frac12}}\right) Y_k^{-n}\, u^{n/2}\ \frac{du}{u} \\
&\ll \int\limits_{Y_k^2}^{1} u^{-\frac12}\left(Y_{k+1}^{-1}Y_k u^{-\frac12}\right)\ \cdots\ \left(Y_{2n}^{-1}Y_k u^{-\frac12}\right) Y_k^{-n}\, u^{n/2}\ \frac{du}{u} \\
&\ll \int\limits_{Y_k^2}^{1} \left(Y_{k+1}^{-1}Y_{k+2}^{-1}\ \cdots\ Y_{2n}^{-1}\right) Y_k^{n-k}\, u^{-\frac32-\frac n2+\frac k2}\ du \\
&\ll \begin{cases} Y_{k+1}^{-1}\cdots Y_{2n}^{-1}\, Y_k^{n-k}\cdot\log(y_1\cdots y_{2n-1}), & \text{if } k=n+1, \\ Y_{k+1}^{-1}\cdots Y_{2n}^{-1}\, Y_k^{-1}, & \text{if } k\neq n+1. \end{cases}
\end{aligned}
\tag{3.7}
$$

We also have

$$
\begin{aligned}
I_2 &= \int\limits_{1}^{\infty} \sum_{a_k\neq 0} e^{-\pi a_k^2 u}\left[1+Y_{k+1}^{-1}Y_k\, u^{-\frac12}\right]\cdots\left[1+Y_{2n}^{-1}Y_k\, u^{-\frac12}\right]\cdot u^{\frac n2}\, Y_k^{-n}\ \frac{du}{u} \\
&\ll Y_{k+1}^{-1}\cdots Y_{2n}^{-1}\, Y_k^{n-k}.
\end{aligned}
\tag{3.8}
$$

<u>Case 3: $k=2n$</u>

Clearly $Y_{2n}\le 1$. We compute

$$
\begin{aligned}
\mathcal{I}_{2n}(z) &= \int\limits_{Y_{2n}^2}^{\infty} \sum_{a_{2n}\neq 0} e^{-\pi a_{2n}^2 u}\, Y_{2n}^{-n}\, u^{n/2}\ \frac{du}{u} \\
&= \int\limits_{Y_{2n}^2}^{1} \ +\ \int\limits_{1}^{\infty}
\end{aligned}
$$

where

$$
\int_{Y_{2n}^2}^{1} \ll \int_{Y_{2n}^2}^{1} u^{-\frac12+\frac n2-1}\, du\cdot Y_{2n}^{-n} \ \ll\ \left(1+Y_{2n}^{n-1}\right)\cdot Y_{2n}^{-1} \ \ll\ Y_{2n}^{-n}
$$

and

$$
\int\limits_{1}^{\infty} \ll\ Y_{2n}^{-n}.
$$

Hence

$$\mathcal{I}_{2n}(z) \ll Y_{2n}^{-n}. \tag{3.9}$$

To complete the proof of Proposition 3.4, we make use of the following two identities.

$$Y_{k+1}^{-1} \cdots Y_{2n}^{-1}\, Y_k^{n-k} = \left(y_1 y_2^2 \cdots y_{2n-k}^{2n-k}\right)^{\frac{1}{2}} \left(y_{2n-k+1}^{k-1}\, y_{2n-k+2}^{k-2} \cdots y_{2n-1}\right)^{\frac{1}{2}}. \tag{3.10}$$

$$Y_{k+1}^{-1} \cdots Y_{2n}^{-1}\, Y_k^{-1} = \left(y_1 y_2^2 \cdots y_{2n-k}^{2n-k}\right)^{\frac{k-1}{2n}} \left(y_{2n-k+1}^{k-1}\, y_{2n-k+2}^{k-2} \cdots y_{2n-1}\right)^{\frac{2n-k+1}{2n}}. \tag{3.11}$$

The proof of Proposition 3.4 then follows form (3.6), (3.7), (3.8), (3.9) in conjunction with (3.10), (3.11). □

§4 Eisenstein series associated to the $P_{n,n}$ parabolic on $GL(2n)$:

We fix an even Maass form (normalized with first Fourier coefficient = 1)

$$f : GL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^\times) \to \mathbb{C}, \tag{4.1}$$

of type $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}$ as in definition 5.1.3 [Goldfeld, 2006]. Associated to f there are Langlands parameters $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^n$ as in (2.3).

Also associated to f we have the L-function

$$L(s,f) = \prod_p \prod_{i=1}^n \left(1 - \frac{\alpha_{p,i}}{p^s}\right)^{-1}, \qquad (\alpha_{p,i} \in \mathbb{C}),$$

and the Rankin-Selberg L-function

$$L(s, f \times \widetilde{f}\,) = \prod_p \prod_{i=1}^n \prod_{j=1}^n \left(1 - \frac{\alpha_{p,i}\overline{\alpha_{p,j}}}{p^s}\right)^{-1}$$

which satisfy the functional equations

(4.2)

$$\Lambda(s,f) = \pi^{-\frac{ns}{2}} \prod_{i=1}^n \Gamma\left(\frac{s+\alpha_i}{2}\right) L(s,f) = \epsilon(f)\cdot\Lambda(1-s,\widetilde{f}),$$

$$\Lambda\left(s,\, f\times\widetilde{f}\right) = \pi^{-\frac{n^2 s}{2}} \prod_{i=1}^n \prod_{j=1}^n \Gamma\left(\frac{s+\alpha_i+\overline{\alpha_j}}{2}\right) L(s, f\times\widetilde{f}\,) = \epsilon(f\times\widetilde{f}\,)\cdot\Lambda(1-s,\widetilde{f}\times f),$$

with root numbers $\epsilon(f)$, $\epsilon(f \times \widetilde{f}\,)$ of absolute value one, and where $\widetilde{f}$ denotes the dual Maass form, i.e., $\widetilde{f}(z) = f(w\ {}^t(z^{-1}))$ with w the long element of the Weyl group.

Remark 4.3: *The completed Rankin-Selberg L-function given by* $\Lambda(s, f \times f)$ *is entire if* f *is not self dual (see* [Jacquet, 1972] *and* [Moeglin-Waldspurger, 1989] *for a proof).*

Let $s \in \mathbb{C}$ with $\Re(s) \gg 1$. For the parabolic $\mathcal{P} = P_{n,n}$, with nilpotent radical $N^{\mathcal{P}}$ and standard Levi $M^{\mathcal{P}}$, define

$$\phi_s(\mathfrak{n}\,\mathfrak{m}\,k) := \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{ns} f(\mathfrak{m}_1)f(\mathfrak{m}_2), \tag{4.4}$$

where

$$\mathfrak{n} \in N^{\mathcal{P}}, \qquad \mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & \\ & \mathfrak{m}_2 \end{pmatrix} \in M^{\mathcal{P}}, \qquad k \in K.$$

Let $z \in GL(2n, \mathbb{R})/KR^{\times}$. By the Iwasawa decomposition, z lies in the minimal parabolic, and hence lies in every standard parabolic, so z has a decomposition of the form $z = \mathfrak{n}\mathfrak{m}k$ as above. We may then define the Eisenstein series

$$E(z, f; s) := \sum_{\gamma \,\in\, (P_{n,n}(\mathbb{Z}) \cap \Gamma)\backslash\Gamma} \phi_s(\gamma z). \tag{4.5}$$

Let

$$C_{\mathcal{P}}\, E(z, f; s) := \phi_s(z) + \frac{\Lambda(2ns - n, f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - n, f \times \widetilde{f}\,)} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n(1-s)} f(\mathfrak{m}_1)\, f(\mathfrak{m}_2) \tag{4.6}$$

denote the constant term of E along the parabolic $\mathcal{P}$ (see proposition 8.5). It is convenient to introduce the height function

$$h(z) := \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|. \tag{4.7}$$

Define

$$\widehat{E}_A(z, f; s) := E(z, f; s) \;-\; \sum_{\substack{\gamma \,\in\, (P_{n,n}(\mathbb{Z}) \cap \Gamma)\backslash\Gamma \\ h(\gamma z) \geq A}} C_{\mathcal{P}}\, E(z, f; s)\,\Big|_{\gamma} \tag{4.8}$$

to be the truncated Eisenstein series in the sense of Arthur [Arthur, 1980]. It appears difficult to compute a useful Fourier series expansion of Arthur's truncation, so we introduce a modified version.

Definition 4.9 (Smoothed Arthur truncation of E): Let $A \geq 1$. We define

$$\begin{aligned}
\widehat{E}^*_A(z, f; s) &:= E(z, f; s) - A^{\frac{n}{2}}\, E(z, f; s - 1/2) + \frac{\Lambda(2ns - 2n, f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - 2n, f \times \widetilde{f}\,)}\, E(z, f; 2 - s) \\
&\quad - \sum_{\substack{\gamma \in P_{n,n}(\mathbb{Z})\backslash\Gamma \\ h(\gamma z) \geq A}} h(z)^{ns} \left(1 - \frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(\mathfrak{m}_1) f(\mathfrak{m}_2)\,\Big|_{\gamma} \\
&\quad - \frac{\Lambda(2ns - 2n,\; f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - 2n,\; f \times \widetilde{f}\,)} \sum_{\substack{\gamma \,\in\, (P_{n,n}(\mathbb{Z}) \cap \Gamma)\backslash\Gamma \\ h(\gamma z) \geq A}} h(z)^{n(2-s)} \left(1 - \frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(\mathfrak{m}_1) f(\mathfrak{m}_2)\,\Big|_{\gamma}
\end{aligned}$$

to be the smoothed Arthur truncation of $E(z, f; s)$.

Proposition 4.10 *Let* $h(z)$ *be given by (4.7). Then for any* $\gamma \in P_{n,n}(\mathbb{Z})\backslash SL(2n, \mathbb{Z})$ *and* $y_i \geq 1,\ (i = 1, 2, \dots, 2n-1)$, *we have* $h(\gamma z) \leq h(z)$.

The proof of proposition 4.10 relies on the following lemma.

Lemma 4.11 *Suppose* M *is an* $m \times m$ *upper triangular matrix. Let* $k \leq m$. *Define*

$$\begin{aligned} I &:= (i_1, \dots, i_k),\ \big(\text{with}\, 1 \leq i_1 < i_2 < \cdots < i_k \leq m\big), \\ J &:= (j_1, \dots, j_k),\ \big(\text{with}\, 1 \leq j_1 < j_2 < \cdots < j_k \leq m\big), \end{aligned}$$

and let $[M]_{I,J}$ *denote the determinant of the* $k \times k$ *sub matrix of* M *that corresponds to the rows with index* I *and the columns with index* J. *If* $i_\ell > j_\ell$, *(for some* $1 \leq \ell \leq k$*), then* $[M]_{I,J} = 0$.

Proof of lemma 4.11: Since $M = \big(m_{i,j}\big)_{\substack{1\leq i\leq m\\ 1\leq j\leq m}}$ is upper triangular we have $m_{i,j} = 0$ if $i > j$. Now $[M]_{I,J}$ is the minor of the matrix M determined by I, J and has the form

$$[M]_{I,J} = \mathrm{Det}\left(\begin{array}{cccccccc} & \overset{j_1}{\downarrow} & \overset{j_2}{\downarrow} & \cdots & \overset{j_\ell}{\downarrow} & \overset{j_r}{\downarrow} & \cdots & \overset{j_k}{\downarrow} \\ i_1 \to & * & * & \cdots & * & * & \cdots & * \\ & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ i_{\ell-1} \to & * & * & \cdots & * & * & \cdots & * \\ i_\ell \to & 0 & 0 & \cdots & 0 & * & \cdots & * \\ & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ i_k \to & 0 & 0 & \cdots & 0 & * & \cdots & * \end{array}\right).$$

Laplace's theorem says that if we select any ℓ columns/ rows of a matrix M and form all possible ℓ-columned/rowed minors from these ℓ columns/rows, multiply each of these minors by its cofactor, and then add the results, we obtain $\mathrm{Det}(M)$. Expand $[M]_{I,J}$ along the first ℓ columns. Since at least one row in the $\ell \times \ell$ minors is zero, we immediately see that $[M]_{I,J} = 0$. $\square$

Proof of proposition 4.10: Let

$$\mu_1 = \cdots = \mu_{n-1} = \mu_{n+1} = \cdots = \mu_{2n-1} = 0, \qquad \mu_n = \frac{1}{n}.$$

With this choice of $\mu = (0, \dots, 0, \frac{1}{n}, 0, \dots, 0)$ we see that (2.2) is given by

$$I_\mu(z) = \prod_{i=1}^{2n-1} y_i^{b_{i,n}\cdot\frac{1}{n}} = h(z).$$

Further, by lemma 5.7.2 [Goldfeld, 2006], it follows that for $\gamma \in GL(2n, \mathbb{Z})$ we have

$$\begin{aligned} I_\mu(\gamma z) = \prod_{i=0}^{2n-2} \big|\big| e_{2n-i}\,\gamma\, z \ \wedge \ \cdots \ \wedge \ e_{2n-1}\,\gamma\, z \ \wedge \ e_{2n}\,\gamma\, z \big|\big|^{-2n\mu_{2n-i-1}} & \\ \cdot \prod_{i=0}^{2n-2} \big|\big| e_{2n-i}\, z \ \wedge \ \cdots \ \wedge \ e_{2n-1}\, z \ \wedge \ e_{2n}\, z \big|\big|^{2n\mu_{2n-i-1}} \, I_\mu(z) & \\ = \big|\big| e_{n+1}\,\gamma\, z \ \wedge \cdots \ \wedge \ e_{2n}\,\gamma\, z \big|\big|^{-2} \cdot \big|\big| e_{n+1}\, z \ \wedge \cdots \ \wedge \ e_{2n}\, z \big|\big|^{2} \, I_\mu(z). & \end{aligned}$$

Here $\big|\big| e_{n+1}\, z \ \wedge \cdots \ \wedge \ e_{2n}\, z \big|\big|^{2}$ is the sum of the squares of all the $n \times n$ minors of the last n rows of z.

Fix $A, B \in GL(2n, \mathbb{R})$, an integer $1 \le k \le 2n$, and

$$\begin{aligned} I &:= (i_1, \dots, i_k), \ \big(\text{with } 1 \le i_1 < i_2 < \cdots < i_k \le 2n\big), \\ J &:= (j_1, \dots, j_k), \ \big(\text{with } 1 \le j_1 < j_2 < \cdots < j_k \le 2n\big). \end{aligned}$$

With these choices we have the Cauchy-Binet formula

$$[AB]_{I,J} = \sum_K [A]_{I,K}\, [B]_{K,J}, \tag{4.12}$$

where the sum goes over all $K = (\kappa_1, \dots, \kappa_k)$, $\big(\text{with } 1 \le \kappa_1 < \kappa_2 < \cdots < \kappa_k \le 2n\big)$.

We will apply (4.12) with the choices $k = n$ and

$$B = z = \begin{pmatrix} y_1 \cdots y_{2n-1} & y_1 \cdots y_{2n-2} x_{1,2} & \cdots & y_1 x_{1,2n-1} & x_{1,2n} \\ & y_1 \cdots y_{2n-2} & \cdots & y_1 x_{2,2n-1} & x_{2,2n} \\ & & \ddots & \vdots & \vdots \\ & & & y_1 & x_{2n-1,2n} \\ & & & 0 & 1 \end{pmatrix},$$

$$A = \gamma = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{2n,1} & \alpha_{2n,2} & \cdots & \alpha_{2n,2n} \end{pmatrix}.$$

Let $C := \gamma z$. Then by (4.12) we see that

$$[C]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} = \sum_K [\gamma]_{(n+1,n+2,\dots,2n),\,K} \cdot [z]_{K,\,(1,2,\dots,n)}.$$

By lemma 4.11, we have $[z]_{K,\,(1,2,\dots,n)} = 0$ unless $K = (1, 2, \dots, n)$. It follows that

$$[C]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} = [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} \cdot [z]_{(1,2,\dots,n),\,(1,2,\dots,n)}$$

Again, by lemma 4.11, we see that

$$\begin{aligned}
[C]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-1,n+1)} &= \sum_{K} [\gamma]_{(n+1,n+2,\dots,2n),\,K} \cdot [z]_{K,\,(1,2,\dots,n-1,n+1)} \\
&= [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} \cdot [z]_{(1,2,\dots,n),\,(1,2,\dots,n-1,n+1)} \\
&\qquad + [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-1,n+1)} \cdot [z]_{(1,2,\dots,n),\,(1,2,\dots n-1,,n+1)}.
\end{aligned}$$

Continuing in the same manner it follows that

$$\begin{aligned}
&[C]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-2,n,n+1)} \\
&\quad = [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} \cdot [z]_{(1,2,\dots,n),\,(1,2,\dots,n-2,n,n+1)} \\
&\qquad\quad + \\
&\qquad\qquad \ddots \\
&\qquad\qquad\quad + [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-1,n+1)} \cdot [z]_{(1,2,\dots,n-1,n+1),\,(1,2,\dots n,n+1)} \\
&\qquad\qquad\qquad + [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n,n+1)} \cdot [z]_{(1,2,\dots,n,n+1),\,(1,2,\dots n,n+1)}, \\
&\qquad\qquad\qquad\quad \vdots
\end{aligned}$$

$$\begin{aligned}
&[C]_{(n+1,n+2,\dots,2n),\,(n+1,\dots,2n)} \\
&\quad = [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} \cdot [z]_{(1,2,\dots,n),\,(n+1,2,\dots,2n)} \\
&\qquad\qquad + [\gamma]_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-1,n+1)} \cdot [z]_{(1,2,\dots n-1,n+1),\,(n+1,\dots 2n)} \\
&\qquad\qquad\quad + \\
&\qquad\qquad\qquad \ddots \\
&\qquad\qquad\qquad\quad + [\gamma]_{(n+1,n+2,\dots,2n),\,(n+1,\dots,2n)} \cdot [z]_{(n+1,\dots,2n),\,(n+1,\dots 2n)}.
\end{aligned}$$

For

$$1 \le i_1 < i_2 < \cdots < i_n \le 2n - 1,$$

we have

$$\begin{aligned}[z]_{(i_1,i_2,\dots,i_n),(i_1,i_2,\dots i_n)} = y_1^n y_2^n \cdots y_{2n-i_n-1}^n \cdot y_{2n-i_n}^n \\ \cdot y_{2n-i_n+1}^{n-1} \cdots y_{2n-i_{n-1}}^{n-1} \\ \cdot y_{2n-i_{n-1}+1}^{n-2} \cdots y_{2n-i_{n-2}}^{n-2} \\ \vdots \\ \cdot y_{2n-i_2+1} \cdots y_{2n-i_1}.\end{aligned}$$

Here, we define $y_0 := 1$ and

$$\begin{aligned}[z]_{(n+1,n+2,\dots,2n),\,(n+1,n+2,\dots,2n)} &= y_1^{n-1} y_2^{n-2} \cdots y_{n-2}^2 y_{n-1} \\ &= ||e_{n+1}z \wedge \cdots \wedge e_{2n-1}z \wedge e_{2n}z||,\end{aligned}$$

where the exponents of the powers of y_j $(1 \le j \le n-1)$ in the expression

$$y_1^{n-1} y_2^{n-2} \cdots y_{n-2}^2 y_{n-1} \tag{4.13}$$

are less than or equal to the exponents of the powers of y_j $(1 \le j \le n-1)$ that occur in $[z]_{(i_1,i_2,\dots,i_n),\,(i_1,i_2,\dots.i_n)}$. This is because the exponent of y_j in (4.13) is equal to $n-j$ while the exponent of y_j in $[z]_{(i_1,i_2,\dots,i_n),\,(i_1,i_2,\dots.i_n)}$ is bigger than $n-j$ because $i_{n-j} \le 2n-j$. Hence

$$[z]_{(n+1,n+2,\dots,2n),\,(n+1,n+2,\dots,2n)} \;\le\; [z]_{(i_1,i_2,\dots,i_n),(i_1,i_2,\dots i_n)}, \tag{4.14}$$

for any $1 \le i_1 < i_2 < \cdots < i_n \le 2n-1$.

Recall that

$$h(\gamma z) = I_\nu(\gamma z) = ||e_{n+1}\gamma z \wedge \cdots \wedge e_{2n-1}\gamma z \wedge e_{2n}\gamma z||^{-2} \cdot ||e_{n+1}z \wedge \cdots \wedge e_{2n-1}z \wedge e_{2n}z||^2 h(z).$$

To prove $h(\gamma z) \le h(z)$ it is enough to show that

$$||e_{n+1}\gamma z \wedge \cdots \wedge e_{2n-1}\gamma z \wedge e_{2n}\gamma z|| \ge ||e_{n+1}z \wedge \cdots \wedge e_{2n-1}z \wedge e_{2n}z||. \tag{4.15}$$

Here

$$||e_{n+1}z \wedge \cdots \wedge e_{2n-1}z \wedge e_{2n}z||^2 = \left(y_1^{n-1} y_2^{n-2} \cdots y_{n-2}^2 y_{n-1}\right)^2$$

and

(4.16)

$$\begin{aligned}||e_{n+1}\gamma z \wedge \cdots \wedge e_{2n-1}\gamma z \wedge e_{2n}\gamma z||^2 &= [C]^2_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n)} \\ &+ [C]^2_{(n+1,n+2,\dots,2n),\,(1,2,\dots,n-1,n+1)} + \cdots + [C]^2_{(n+1,n+2,\dots,2n),\,(n+1,\dots,2n)}.\end{aligned}$$

Now

$$[C]_{(n+1,n+2,\dots,2n),(1,2,\dots,n)} = [\gamma]_{(n+1,n+2,\dots,2n),(1,2,\dots,n)} \cdot [z]_{(1,2,\dots,n),(1,2,\dots,n)}.$$

If $[\gamma]_{(n+1,n+2,\dots,2n),(1,2,\dots,n)} \neq 0$ then $[\gamma]^2_{(n+1,n+2,\dots,2n),(1,2,\dots,n)} \geq 1$ since it is an integer. It immediately follows from (4.14) that

$$\begin{aligned}[C]^2_{(n+1,n+2,\dots,2n),\ (1,2,\dots,n)} &\geq [z]^2_{(n+1,n+2,\dots,2n),\ (n+1,n+2,\dots,2n)}\\ &= ||e_{n+1}z \wedge \cdots \wedge e_{2n-1}z \wedge e_{2n}z||^2.\end{aligned}$$

which proves (4.15), and, hence, also proves proposition 4.10. If

$$[\gamma]_{(n+1,n+2,\dots,2n),(1,2,\dots,n)} = 0,$$

then we use the next term, (i.e. $[C]^2_{(n+1,n+2,\dots,2n),\ (1,2,\dots,n-1,n+1)}$) in equation (4.16) and repeat the previous argument. This process can be further continued and we eventually prove proposition 4.10. □

For $s \in \mathbb{C}$ with $\Re(s) \gg 1$, let $\widehat{E}_A(z,f;s)$ denote Arthur's truncated Eisenstein series attached to the Maass cusp form f defined in (4.8). Define

$$c_s(f) := \frac{\Lambda(2ns-n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-n,\ f\times\widetilde{f}\,)}. \tag{4.18}$$

Corollary 4.19 *Let $t \in \mathbb{R}$ with $t>1$. Let $\beta = t^{n^{10}}$. Then for $1 \leq y_i \leq \left(t^{1+\epsilon}\right)^{\frac{2n(2n-1)}{2}}$ with $1 \leq i \leq 2n-1$ and $\beta \leq A \leq 2\beta$, we have*

$$\widehat{E}^*_A(z,f;s) \ = \ \widehat{E}_A(z,f;s) \ - \ A^{\frac{n}{2}}\widehat{E}_A(z,f;s-1/2) \ + \ c_{s-\frac12}(f)\widehat{E}_A(z,f;2-s).$$

Proof: Recall from (4.6), (4.7), (4.8) that

$$\begin{aligned}\widehat{E}^*_A(z,f;s) &= E(z,f;s) - A^{\frac{n}{2}}E(z,f;s-1/2) + \frac{\Lambda(2ns-2n,f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,f\times\widetilde{f}\,)}\,E(z,f;2-s)\\ &\quad - \sum_{\substack{\gamma\in P_{n,n}(\mathbb{Z})\backslash\Gamma\\ h(\gamma z)\geq A}} h(z)^{ns}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(\mathfrak{m}_1)f(\mathfrak{m}_2)\Bigg|_{\gamma}\\ &\quad - \frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)} \sum_{\substack{\gamma\in P_{n,n}(\mathbb{Z})\backslash\Gamma\\ h(\gamma z)\geq A}} h(z)^{n(2-s)}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(\mathfrak{m}_1)f(\mathfrak{m}_2)\Bigg|_{\gamma}.\end{aligned}$$

By the assumptions on the $y_i, (1 \le i \le 2n-1)$ and proposition 4.10, we see that

$$h(\gamma z) \ \le \ h(z) = \prod_{i=1}^{2n-1} y_i^{b_{i,n} \cdot \frac{1}{n}} = y_1 y_2^2 y_3^3 \cdots y_n^n \cdot y_{n+1}^{n-1} y_{n+2}^{n-2} \cdots y_{2n-1} \le \left(t^{(2+\epsilon)n^2}\right)^{n^2} < A.$$

Since $c_{s-\frac{1}{2}}(f) = \frac{\Lambda(2ns-2n, f\times \widetilde{f}\,)}{\Lambda(1+2ns-2n, f\times\widetilde{f}\,)}$, it immediately follows that

$$\widehat{E}_A^*(z,f;s) = E(z,f;s) - A^{\frac{n}{2}} E(z,f;s-1/2) + c_{s-\frac{1}{2}}(f)\, E(z,f;2-s).$$

A similar argument can be applied to $\widehat{E}_A(z,f;s)$, using (4.8).
□

§5 Upper Bounds for Rankin-Selberg L-functions on the Line $\Re(s) = 1$:

The main result of this section is the following lemma.

Lemma 5.1 *Let $t \in \mathbb{R}$ and f be the Maass form on $GL(n,\mathbb{Z})$ defined in (4.1). Then for $|t| \ge 1$, we have the bounds*

$$L(1+2int,\ f\times\widetilde{f}\,) \ \ll_{n,f} \log|t|,$$
$$L'(1+2int,\ f\times\widetilde{f}\,) \ \ll_{n,f} (\log|t|)^2.,$$

Proof of lemma 5.1: It follows from theorem 5.3 in [Iwaniec-Kowalski, 2004] that

(5.2)
$$L(1+2int,\ f\times\widetilde{f}\,) \ = \ \sum_{m=1}^{\infty} \frac{\lambda_{f\times\widetilde{f}}(m)}{m^{1+2int}} V_{1+2int}\left(\frac{m}{X}\right) + \epsilon(f\times\widetilde{f}\,) \sum_{m=1}^{\infty} \frac{\overline{\lambda}_{f\times\widetilde{f}}(m)}{m^{-2int}} V_{-2int}(mX)$$
$$+ \left(\operatorname*{Res}_{u=-2int} + \operatorname*{Res}_{u=-1-2int}\right) \frac{\Lambda(s+u,\ f\times\widetilde{f}\,)}{\gamma(s,\ f\times\widetilde{f}\,)} \frac{G(u)}{u} X^u,$$

where $X = (|t|+1)^{\frac{n^2}{2}}$. By proposition 5.4 in [Iwaniec-Kowalski, 2004], we have

$$V_{-2int},\ V_{1+2int}(y) \ \ll \ \left(1 + \frac{y}{\sqrt{q_\infty}}\right)^{-100},$$

with $q_\infty = (|t|+1)^{n^2} = X^2$.

Consequently, $L(1+2int,\ f\times\widetilde{f}\,)$ is essentially approximated by the first piece in the approximate functional equation (5.2). Since

$$\sum_{m\le N} \lambda_{f\times\widetilde{f}}(m) \ll_f N,$$

the bound for $L(1+2int,\ f\times\widetilde{f},)$ follows immediately.

Next, we consider the derivative. Let $G(u) := \left(\cos\left(\frac{\pi u}{400}\right)\right)^{-4n^3}$, as in [Iwaniec-Kowalski, 2004, p. 99], and define

$$I(X, f\times\widetilde{f},\, s) := \frac{1}{2\pi i}\int\limits_{3-i\infty}^{3+i\infty} X^u \Lambda(s+u, f\times\widetilde{f}\,)\, G(u)\, \frac{du}{u^2},$$

where $s = 1+2int$. If we shift the line of integration to $\Re(u) = -3$, we pick up a double pole at $u = 0$ and simple poles at $u = 1-s,\ -s$. Note that

$$G(0) = 1, \qquad G'(0) = 0.$$

Set $\Lambda(s, f\times\widetilde{f}\,) = \gamma(s, f\times\widetilde{f}\,)L(s, f\times\widetilde{f}\,)$. It follows that

$$\begin{aligned}\operatorname*{Res}_{u=0} X^u\Lambda(s+u, f\times\widetilde{f}\,)\,\frac{G(u)}{u^2} &= \Big[X^u\Lambda(s+u, f\times\widetilde{f}\,)\,G(u)\Big]'_{u=o}\\ &= \log X\,\Lambda(s, f\times\widetilde{f}\,)\ +\ \gamma'(s, f\times\widetilde{f}\,)L(s, f\times\widetilde{f}\,)\ +\ \gamma(s, f\times\widetilde{f}\,)L'(s, f\times\widetilde{f}\,).\end{aligned}$$

Applying the functional equation (4.2), we obtain

(5.3)
$$\begin{aligned}&(\log X)\cdot\gamma(s, f\times\widetilde{f}\,)L(s, f\times\widetilde{f}\,) + \gamma'(s, f\times\widetilde{f}\,)L(s, f\times\widetilde{f}\,)\\ &\quad + \gamma(s, f\times\widetilde{f}\,)L'(s, f\times\widetilde{f}\,)\ +\ \left(\operatorname*{Res}_{u=1-s} + \operatorname*{Res}_{u=-s}\right)\Lambda(s+u,\, f\times\widetilde{f}\,)\frac{G(u)}{u^2}X^u\\ &\qquad = I(X,\, f\times\widetilde{f}, s)\ +\ \epsilon(f\times\widetilde{f}\,)I(X^{-1},\, \widetilde{f}\times f,\, 1-s).\end{aligned}$$

Now, we may expand $I(X, f\times f, s)$ into an absolutely convergent series. Let $L(s, f\times\widetilde{f}\,) = \sum\limits_{m=1}^{\infty}\lambda_{f\times\widetilde{f}}(m)m^{-s}$. Hence

$$\begin{aligned}I(X, f\times\widetilde{f}, s) &= \sum_{m=1}^{\infty}\lambda_{f\times\widetilde{f}}(m)m^{-s}\cdot\frac{1}{2\pi i}\int\limits_{3-i\infty}^{3+i\infty}\gamma(s+u, f\times\widetilde{f}\,)\left(\frac{X}{m}\right)^u G(u)\,\frac{du}{u^2}\\ &= \gamma(s, f\times\widetilde{f}\,)\sum_{m=1}^{\infty}\frac{\lambda_{f\times\widetilde{f}}(m)}{m^s}V_s^*\left(\frac{m}{X}\right),\end{aligned}$$

where

$$V_s^*(y) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} y^{-u} G(u) \frac{\gamma(s+u, f\times\widetilde{f}\,)}{\gamma(s, f\times\widetilde{f}\,)}\,\frac{du}{u^2}. \tag{5.4}$$

Next, we do the same for $I\left(X^{-1}, \widetilde{f}\times f,\, 1-s\right)$ and then divide both sides of (5.3) by $\gamma(s, f\times\widetilde{f}\,)$. It follows that

(5.5)

$$\begin{aligned}(\log X)\cdot L(s, f\times\widetilde{f}\,) \;+\; &\frac{\gamma'(s, f\times\widetilde{f}\,)}{\gamma(s, f\times\widetilde{f}\,)} L(s, f\times\widetilde{f}\,) \;+\; L'(s, f\times\widetilde{f}\,)\\ &+\left(\operatorname*{Res}_{u=1-s} \;+\; \operatorname*{Res}_{u=-s}\right)\frac{\Lambda(s+u,\, f\times\widetilde{f}\,)}{\gamma(s,\, f\times\widetilde{f}\,)}\frac{G(u)}{u^2}X^u\\ = \sum_{m=1}^{\infty}\frac{\lambda_{f\times\widetilde{f}}(m)}{m^{1+2int}}V_s^*\left(\frac{m}{X}\right) \;+\; &\epsilon(f\times\widetilde{f}\,)\sum_{m=1}^{\infty}\frac{\overline{\lambda}_{f\times\widetilde{f}}(m)}{m^{-2int}}V_s^*\left(mX\right).\end{aligned}$$

Recall that $q_\infty = (|t|+1)^{n^2}$.

Claim: For $0 < \Re(s) \le 1$ and $|\Im(s)| \ge 1$, we have

$$V_s^*(y) \ll \begin{cases} \left(\frac{y}{\sqrt{q_\infty}}\right)^{-100}, & \text{if } y \ge \sqrt{q_\infty},\\ \log y + \mathcal{O}\left(\frac{y}{\sqrt{q_\infty}}\right)^{\frac{\Re(s)}{2}}, & \text{if } y \le \sqrt{q_\infty}.\end{cases}$$

Stirling's asymptotic formula for the Gamma function says that for fixed σ, as $|t|\to\infty$,

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}\,|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

If we apply the above asymptotic formula to the n^2 Gamma functions in the numerator and denominator of $\frac{\gamma(s+u, f\times\widetilde{f}\,)}{\gamma(s, f\times\widetilde{f}\,)}$ (see (4.2)) it follows that

$$\frac{\gamma(s+u, f\times\widetilde{f}\,)}{\gamma(s, f\times\widetilde{f}\,)} \ll q_\infty^{\frac{\Re(u)}{2}} e^{\frac{\pi}{2}n^2|u|}. \tag{5.6}$$

The first bound in the claim follows upon shifting the line of integration in (5.4) to $\Re(u) = 100$ and then using the estimate (5.6). If we shift the line of integration to $\Re(u) = -\Re(s)/2$, we derive the second bound.

It is clear that $V^*_{-2it}(nX) \ll 1$. The second bound in lemma 5.1 now follows from (5.5) and the above claim.

$\square$

§6 Maass-Selberg Relation:

The Maass-Selberg relation for $GL(n)$ with $n > 2$ (due to Langlands) is an identity for the inner product of two Arthur truncated Eisenstein series (see [Garrett, 2005], [Labesse-Waldspurger, 2013]).

Theorem 6.1 (Maass-Selberg relation) *Let $r, s \in \mathbb{C}$ with $r \neq \bar{s}$ and $r + \bar{s} \neq 1$. Then we have*

$$\Big\langle \widehat{E}_A(*, f;\, r),\ \widehat{E}_A(*, f;\, s)\Big\rangle \;=\; \langle f,f\rangle\langle f,f\rangle \cdot \frac{A^{n(r+\bar{s}-1)}}{r+\bar{s}-1} \;+\; \left|\langle f,f\rangle\right|^2 \cdot \bar{c}_s\,\frac{A^{n(r-\bar{s})}}{r-\bar{s}}$$
$$+\left|\langle f,f\rangle\right|^2 \cdot c_r\frac{A^{n(\bar{s}-r)}}{\bar{s}-r} \;+\; \langle f,f\rangle\langle f,f\rangle \cdot c_r\,\bar{c}_s\frac{A^{n(1-r-\bar{s}}}{1-r-\bar{s})},$$

where $c_s := \frac{\Lambda(2ns-n,\ f\times \widetilde{f}\,)}{\Lambda(1+2ns-n,\ f\times\widetilde{f}\,)}$. Here $\langle f, f\rangle$ is the inner product on $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$, while the inner product of Eisenstein series is on $SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}$.

Next, we prove the following corollary.

Corollary 6.2 *Define ${c_s}' := \frac{d}{ds}c_s$. Then for $s = \frac{1}{2} + it$ with $0 \neq t \in \mathbb{R}$, we have*

$$\Big\langle \widehat{E}_A(*, f;\ s),\ \widehat{E}_A(*, f;\ s)\Big\rangle \;=\; \left|\langle f,f\rangle\right|^2 \cdot \bar{c}_s\frac{A^{2nit}}{2it} \;+\; \left|\langle f,f\rangle\right|^2 \cdot c_s\frac{A^{-2nit}}{-2it}$$
$$-\;\langle f,f\rangle\langle f,f\rangle \cdot \frac{c_s'}{c_s} \;+\; \langle f,f\rangle\langle f,f\rangle \cdot 2n\log A.$$

Proof of corollary 6.2: Let

$$s = \frac{1}{2} + it, \qquad r = s + \epsilon, \qquad (t \in \mathbb{R},\ \epsilon > 0),$$

for $\epsilon \to 0$. Then Theorem 6.1 takes the form

$$\Big\langle \widehat{E}_A(*, f;\ r),\ \widehat{E}_A(*, f;\ s)\Big\rangle \;=\; \langle f,f\rangle\langle f,f\rangle \cdot \frac{A^{n\epsilon}}{\epsilon} \;+\; \left|\langle f,f\rangle\right|^2 \cdot \bar{c}_s\,\frac{A^{2nit+\epsilon}}{2it+\epsilon}$$

(6.3)

$$+\;\left|\langle f,f\rangle\right|^2 \cdot c_r\,\frac{A^{-n\epsilon-2nit}}{-\epsilon-2it} \;-\; \langle f,f\rangle\langle f,f\rangle \cdot c_r\,\bar{c}_s\,\frac{A^{-n\epsilon}}{\epsilon}.$$

On the line $\Re(s) = \frac{1}{2}$, we have $|c_s| = 1$. It follows by the Taylor expansion around $s = 0$ that

$$\begin{aligned} c_r \cdot \overline{c}_s &= \overline{c}_s \left[c_s + {c_s}' \epsilon + o(\epsilon^2) \right] \\ &= 1 + \overline{c}_s {c_s}' \epsilon + o(\epsilon^2). \end{aligned}$$

Further,

$$A^{n\epsilon} = 1 + (\log A) \cdot n\epsilon + o(\epsilon^2).$$

$$A^{-n\epsilon} = 1 - (\log A) \cdot n\epsilon + o(\epsilon^2).$$

Corollary 6.2 immediately follows upon letting $\epsilon \to 0$ in (6.3). $\square$

Proposition 6.4 Let $\mathcal{F} := SL(2n, \mathbb{Z})\backslash GL(2n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$. Then we have

$$\left| L(1 + 2int,\, f \times \widetilde{f}\,) \right| \int\limits_{\substack{\mathcal{F} \\ 1 \le y_i \le (t^{1+\epsilon})^{n(2n-1)} \\ 1 \le i \le 2n-1}} \left| \widehat{E}_A^*(z, f;\, 1 + it) \right|^2 \, d^\star z \;\ll\; A^n (\log A)(\log |t|).$$

Proof: This follows immediately from corollary 4.19, the Maass-Selberg relation in theorem 6.1 and corollary 6.2, and the upper bounds for Rankin-Selberg L-functions given in lemma 5.1.

$\square$

§7 Stade's formula for Whittaker functions:

Let $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, denote spectral parameters for $GL(n, \mathbb{R})$ with $n \ge 2$. The Whittaker function $W_{n,\nu}$ on $GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$ is defined to be

$$W_{n,\nu}(z) := \int\limits_{\mathbb{R}^{\frac{(n-1)n}{2}}} I_\nu(w_n u z) e^{-2\pi i (u_{1,2} + \cdots + u_{n-1,n})} \prod_{1 \le i < j \le n} du_{i,j}, \tag{7.1}$$

where $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ is the long element of the Weyl group of $GL(n, \mathbb{R})$ and

$$\nu_{j,k} = \sum_{i=0}^{j-1} \frac{n\nu_{n-k+i} - 1}{2}.$$

The completed Whittaker function is defined to be

$$W_{n,\nu}^*(z) := W_{n,\nu}(z) \prod_{j=1}^{n-1} \prod_{k=j}^{n-1} \frac{\Gamma\left(\frac{1}{2} + \nu_{j,k}\right)}{\pi^{\frac{1}{2} + \nu_{j,k}}}.$$

Stade found a beautiful formula which expresses the Whittaker function as an integral of K-Bessel functions. For $\nu \in \mathbb{C}$, the Bessel function K_ν is defined as

$$K_\nu(2\pi y) := \frac{1}{2} \int\limits_0^\infty e^{-\pi y(t+1/t)}\, t^\nu \, \frac{dt}{t}. \tag{7.2}$$

For $y, a, b > 0$, define

$$K_\nu^*(y; a, b) := 2\left(\frac{a}{b}\right)^{\frac{\nu}{2}} \cdot K_\nu\left(2\pi y\sqrt{ab}\right).$$

For any array $\{u_{k,\ell}\}$, define

$$S(j,i) := 1 + \frac{1}{u_{j-1,i}}\left(1 + \frac{u_{j-2,i-1}}{u_{j-2,i}}\left(1 + \frac{u_{j-3,i-1}}{u_{j-3,i}}\left(1 + \cdots + \frac{u_{2,i-1}}{u_{2,i}}\left(1 + \frac{u_{1,i-1}}{u_{1,i}}\right)\cdots\right)\right)\right)$$

for $1 \le j \le i$.

Proposition 7.3 (Stade, 1990) *Let $n \ge 2$ and $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ with $\Re(\nu_i) > \frac{1}{n}$ for $1 \le i \le n-1$. Set $\mu_j := \sum\limits_{k=j}^{n-1} \nu_{j.k}$. Then*

$$W^*_{n,\nu}(y) = \prod_{i=1}^{n-1} y_{n-i}^{\sum\limits_{j=1}^{n-1} b_{i,j}\nu_j - \mu_i} \int\limits_{(\mathbb{R}^+)^{\frac{(n-2)(n-1)}{2}}} \prod_{i=1}^{n-1} \left\{ K^*_{\mu_i}\left(y_i\, ;\, 1 + \sum_{k=i+1}^{n-1} u_{1,k},\, S(i,i)\right) \cdot \left(\prod_{k=i+1}^{n-1} u_{i,k}^{-\nu_{i,k-i+1}}\right)\right\} \prod_{1 \le i < j \le n-1} \frac{du_{i,j}}{u_{i,j}}.$$

Corollary 7.4 *Let $n \ge 2$, and $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ with associated Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ as in (2.3). Assume $\Im(\alpha_i) \le C|t|$ for $t \in \mathbb{R}$ with $|t| \to \infty$ and $C > 0$ a fixed constant. Further assume that*

$$\max_{1 \le i \le n-1} y_i \;\ge\; |t|^{1+\epsilon}, \qquad \text{(for some fixed } \epsilon > 0).$$

Then

$$W_{n,\nu}(y) \ll \Big(\max_{1 \le i \le n-1} y_i\Big)^{-N}$$

for any fixed large $N > 1$. The implied constant in the $\ll$-symbol above does not depend on t, y.

Proof: The proof follows from proposition 7.3 and the bound for the K-Bessel function

$$e^{\frac{\pi}{2}|\Im\nu|}K_\nu(y) \ll y^{-N}, \qquad \Big(\text{for } y \geq (|\Im(\nu)|+1)^{1+\epsilon},\ |\Re(\nu)| \ll 1\Big),$$

which holds for any fixed $N > 1$. $\square$

Assuming $\Re(\alpha_i) = 0$ for $1 \leq i \leq n$, [Brumley-Templier] proved corollary 7.4 using a different method.

§8 Fourier Expansions of Eisenstein Series:

Definition 8.1 (Automorphic function) *Let $k \in \mathbb{Z}$ with $k \geq 2$. Let $z = xy \in GL(k, \mathbb{R})$ as in (2.1). A function*

$$F : SL(k,\mathbb{Z})\backslash GL(k,\mathbb{R})/O(k,\mathbb{R})\cdot\mathbb{R}^\times \ \to\ \mathbb{C}$$

is called an automorphic function if it is smooth and $F(z)$ has polynomial growth in

$$Y_{_{k_1,\dots,k_r}} = \prod_{i=1}^{r} y_{k_i}, \qquad (Y_{_{k_1,\dots,k_r}} \to \infty),$$

for all $1 \leq k_1 < k_2, \cdots < k_r \leq n-1$, and where y_j is fixed if $j \neq k_i$ $(1 \leq i \leq r)$.

Definition 8.2 Let $m, n \in \mathbb{Z}$ with $n \geq 2$ and $1 \leq m < n$. We define the parabolic subgroup

$$\widetilde{P}_{n,m} := \left(\begin{pmatrix} SL(n-m) & \begin{matrix} * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ * & * & \cdots & * \end{matrix} \\ & \begin{matrix} 1 & * & \cdots & * \\ & \ddots & & \vdots \\ & & 1 & * \\ & & & 1 \end{matrix} \end{pmatrix}\right) \subset SL(n).$$

Theorem 8.3 *Suppose F is an automorphic function for $SL(k,\mathbb{Z})$ with $k \geq 2$. Then F has the following Fourier expansion.*

$$F(z) = \int_0^1 \cdots \int_0^1 F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k} \\ & 1 & \cdots & 0 & \vdots \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1 \end{pmatrix} z \right) du_{1,k}\, du_{2,k} \cdots du_{k-1,k}$$

$$+ \sum_{1 \leq \ell \leq k-2} \sum_{m_{k-1} \neq 0} \cdots \sum_{m_{k-\ell} \neq 0} \sum_{\gamma_{k-1} \in \widetilde{P}_{k-1,\ell}(\mathbb{Z}) \backslash SL(k-1,\mathbb{Z})}$$

$$\cdot \int_0^1 \cdots \int_0^1 F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k-\ell} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & 0 & u_{2,k-\ell} & \cdots & u_{2,k-1} & u_{2,k} \\ & & \ddots & & & & \vdots & \vdots \\ & & & & & & \vdots & \vdots \\ & & & & & & 1 & u_{k-1,k} \\ & & & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z \right)$$

$$\cdot\, e^{-2\pi i \left(m_{k-1} u_{k-1,k} \,+\, \cdots \,+\, m_{k-\ell} u_{k-\ell,\; k-\ell+1} \right)} \prod_{i=1}^{k-1} \prod_{j=\max(k-\ell,\, i+1)}^{k} du_{i,j}$$

$$+ \sum_{m_{k-1} \neq 0} \cdots \sum_{m_1 \neq 0} \sum_{\gamma_{k-1} \in U_{k-1}(\mathbb{Z}) \backslash SL(k-1,\mathbb{Z})}$$

$$\cdot \int_0^1 \cdots \int_0^1 F\left(\begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & u_{2,k-1} & u_{2,k} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z \right)$$

$$\cdot\, e^{-2\pi i \left(m_{k-1} u_{k-1,k} \,+\, \cdots \,+\, m_1 u_{1,2} \right)} \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} du_{i,j}.$$

Proof: See [Goldfeld, 2006], [Ichino, Yamana].

To simplify the later exposition, we define the last sum above as the non degenerate term associated to the automorphic function F. Here is a formal definition of both the degenerate and the non degenerate terms..

Definition 8.4 (degenerate and non degenerate term) Let F be an automorphic function for $SL(k,\mathbb{Z})$ with $k \geq 2$. We define the non degenerate term associated to F to be

$$\mathrm{ND}(F) := \sum_{m_{k-1}\neq 0} \cdots \sum_{m_1 \neq 0} \sum_{\gamma \in U_{k-1}(\mathbb{Z})\backslash SL(k-1,\mathbb{Z})}$$

$$\cdot \int_0^1 \cdots \int_0^1 F\left(\begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & u_{2,k-1} & u_{2,k} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z\right)$$

$$\cdot\, e^{-2\pi i\left(m_{k-1}u_{k-1,k} \,+\, \cdots \,+\, m_1 u_{1,2}\right)} \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} du_{i,j}.$$

The other terms in the expansion for $F(z)$ given in theorem 8.3 are defined to be the degenerate term which is denoted $D(F)$.

Proposition 8.5 (Langlands) *The Eisenstein series $E(z,f;s)$ has constant term $C_{\mathcal{P}}$ (as defined in (4.6)) along the parabolic $\mathcal{P} = P_{n,n}$. Along all other parabolics, the constant term is 0.*

Proof: See theorems 6.2.1, 6.3.1 in [Shahidi, 2010] and proposition 10.10.3 in [Goldfeld, 2006]. $\square$

Proposition 8.6 *Let $z \in GL(2n,\mathbb{R})$ with Iwasawa decomposition*

$$z = \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,2n} \\ & 1 & \cdots & x_{2,2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix},$$

where Y_1, Y_2 are diagonal matrices in $GL(n,\mathbb{R})$. Let $E(z,f;s)$ be the Eisenstein series defined in (4.5). Then we have

$$E(z,f;s) = \sum_{\gamma \in \widetilde{P}_{2n-1,\,n-1}(\mathbb{Z})\backslash SL(2n-1,\mathbb{Z})} \left[\left|\frac{\mathrm{Det}(\mathfrak{m}_1)}{\mathrm{Det}(\mathfrak{m}_2)}\right|^{ns} f(\mathfrak{m}_1) f^*(\mathfrak{m}_2) \right.$$

$$\left. +\; c_s(f)\cdot \left|\frac{\mathrm{Det}(\mathfrak{m}_1)}{\mathrm{Det}(\mathfrak{m}_2)}\right|^{n(1-s)} f(\mathfrak{m}_1) f^*(\mathfrak{m}_2)\right] \cdot \Bigg|_{\begin{pmatrix} \gamma & \\ & 1\end{pmatrix}} \;+\; \mathrm{ND}(E).$$

Here

$$f^*(\mathfrak{m}_2) = \sum_{m_{2n-1}\neq 0} \cdots \sum_{m_{n+1}\neq 0} \frac{B(m_{2n-1},\dots,m_{n+1})}{\prod\limits_{k=1}^{n-1} |m_{2n-k}|^{\frac{k(n-k)}{2}}} \cdot W_{n,\nu}\left(MY_2\right)$$

$$\cdot\, e^{2\pi i\left(m_{2n-1}x_{2n-1,2n} + \quad \cdots \quad + m_{n+1}x_{n+1,n+2}\right)},$$

where B denotes the Fourier coefficient of f, the matrix M is the diagonal matrix

$$M = \operatorname{diag}\big((m_{2n-1}m_{2n-2}\cdots|m_{n+1}|, \ \ldots, \ m_{2n-1}m_{2n-2}, m_{2n-1}, 1)\big),$$

and $W_{n,\nu}$, is the Whittaker function (see (7.1)) in the Fourier expansion of f.

Proof of Proposition 8.6: There are two cases to consider.

CASE 1, $\ell \neq n-1$: Let

$$U_\ell := \begin{pmatrix} 1 & u_{2n-\ell,2n-\ell+1} & \cdots & u_{2n-\ell,\,2n-1} & u_{2n-\ell,\,2n} \\ & 1 & \cdots & u_{2n-\ell+1,\,2n-1} & u_{2n-\ell+1,\,2n} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{2n-1,\,2n} \\ & & & & 1 \end{pmatrix}.$$

Then we see that for $L = 2n-\ell-1$, we have

$$\begin{pmatrix} u_{1,\,2n-\ell} & \cdots & u_{1,\,2n} \\ u_{2,\,2n-\ell} & \cdots & u_{2,\,2n} \\ \vdots & \cdots & \vdots \\ u_{2n-\ell-1,\,2n-\ell} & \cdots & u_{2n-\ell-1,\,2n} \end{pmatrix} \cdot U_\ell$$

$$= \begin{pmatrix} u_{1,\,2n-\ell} & u_{1,\,2n-\ell}\, u_{2n-\ell,\,2n-\ell+1} + u_{1,\,2n-\ell+1} & \cdots & u_{1,\,2n-\ell}\, u_{2n-\ell,2n} + \cdots + u_{1,2n} \\ u_{2,\,2n-\ell} & u_{2,\,2n-\ell}\, u_{2n-\ell,\,2n-\ell+1} + u_{2,\,2n-\ell+1} & \cdots & u_{2,\,2n-\ell}\, u_{2n-\ell,2n} + \cdots + u_{2,2n} \\ \vdots & \vdots & \cdots & \vdots \\ u_{L,2n-\ell} & u_{L,2n-\ell}\, u_{2n-\ell,\,2n-\ell+1} + u_{L,\,2n-\ell+1} & \cdots & u_{L,2n-\ell}\, u_{2n-\ell,2n} + \cdots + u_{L,2n} \end{pmatrix}.$$

In the above, we make the change of variables:

$$\begin{array}{c|c} u_{1,2n-\ell}\, u_{2n-\ell,2n-\ell+1} \ + \ u_{1,2n-\ell+1} \ \to \ u_{1,2n-\ell+1} & u_{1,2n-\ell}\, u_{2n-\ell,2n} + \cdots + u_{1,2n} \ \to \ u_{1,2n} \\ \vdots & \vdots \\ u_{L,2n-\ell}\, u_{2n-\ell,2n-\ell+1} + u_{L,2n-\ell+1} \ \to \ u_{L,2n-\ell+1} & u_{L,2n-\ell}\, u_{2n-\ell,2n} + \cdots + u_{L,2n} \ \to \ u_{L,2n} \end{array}$$

It follows from the above calculations that with the notation $E(z) := E(z, f; s)$, we have

$$\int_0^1 \cdots \int_0^1 E\left(\begin{array}{ccccccc} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{L=2n-\ell-1}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{\ell+1}} \\ \end{array} \right.$$

$$\int_0^1 \cdots \int_0^1 E\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,2n-\ell} & \cdots & u_{1,2n} \\ & 1 & \cdots & 0 & u_{2,2n-\ell} & \cdots & u_{2,2n} \\ & & \ddots & \vdots & \vdots & \cdots & \vdots \\ & & & 1 & u_{L,2n-\ell} & \cdots & u_{L,2n} \\ & & & & 1 & \cdots & u_{L+1,2n} \\ & & & & & \ddots \ \vdots & \vdots \\ & & & & & 1 & u_{2n-1,2n} \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z \right)$$

$$\cdot e^{-2\pi i (m_{2n-1} u_{2n-1,2n} + \cdots + m_{2n-\ell} u_{2n-\ell, 2n-\ell+1})} \, d^{\times} u$$

is equal to (for $I_{L\times L}$, the $L \times L$ identity matrix)

$$\int_0^1 \cdots \int_0^1 E\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,2n-\ell} & \cdots & u_{1,2n} \\ & 1 & \cdots & 0 & u_{2,2n-\ell} & \cdots & u_{2,2n} \\ & & \ddots & \vdots & \vdots & \cdots & \vdots \\ & & & 1 & u_{L,2n-\ell} & \cdots & u_{L,2n} \\ & & & & 1 & \cdots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} I_{L\times L} & 0 \\ 0 & U_\ell \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z \right)$$

$$\cdot \, e^{-2\pi i (m_{2n-1} u_{2n-1,2n} + \cdots + m_{2n-\ell} u_{2n-\ell, 2n-\ell+1})} \, d^{\times} u,$$

which is 0 by proposition 8.5.

CASE 2, $\ell = n-1$: In this case, define

$$\mathcal{K}_n := \int_0^1 \cdots \int_0^1 E\left(\begin{pmatrix} I_{n\times n} & \begin{pmatrix} u_{1,n+1} & \cdots & u_{1,2n} \\ \vdots & \cdots & \vdots \\ u_{n,n+1} & \cdots & u_{n,2n} \end{pmatrix} \\ & I_{n\times n} \end{pmatrix} \begin{pmatrix} I_{n\times n} & 0 \\ 0 & U_{n-1} \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z \right)$$

$$\cdot \, e^{-2\pi i (m_{2n-1} u_{2n-1,2n} + \cdots + m_{n+1} u_{n+1, n+2})} \, d^{\times} u.$$

Let $z' = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z$. We write z' in Iwasawa form, so that

$$z' = \begin{pmatrix} X_1' & X_2' \\ & X_3' \end{pmatrix} \begin{pmatrix} Y_1' & \\ & Y_2' \end{pmatrix} \subset GL(2n, \mathbb{R}).$$

Then

$$\begin{pmatrix} I_{n\times n} & \\ & U_{n-1} \end{pmatrix} \cdot z' \ = \begin{pmatrix} X_1' & X_2' \\ & U_{n-1}X_3' \end{pmatrix} \begin{pmatrix} Y_1' & \\ & Y_2' \end{pmatrix}.$$

Here X_3' is a unipotent matrix with off diagonal entries $x_{i,j}'$ and $U_{n-1}X_3'$ is equal to

$$\begin{pmatrix} 1 & u_{n+1,n+2} & \cdots & u_{n+1,2n-1} & u_{n+1,2n} \\ & 1 & \cdots & u_{n+2,2n-1} & u_{n+2,2n} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{2n-1,2n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x'_{n+1,n+2} & \cdots & x'_{n+1,2n-1} & x'_{n+1,2n} \\ & 1 & \cdots & x'_{n+2,2n-1} & x'_{n+2,2n} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & x'_{2n-1,2n} \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} u_{n+1,n+2} + x'_{n+1,n+2} & & \cdots & u_{n+1,2n} + \cdots + x'_{n+1,2n} \\ & 1 & \cdots & u_{n+2,2n} + \cdots + x'_{n+2,2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix},$$

which becomes

$$\begin{pmatrix} 1 & u_{n+1,n+2} & \cdots & u_{n+1,2n-1} & u_{n+1,2n} \\ & 1 & \cdots & u_{n+2,2n-1} & u_{n+2,2n} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{2n-1,2n} \\ & & & & 1 \end{pmatrix}$$

after making the following change of variables:

$$u_{n+1,n+2} + x'_{n+1,n+2} \ \rightarrow \ u_{n+1,n+2}, \qquad \ldots \qquad , u_{2n-1,2n} + x'_{2n-1,2n} \ \rightarrow \ u_{2n-1,2n}.$$

Consequently,

$$\mathcal{K}_n = \int_0^1 \cdots \int_0^1 \left[\int_0^1 \cdots \int_0^1 E\left(\begin{pmatrix} I_{n\times n} & \begin{matrix} u_{1,n+1} & \cdots & u_{1,2n} \\ \vdots & \cdots & \vdots \\ u_{n,n+1} & \cdots & u_{n,2n} \end{matrix} \\ & I_{n\times n} \end{pmatrix} \begin{pmatrix} X_1'Y_1' & X_2'Y_2' \\ & \\ 0 & U_{n-1}Y_2' \end{pmatrix} \right) \prod_{i=1}^{n} \prod_{j=n+1}^{2n} du_{i,j} \right]$$

$$\cdot\ e^{-2\pi i(m_{2n-1}u_{2n-1,2n} + \cdots + m_{n+1}u_{n+1,n+2})} \prod_{k=n+1}^{2n-1} \prod_{\ell=k+1}^{2n} du_{k,\ell}$$

$$\cdot\ e^{2\pi i(m_{2n-1}x'_{2n-1,2n} + \cdots + m_{n+1}x'_{n+1,n+2})}.$$

By proposition 8.5, the inner integral above is

$$\left|\frac{\det(Y_1')}{\det(Y_2')}\right|^{ns} f\Big(X_1'Y_1'\Big) f\Big(U_{n-1}Y_2'\Big) \ +\ c_s(f)\cdot \left|\frac{\det(Y_1')}{\det(Y_2')}\right|^{n(1-s)} f\Big(X_1'Y_1'\Big) f\Big(U_{n-1}Y_2'\Big).$$

Further, the outer integral picks up the Fourier coefficient of f and $\widetilde{f}$, respectively:

$$\int_0^1 \cdots \int_0^1 f\Big(U_{n-1}Y_2'\Big) e^{-2\pi i\left(m_{2n-1}u_{2n-1,2n} + \ \cdots \ m_{n+1}u_{n+1,n+2}\right)} \prod_{k=n+1}^{2n-1} \prod_{\ell=k+1}^{2n} du_{k,\ell}$$

$$= \frac{B(m_{2n-1}, \ldots, m_{n+1})}{\prod\limits_{k=1}^{n-1} |m_{2n-k}|^{\frac{k(n+k)}{2}}} \, W_{n,\nu}\left(MY_2'\right)$$

where $B(m_{2n-1}, \ldots, m_{n+1})$ is the Fourier coefficient of f.

□

Corollary 8.7 *Let $z \in GL(2n, \mathbb{R})$ as in proposition 8.6. Let $\widehat{E}_A^*(z, f; s)$ be the smoothed truncated Eisenstein series in definition 4.9, and set*

$$\widetilde{\Gamma}(2n-1) := \widetilde{P}_{2n-1,n-1}(\mathbb{Z})\backslash SL(2n-1, \mathbb{Z}).$$

Then we have

$$\widehat{E}_A^*(z,f;s) = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h\left(\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)z\right)\,<\,A}} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{ns}\left[1-\frac{A^{n/2}}{h(y)^{n/2}}\right] f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$+\frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h\left(\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)z\right)\,<\,A}} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n(2-s)}\left[1-\frac{A^{n/2}}{h(y)^{n/2}}\right] f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$+\sum_{\gamma\in\widetilde{\Gamma}(2n-1)}\left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n(1-s)} c_s(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$+\frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\sum_{\gamma\in\widetilde{\Gamma}(2n-1)}\left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n(s-1)} c_{2-s}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$-\sum_{\gamma\in\widetilde{\Gamma}(2n-1)}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{A^{-\frac{n}{2}w}}{w(w+1)}\left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n\left(1-s-\frac{w}{2}\right)} c_{s+\frac{w}{2}}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)} dw$$

$$-\frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\sum_{\gamma\in\widetilde{\Gamma}(2n-1)}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{A^{-\frac{n}{2}w}}{w(w+1)}\left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n\left(-1+s-\frac{w}{2}\right)}$$

$$\cdot\, c_{2-s+\frac{w}{2}}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\,\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)} dw$$

$$+\ ND\left(\widehat{E}_A^*\right).$$

Proof of corollary 8.7: The proof follows from the well known identity

$$\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{x^w}{w(w+1)}\,dw = \begin{cases} 1-\frac{1}{x}, & \text{if } x\ge 1,\\ 0, & \text{otherwise,}\end{cases} \tag{8.8}$$

together with the identity

$$\widehat{E}^*_A(z,f;s) = E(z,f;s) - A^{\frac{n}{2}}E(z,f;s-1/2) + \frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\cdot E(z,f;\,2-s)$$

(8.9)

$$-\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{A^{-\frac{n}{2}w}}{w(w+1)}E\left(z,f;\ s+\frac{w}{2}\right)\,dw$$

$$-\ \frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\cdot\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{A^{-\frac{n}{2}w}}{w(w+1)}E\left(z,f;\ 2-s+\frac{w}{2}\right)\,dw.$$

and proposition 8.6. □

§9 Coset representatives for $\widetilde{P}_{n,m}\backslash SL(n,\mathbb{Z})$:

Recall definition 8.2 which states that

$$\widetilde{P}_{n,m}(\mathbb{Z}) := \left(\begin{pmatrix} & * & * & \cdots & * \\ SL(n-m,\mathbb{Z}) & \vdots & \vdots & \cdots & \vdots \\ & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}\right) \subset SL(n,\mathbb{Z}). \tag{9.1}$$

The main goal of this section is the proof of the following proposition.

Proposition 9.2 *Let $A, A' \in SL(n,\mathbb{Z})$. Then there exists a matrix $X \in \widetilde{P}_{n,m}(\mathbb{Z})$ such that*

$$A = XA'$$

if and only if for each $1 \le k \le m$, the matrices A, A' have the same $k\times k$ minors on the bottom k rows.

The proof of proposition 9.2 is based on the following lemma.

Lemma 9.3 *Suppose $n \ge m \ge 1$ and $a_{i,j}, a'_{i,j} \in \mathbb{Z}$ (for $1 \le i \le m$, $1 \le j \le n$). Define matrices*

$$A = \Big(a_{i,j}\Big)_{1\le i\le m,\ 1\le j\le n}, \qquad A' = \Big(a'_{i,j}\Big)_{1\le i\le m,\ 1\le j\le n}.$$

If $\mathrm{rank}(A) = \mathrm{rank}(A') = m$*, then there exists a rational $m\times m$ matrix*

$$U = \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

such that

$$A = UA'$$

if and only if, for each $1 \le k \le m$*, the matrices* A, A' *have the same* $k \times k$ *minors in the bottom* k *rows.*

Proof of lemma 9.3: It is easy to show that if $A = UA'$ then the matrices A, A' have the same $k \times k$ minors in the bottom k rows. In the other direction, we use induction. Before proceeding with the induction we establish two claims.

Claim 9.4: Let $k \in \mathbb{Z}$ with $k \ge 2$. Let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix}, \quad A' = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix}$$

be two matrices in $GL(k, \mathbb{Z})$ satisfying $\det(A) = \det(A') \neq 0$. Then there exists

$$U = \begin{pmatrix} 1 & u_1 & u_2 & \cdots & u_{k-1} \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \in SL(k, \mathbb{Q}) \tag{9.5}$$

such that $A' = U \cdot A$.

To prove the claim note that $\det(A) = \det(A') \neq 0$ implies (by expanding in terms of cofactors $C_{i,1}$ along the first row) that $\det(A) = \sum_{i=1}^{k} \alpha_i C_{i,1} = \sum_{i=1}^{k} \alpha'_i C_{i,1} = \det(A')$ which implies that

$$\sum_{i=1}^{k} (\alpha_i - \alpha'_i) C_{i,1} = 0. \tag{9.6}$$

It immediately follows that if we set

$$\alpha'_i = \alpha_i + a_{2,i} u_1 + a_{3,i} u_2 + \cdots a_{k,i} u_{k-1}, \qquad (i = 1, 2, \ldots, k),$$

then we can find rational numbers $u_1, \ldots, u_{k-1}$ so that (9.6) is satisfied. Since $\det(A) \neq 0$, one then concludes that the matrix $U = A'A^{-1}$ must be of the form (9.5).

Claim 9.7 Let $n, k \in \mathbb{Z}$ with $2 \le k \le n$. Let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix}, \quad A' = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_n \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix},$$

be two matrices with integer coefficients. Assume that the $k \times k$ minors of A are the same as the $k \times k$ minors of A', and that at least one of these minors is non-zero. Then there exists

$$U = \begin{pmatrix} 1 & u_1 & u_2 & \cdots & u_{k-1} \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \in SL(k, \mathbb{Q}) \tag{9.8}$$

such that $A' = U \cdot A$.

To prove claim 9.7, note that, without loss of generality, we may assume the first minor

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix} = \det \begin{pmatrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix} \neq 0.$$

It immediately follows from claim 9.4 that there exists U of the form (9.8) such that

$$\begin{pmatrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix} = U \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix}$$

Similarly, for any $k < \ell \leq n$ we must have

$$\begin{pmatrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_{k-1} & \alpha'_\ell \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k-1} & a_{2,\ell} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & a_{k,\ell} \end{pmatrix} = U \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{k-1} & \alpha_\ell \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k-1} & a_{2,\ell} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & a_{k,\ell} \end{pmatrix}.$$

This is because a solution U must exist even if the determinants of both minors are zero, and $u_1, \ldots, u_{k-1}$ are already determined by the first $k-1$ columns and the previous computation. This implies that $\alpha'_\ell = \alpha_\ell + a_{2,\ell}u_1 + \cdots a_{k,\ell}u_{k-1}$ for all $a \leq \ell \leq n$. This completes the proof of claim 9.7.

Let's assume that

$$A = \Big(a_{i,j}\Big)_{1 \leq i \leq m,\ 1 \leq j \leq n}, \qquad A' = \Big(a'_{i,j}\Big)_{1 \leq i \leq m,\ 1 \leq j \leq n}$$

have the same $k \times k$ minors in the bottom k rows. We want to prove there exists a unipotent matrix U such that $A = UA'$. For the 1×1 minors on the bottom row this implies that $a_{m,i} = a'_{m,i}$ for $i = 1, 2, \ldots n$. For the 2×2 minors on the bottom two rows we may use claim 9.7 to conclude that

$$\begin{pmatrix} a'_{m-1,1} & \cdots & a'_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} = \begin{pmatrix} 1 & u_0 \\ & 1 \end{pmatrix} \begin{pmatrix} a_{m-1,1} & \cdots & a_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix},$$

for some rational number u_0.

Next, we consider the 3×3 minors on the bottom 3 rows. We want to prove there exists an upper triangular unipotent 3×3 matrix U such that

$$\begin{pmatrix} a'_{m-2,1} & \cdots & a'_{m-2,n} \\ a'_{m-1,1} & \cdots & a'_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} = U \cdot \begin{pmatrix} a_{m-2,1} & \cdots & a_{m-2,n} \\ a_{m-1,1} & \cdots & a_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}.$$

We may set

$$U = \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & u_0 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & u_0 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

which implies that we need to prove

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ & 1 & u_0 \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} a'_{m-2,1} & \cdots & a'_{m-2,n} \\ a'_{m-1,1} & \cdots & a'_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} &= \begin{pmatrix} a'_{m-2,1} & \cdots & a'_{m-2,n} \\ a_{m-1,1} & \cdots & a_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \\ &= \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{m-2,1} & \cdots & a_{m-2,n} \\ a_{m-1,1} & \cdots & a_{m-1,n} \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}. \end{aligned}$$

But this again follows from claim 9.7. It is clear that we may proceed by induction to complete the proof of lemma 9.3.

□

Proof of proposition 9.2: Let $A, A' \in SL(n,\mathbb{Z})$. Assume there exists $X \in \widetilde{P}_{n,m}(\mathbb{Z})$ such that $A = XA'$. Then it is obvious that for each $1 \le k \le m$ the matrices A, A' have the same $k \times k$ minors in the bottom k rows.

In the other direction, let

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{n-m,1} & \cdots & a_{n-m,n} \\ a_{n-m+1,1} & \cdots & a_{n-m+1,n} \\ \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} := \begin{pmatrix} M_{(n-m)\times n} \\ \\ N_{m\times n} \end{pmatrix},$$

$$A' = \begin{pmatrix} a'_{1,1} & \cdots & a'_{1,n} \\ \vdots & \cdots & \vdots \\ a'_{n-m,1} & \cdots & a'_{n-m,n} \\ a'_{n-m+1,1} & \cdots & a'_{n-m+1,n} \\ \vdots & \cdots & \vdots \\ a'_{n,1} & \cdots & a'_{n,n} \end{pmatrix} := \begin{pmatrix} M'_{(n-m)\times n} \\ \\ N'_{m\times n} \end{pmatrix}.$$

By lemma 9.3, there exists a rational $m \times m$ matrix

$$U = \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & \vdots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

such that

$$N = UN'.$$

Let

$$\big(X_1,\ X_2\big) := MA'^{-1}.$$

Clearly

$$X := \begin{pmatrix} X_1 & X_2 \\ & U \end{pmatrix} \in \widetilde{P}_{n,m}(\mathbb{Z}).$$

$\square$

§10 Bounds for Eisenstein series:

Fix the parabolic $\mathcal{P} = P_{n,n}$ in $GL(2n, \mathbb{R})$. Let $z = \mathfrak{n}\mathfrak{m}k \in GL(2n, \mathbb{R})$ with

$$\mathfrak{n} \in N^{\mathcal{P}}, \qquad \mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & \\ & \mathfrak{m}_2 \end{pmatrix} \in M^{\mathcal{P}}, \qquad k \in K.$$

Proposition 10.1 *Let f be a Maass form on $GL(n, \mathbb{Z})$ as in (4.1) with Whittaker function $W_{n,\nu}$, as in proposition 8.6, and set*

$$M = \operatorname{diag}\big(m_{2n-1}m_{2n-2}\cdots|m_{n+1}|,\ \ \ldots,\ \ m_{2n-1}m_{2n-2}, m_{2n-1}, 1\big).$$

Then

$$\sum_{\gamma \in \widetilde{P}_{2n-1,n-1}(\mathbb{Z})\backslash SL(2n-1,\mathbb{Z})} f(\mathfrak{m}_1) W_{n,\nu}\left(MY_2\right) \Bigg|_{\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}} \ll 1$$

for any $z \in GL(2n, \mathbb{R})$ for which $|x_{i,j}| \le 1$, and $y_i \ge 1$ with $(1 \le i \le j \le 2n)$. The upper bound above does not depend on M.

Proof of proposition 10.1: For $1 \le i \le n-1$, set

$$\mu_j^i = \begin{cases} -\frac{1}{i+1}, & \text{if } j = 2n - i - 1, \\ 1, & \text{if } j = 2n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mu^i = (\mu^i_1, \dots, \mu^i_{2n-1})$. Then

$$I_{\mu^i}(z) = \prod_{\ell=1}^{i} y_\ell^{\frac{2n}{i+1}(i+1-\ell)}.$$

It follows from lemma 5.7.2 [Goldfeld, 2006] that for

$$\gamma' = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \in SL(2n, \mathbb{Z}),$$

with $\gamma \in SL(2n-1, \mathbb{Z})$, that we have

$$I_{\mu^i}(\gamma' z) = \big|\big| e_{2n-i}\gamma' z \wedge \cdots \wedge e_{2n-1}\gamma' z \wedge e_{2n}\gamma' z \big|\big|^{\frac{2n}{i+1}} \cdot \big|\big| e_{2n-i} z \wedge \cdots \wedge e_{2n} z \big|\big|^{\frac{-2n}{i+1}} I_{\mu^i}(z). \tag{10.2}$$

Remark 10.3 *For any $2n \times 2n$ matrix M, the wedge product $\big|\big| e_{2n-i}M \wedge \cdots \wedge e_{2n}M \big|\big|^2$ is the sum of the squares of all the $(i+1) \times (i+1)$ minors of the last $i+1$ rows of M.*

Consequently, for $z \in GL(2n, \mathbb{R})$ in upper triangular Iwasawa form, we have

$$\big|\big| e_{2n-i} z \wedge \cdots \wedge e_{2n} z \big|\big| \;=\; \big[z\big]_{(2n-i,\dots,2n),\,(2n-i,\dots,2n)} = \prod_{\ell=1}^{i} y_\ell^{i+1-\ell}, \tag{10.4}$$

where for a matrix M, and for vectors of integers I, J, we denote the minor of M determined by the rows I and columns J as $[M]_{I,J}$ (see lemma 4.11).

It immediately follows from (10.2), (10.4) and remark 10.3 (with $M = \gamma' z$) that

$$\begin{aligned} I_{\mu^i}(\gamma' z) &= \big|\big| e_{2n-i}\gamma' z \wedge \cdots \wedge e_{2n-1}\gamma' z \wedge e_{2n}\gamma' z \big|\big|^{\frac{2n}{i+1}} \\ &= \Big(\big[C\big]^2_{\alpha_i,\,\beta_{i,1}} + \big[C\big]^2_{\alpha_i,\,\beta_{i,2}} + \cdots + \big[C\big]^2_{\alpha_i,\,\beta_{i,L}} \Big)^{\frac{n}{i+1}}, \end{aligned}$$

where

$$\begin{aligned} &C := \gamma' z, \\ &\alpha_i := (2n-i, 2n-i+1, \cdots, 2n), \\ &\beta_{i,1} := (1, 2, \dots, i+1), \\ &\beta_{i,2} := (1, 2, \dots, i,\, i+2), \\ &\qquad\vdots \\ &\beta_{i,L} := \alpha_i = (2n-i, 2n-i+1, \dots, 2n). \end{aligned}$$

By the Cauchy-Binet formula (4.12) and lemma 4.11 it follows that

$$
\begin{aligned}
\big[C\big]_{\alpha_i,\,\beta_{i,1}} &= \big[\gamma'\big]_{\alpha_i,\beta_{i,1}}\ \big[z\big]_{\beta_{i,1},\beta_{i,1}}\\
\big[C\big]_{\alpha_i,\,\beta_{i,2}} &= \big[\gamma'\big]_{\alpha_i,\beta_{i,1}}\ \big[z\big]_{\beta_{i,1},\beta_{i,2}}\ +\ \big[\gamma'\big]_{\alpha_i,\beta_{i,2}}\ \big[z\big]_{\beta_{i,2},\beta_{i,2}}\\
&\ \ \vdots\\
\big[C\big]_{\alpha_i,\alpha_i} &= \big[\gamma'\big]_{\alpha_i,\beta_{i,1}}\big[z\big]_{\beta_{i,1},\alpha_i}\ +\ \big[\gamma'\big]_{\alpha_i,\beta_{i,2}}\big[z\big]_{\beta_{i,2},\alpha_i}\\
&\qquad\qquad +\ \cdots\ +\ \big[\gamma'\big]_{\alpha_i,\alpha_i}\big[z\big]_{\alpha_i,\alpha_i}
\end{aligned}
$$

If $\big[\gamma'\big]_{\alpha_i,\beta_{i,1}} \neq 0$ then

$$\big[C\big]^2_{\alpha_i,\beta_{i,1}}\ \geq\ \big[z\big]_{\beta_{i,1},\beta_{i,1}}\ \geq\ 1.$$

If $\big[\gamma'\big]_{\alpha_i,\beta_{i,1}} = 0$ then

$$\big[C\big]^2_{\alpha_i,\beta_{i,2}}\ \geq\ \big[z\big]_{\beta_{i,2},\beta_{i,2}}\ \geq 1,$$

and by repeating the previous argument, we have

$$I_{\mu^i}\left(\gamma' z\right)\ \geq\ 1.$$

Hence

$$\max_{1\leq \ell\leq i} y_\ell\left(\gamma'\right) \geq 1.$$

We adopt the notation

$$y_i(\gamma') := y_i\big|_{\gamma'}, \qquad (\text{for } 1 \leq i \leq n-1).$$

Now it follows from corollary 7.4 that

$$
\begin{aligned}
W_{n,\nu}(MY_2)\Big|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)} &\ll \left(\max_{1\leq j<n} y_j(\gamma')\right)^{-Nn^3} \ll \left|I_{\mu^i}\left(\gamma' z\right)\right|^{-N}\\
&\ll \left(\big[C\big]^2_{\alpha_i,\beta_{i,1}}\ +\ \big[C\big]^2_{\alpha_i,\beta_{i,2}}\ +\ \cdots\ +\big[C\big]^2_{\alpha_i,\beta_{i,L}}\right)^{-N},
\end{aligned}
$$

for arbitrary large fixed $N > 1$, and every $1 \leq i < n$.

Since f is a cusp form, it is absolutely bounded, and it immediately follows from the above calculations and proposition 9.2 that

$$\sum_{\gamma\in\widetilde{P}_{2n-1,n-1}(\mathbb{Z})\backslash SL(2n-1,\mathbb{Z})} f(\mathfrak{m}_1)\,W_{n,\nu}\left(MY_2\right)\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$\ll \sum_{\gamma\in\widetilde{P}_{2n-1,n-1}(\mathbb{Z})\backslash SL(2n-1,\mathbb{Z})} W_{n,\nu}\left(MY_2\right)\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}$$

$$\ll \sum_{\gamma\in\widetilde{P}_{2n-1,n-1}(\mathbb{Z})\backslash SL(2n-1,\mathbb{Z})} \Big([C]^2_{\alpha_i,\beta_{i,1}} + [C]^2_{\alpha_i,\beta_{i,2}} + \cdots + [C]^2_{\alpha_i,\beta_{i,L}}\Big)^{-N}$$

$$\ll 1.$$

□

Proposition 10.5 *Let $z \in GL(k,\mathbb{R})/O(k,\mathbb{R})\cdot\mathbb{R}^\times$ with Iwasawa decomposition $z = xy$ as in (2.1) where $-\frac{1}{2} \le x_{i.j} < \frac{1}{2}$, $y_i \ge 1$, $(1 \le i < j \le k)$. Let*

$$M = \mathrm{diag}\big(m_{2n-1}m_{2n-2}\cdots|m_{n-1}|,\ \ldots,\ m_{2n-1}m_{2n-2}, m_{2n-1}, 1\big),$$

and for spectral parameters $\nu = (\nu_1, \ldots, \nu_{k-1}) \in \mathbb{C}^{k-1}$, define

$$F(z) := \sum_{\gamma\in U_{k-1}(\mathbb{Z})\backslash SL(k,\mathbb{Z})} \sum_{m_1=1}^{\infty}\cdots\sum_{m_{k-1}\neq 0} \frac{A(m_1,\ldots,m_{k-1})}{\prod\limits_{\ell=1}^{k-1}|m_\ell|^{\frac{\ell(k-\ell)}{2}}}\,W_{k,\nu}\left(My\right) \cdot e^{2\pi i(m_1x_{1,2}+m_2x_{2,3}+\cdots+m_{k-1}x_{k-1,k})}\Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)},$$

where the coefficients $A(m_1,\ldots,m_{k-1}) \in \mathbb{C}$ satisfy

$$\sum_{\left|m_1^{k-1}\cdots m_{k-1}\right|\le N} |A(m_1,\ldots,m_{k-1})|^2 \ll N.$$

Assume the Langlands parameters $(\alpha_1,\ldots,\alpha_k)$ attached to $W_{k,\nu}$ (see definition (2.3)) satisfy $|\Im(\alpha_i)| \ll |t|$ as $|t| \to \infty$, for all $1 \le i \le k$. Then for

$$\max_{1\le i\le k-1} y_i \ \ge\ \left(|t|^{1+\epsilon}\right)^{\frac{k(k-1)}{2}}$$

we have

$$F(z) \ll \left(\max_{1\le i\le k-1} y_i\right)^{-N}$$

for any fixed constant $N > 1$.

The proof of proposition 10.5 requires the following lemma.

Lemma 10.6 *Let* $\gamma' = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \in SL(k,\mathbb{Z})$ *for* $k \ge 2$. *Assume that* $\min\limits_{1\le i<k} y_i \ge 1$. *Then*

$$\max_{1\le i<k} y_i(\gamma') \;\ge\; \left(\max_{1\le i<k} y_i\right)^{\frac{2}{(k-1)k}}.$$

Proof of lemma 10.6: Set $\mu = (0,\dots,0,1)$ and let $z = xy \in GL(k,\mathbb{R})/(O(k,\mathbb{R})\cdot\mathbb{R}^\times)$. It follows from (2.2) that

$$I_\mu(z) = \prod_{\ell=1}^{k-1} y_\ell^{k-\ell} \;\ge\; \max_{1\le i\le k-1} y_i.$$

On the other hand,

$$I_\mu(\gamma' z) = \prod_{\ell=1}^{k-1} y_\ell(\gamma')^{k-\ell} \;\le\; \left(\max_{1\le i<k} y_i(\gamma')\right)^{\frac{(k-1)k}{2}}.$$

Claim: $I_\mu(\gamma' z) = I_\mu(z)$.

The claim is a consequence of the fact that

$$I_\mu(\gamma' z) = \prod_{i=0}^{k-2} \left|\left|e_{k-i}\gamma' z \wedge \cdots \wedge e_k \gamma' z\right|\right|^{-k\mu_{k-i-1}} \cdot \left|\left|e_{k-i} z \wedge \cdots \wedge e_k z\right|\right|^{k\mu_{k-i-1}} I_\mu(z),$$

and that $\mu_{k-1} = 1$ only if $i = 0$, (the other $\mu_i{}'$s are zero). Hence, only the last row of z contributes which establishes the claim.

It follows that

$$\max_{1\le i<k} y_i(\gamma') \ge \left(\max_{1\le i<k} y_i\right)^{\frac{2}{(k-1)k}}. \quad \Box$$

When $k = 3$, the above lemma 10.6 and proposition 10.5 are due to [Brumley-Templier] with a better power.

Proof of proposition 10.5: For every $1 \le i \le k-2$, we choose $\mu^i := (\mu_1^i, \dots, \mu_{k-1}^i)$ where

$$\mu_j^i = \begin{cases} -\frac{1}{i+1}, & \text{if } j = k-i-1, \\ 1, & \text{if } j = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

With this choice, we see that

$$(*) \qquad I_{\mu^i}(z) = \prod_{\ell=1}^{i} y_\ell^{\frac{i-\ell+1}{i+1}k}.$$

It follows from lemma 5.7.2 in [Goldfeld, 2006] that for

$$\gamma' = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \in SL(k, \mathbb{Z})$$

we have

$$I_{\mu^i}(\gamma' z) = \big|\big|e_{k-i}\gamma' z \wedge \cdots \wedge e_k\gamma' z\big|\big|^{\frac{k}{i+1}} \cdot \big|\big|e_{k-i} z \wedge \cdots \wedge e_k z\big|\big|^{-\frac{k}{i+1}} \cdot I_{\mu^i}(z).$$

Since

$$\big|\big|e_{k-i} z \wedge \cdots \wedge e_k z\big|\big| = \big[z\big]_{(k-i,\dots,k)(k-i,\dots,k)} = \prod_{\ell=1}^{i} y_\ell^{i-\ell+1},$$

if we combine the above with (*), we obtain

$$I_{\mu^i}(\gamma' z) = \big|\big|e_{k-i}\gamma' z \wedge \cdots \wedge e_k\gamma' z\big|\big|^{\frac{k}{i+1}}.$$

Since

$$I_{\mu^i}(\gamma' z) \ll \left(\max_{1\le i\le k-1} y_i(\gamma')\right)^{2k^2}$$

and

$$\max_{1\le i\le k-1} y_i \ge \big(t^{1+\epsilon}\big)^{\frac{k(k-1)}{2}},$$

it follows from corollary 7.4 and the above that

$$W_{k,\nu}\big(M\cdot y(\gamma)\big) \ll \max_{1\le i\le k-1} \big(m_i\, y_i(\gamma')\big)^{-N_1} \ll I_{\mu^i}(\gamma' z)^{\frac{-N_1}{4k^2}} \left(\max_{1\le i\le k-1} y_i\right)^{\frac{-N_1}{k(k-1)}}$$
$$= \big|\big|e_{k-i}\gamma' z \wedge \cdots e_k\gamma' z\big|\big|^{-N'} \cdot \left(\max_{1\le i\le k-1} y_i\right)^{-N}$$

where $N' = \frac{N_1}{4k(i+1)}$ and $N = \frac{N_1}{k(k-1)}$ are very large.

Hence, for $k \ge 4$, we obtain

$$F(z) \ll \sum_{i=1}^{k-2} \sum_{\gamma\in U_{k-1}(\mathbb{Z})\backslash SL(k-1,\mathbb{Z})} \sum_{m_1\ge 1} \cdots \sum_{m_{\ell-1}\ne 0} \frac{|A(m_1,\dots,m_{k-1})|}{\prod_{\ell=1}^{k-1} |m_\ell|^{\frac{\ell(k-\ell)}{2}}}$$
$$\cdot \big|\big|e_{k-i}\gamma' z \wedge \cdots \wedge e_k\gamma' z\big|\big|^{-N'} (\max y_i)^{-N}$$
$$\ll \left(\max_{1\le i\le k-1} y_i\right)^{-N},$$

by theorem 11.3.2 in [Goldfeld, 2006] and similar arguments as in the proof of proposition 10.1.

For $k = 3$ and

$$\gamma' = \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix}$$

we have

$$y_1(\gamma') = |cz_2 + d| y_1, \qquad y_2(\gamma') = \frac{y_2}{|cz_2 + d|^2}.$$

Hence

$$y_1(\gamma')^2\, y_2(\gamma') = y_1^2 y_2.$$

It follows that

$$\begin{aligned} \max_{1\le j\le 2} m_j y_j(\gamma') \; &\ge \; \Big(m_1^2\, m_2\, y_1(\gamma')^2\, y_2(\gamma')\Big)^{\frac{1}{3}} \; = \; \big(m_1^2 m_2\, y_1^2\, y_2\big)^{\frac{1}{3}} \\ &\ge \big(m_1^2 m_2\big)^{\frac{1}{3}} \end{aligned}$$

and

$$\begin{aligned} W\left(M\, y(\gamma')\right) &\ll \Big(\max_{1\le j\le 2} m_j\, y_j(\gamma')\Big)^{-1} \cdot \Big(\max_{1\le j\le 2} y_j(\gamma')\Big)^{-N_1} \\ &\ll \big(m_1^2 m_2\big)^{-\frac{1}{3}} \cdot \big|\big|e_2\gamma' z \wedge e_3 \gamma' z\big|\big|^{-N'} \Big(\max_{1\le j\le 2} y_j\Big)^{-N}, \end{aligned}$$

for any large N, N'. Hence

$$\begin{aligned} F(z) \ll &\sum_{\gamma\in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \; \sum_{m_1\ge 1} \; \sum_{m_2\ne 0} \frac{|A(m_1,m_2)|}{|m_1|^{\frac{5}{3}}\, |m_2|^{\frac{4}{3}}} \; \big|\big|e_2\gamma' z \wedge e_3\gamma' z\big|\big|^{-N'} \cdot \Big(\max_{1\le j\le 2} y_j\Big)^{-N} \\ &\ll \Big(\max_{1\le j\le 2} y_j\Big)^{-N}. \end{aligned}$$

☐

Proposition 10.7 *Let us rewrite the identity for* $\widehat{E}_A^*$ *given in corollary 8.7 as*

$$\widehat{E}_A^*(z, f; s) = A_1 + A_2 + A_3 + A_4 - A_5 - A_6 + ND\left(\widehat{E}_A^\star\right),$$

where

$$A_1 = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1) \\ h\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z\right) < A}} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{ns} \left[1 - \frac{A^{n/2}}{h(y)^{n/2}}\right] f(\mathfrak{m}_1) f^*(\mathfrak{m}_2)\, \Bigg|_{\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}}$$

and A_2, A_3, A_4, A_5, A_6 *are defined similarly. Recall that* $h(z) = \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|$. *The following bounds hold.*

(i) *If* $h(z) \ge A$ *and* $\min\limits_{1\le i<2n} y_i \ge \frac{\sqrt{3}}{2}$, *then we have* $A_1, A_2 \ll A^n \ll A^{\frac{n}{2}} h(z)^{\frac{n}{2}} \left(\frac{h(z)}{A}\right)^{-\frac{1}{2}}$,
$A_i \ll 1$ *for* $3 \le i \le 6$.

(ii) *If* $h(z) \le A$ *and* $\min\limits_{1\le i<2n} y_i \ge \frac{\sqrt{3}}{2}$, *then*

$$A_1 \; \ll \; A^{\frac{n}{2}} \cdot \left(y_n^n y_{n+1}^{n-1} y_{n+2}^{n-2} \cdots y_{2n-1}\right)^{\frac{n}{2}} := A', \qquad A_2 \ll A', \qquad A_3, A_4, A_5, A_6 \ll 1.$$

(iii) *If* $\max\limits_{1\le i<2n} y_i \ge \left(|t|^{1+\epsilon}\right)^{n(2n-1)}$ *and* $\min\limits_{1\le i<2n} y_i \ge \frac{\sqrt{3}}{2}$, *and* $g \in \mathbb{C}_0^\infty([1,2])$, *then*

$$\int_0^\infty g\left(\frac{A}{\beta}\right) ND\left(\widehat{E}_A^*(z, f; 1+it)\right) \frac{dA}{A} \ll |t|^{-N}$$

for any fixed $N > 1$.

Proof of proposition 10.7:

Part (i) We first prove part (i) for A_1. It follows from proposition 10.1 that

$$A_1 \ll A^n \sum_{\gamma \in \widetilde{\Gamma}(2n-1)} |f(\mathfrak{m}_1) f^*(\mathfrak{m}_2)| \Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)} \ll \; A^n \ll A^{\frac{n}{2}} h(z)^{\frac{n}{2}} \left(\frac{h(z)}{A}\right)^{-\frac{1}{2}}.$$

Next, we prove the bounds for A_2. Recall that $s = 1 + it$. Then

$$\left|\frac{\Lambda(2ns - 2n,\ f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - 2n,\ f \times \widetilde{f}\,)}\right| \;=\; 1,$$

the same argument as the above shows $A_2 \ll \; A^n \ll A^{\frac{n}{2}} h(z)^{\frac{n}{2}} \left(\frac{h(z)}{A}\right)^{-\frac{1}{2}}$.

To obtain the bound for A_3 we note that

$$|c_s(f)| = \left|\frac{\Lambda(2ns - n,\ f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - n,\ f \times \widetilde{f}\,)}\right| \ll 1,$$

from which it follows that

$$A_3 \ll \sum_{\gamma \in \widetilde{\Gamma}(2n-1)} \left|f(\mathfrak{m}_1) f^*(\mathfrak{m}_2)\right| \Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)} \ll 1.$$

by proposition 10.1. To obtain the bound for A_4, note that

$$\left|\frac{\Lambda(2ns - 2n,\ f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - 2n,\ f \times \widetilde{f}\,)}\right| \;=\; 1, \qquad |c_{2-s}(f)| \ll 1,$$

so by the same argument as for A_3, we obtain $A_4 \ll 1$.

To obtain the bound for A_5 we write $A_5 = A_{5,1} + A_{5,2}$ where (for $\gamma' = \left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)$)

$$A_{5,1} = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h(\gamma' z)<\frac{1}{A}}} \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n\left(1-s-\frac{w}{2}\right)} c_{s+\frac{w}{2}}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\Bigg|_{\gamma'}\, dw$$

and

$$A_{5,2} = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h(\gamma' z)\geq\frac{1}{A}}} \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n\left(1-s-\frac{w}{2}\right)} c_{s+\frac{w}{2}}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\Bigg|_{\gamma'}\, dw.$$

It is clear that $A_{5,2} \ll 1$ by proposition 10.1. For $A_{5,1}$ shift the line of integration to $\Re(w) = -\frac{1}{2}$, and write $A_{5,1} = A_{5,1,1} + A_{5,1,2}$, where $A_{5,1,1}$ is the residue at $w = 0$ given by

$$A_{5,1,1} = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h(\gamma' z)<\frac{1}{A}}} h(z)^{n(1-s)} c_s(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\Bigg|_{\gamma'} \ll 1,$$

by proposition 10.1. Further

$$A_{5,1,2} = \sum_{\substack{\gamma\in\widetilde{\Gamma}(2n-1)\\ h(\gamma' z)<\frac{1}{A}}} \frac{1}{2\pi i}\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} \left|\frac{\det(\mathfrak{m}_1)}{\det(\mathfrak{m}_2)}\right|^{n\left(1-s-\frac{w}{2}\right)} c_{s+\frac{w}{2}}(f)\cdot f(\mathfrak{m}_1)f^*(\mathfrak{m}_2)\Bigg|_{\gamma'}\, dw,$$

and again $A_{5,1,2} \ll 1$ by proposition 10.1.

The case of A_6 is similar to A_5. The only difference is that

$$\left|\frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\right| = 1.$$

The same arguments as for A_5 then give $A_6 \ll 1$.

Part (ii) Next we prove the bounds in part (ii) of proposition 10.7. As in the proof of part (i), let

$$\mu' = \Big(0,\ldots,0,\ \underbrace{-\frac{1}{2n}}_{n^{th}\text{ position}},\ 0,\ldots,\ \underbrace{\frac{1}{2}}_{(2n-1)^{th}\text{ position}}\Big).$$

Then $I_{\mu'}(z) = y_1^{n-1}y_2^{n-2}\cdots y_{n-1} \gg 1$ and $I_{\mu'}(\gamma' z) = Z\cdot I_{\mu'}(z)$. The proof of proposition 4.10 shows that $Z \geq 1$. Hence, $I_{\mu'}(\gamma' z) \geq I_{\mu'}(z)$.

Now

$$\left(\max_{1\le i<n-1} y_i(\gamma' z)\right)^{n^3} \ge I_{\mu'}(\gamma' z),$$

and

$$A_1 \ll A^{\frac{n}{2}} \sum_{\gamma\in\widetilde{\Gamma}(2n-1)} h(z)^{\frac{n}{2}} \left|f^*(\mathfrak{m}_2)\right| \Bigg|_{\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)}.$$

By proposition 4.10

$$h(\gamma' z)^{\frac{n}{2}} \le h(z)^{\frac{n}{2}} = \left(y_n^n y_{n+1}^{n-1} \cdots y_{2n-1}\right)^{\frac{n}{2}} \left(y_1 y_2^2 \cdots y_{n-1}^{n-1}\right)^{\frac{n}{2}},$$

and since

$$f^*\left(\mathfrak{m}_2(\gamma')\right) \ll \left(\max_{1\le i<n} y_i\left(\gamma' z\right)\right)^{-N-n^6}$$

we have

$$\begin{aligned} A_1 &\ll A' \left(y_1 y_2^2 \cdots y_{n-1}^{n-1}\right)^{\frac{n}{2}} \cdot \sum_{\gamma\in\widetilde{\Gamma}(2n-1)} \left(\max_{1\le i<n} y_i\left(\gamma' z\right)\right)^{-N-n^6} \\ &\ll A' \left(y_1 y_2^2 \cdots y_{n-1}^{n-1}\right)^{\frac{n}{2}} I_{\mu'}(z)^{-n^3} \cdot \sum_{\gamma\in\widetilde{\Gamma}(2n-1)} \left(\max_{1\le i<n} y_i\left(\gamma' z\right)\right)^{-N} \\ &\ll A' \end{aligned}$$

The same argument also shows that $A_2 \ll A'$. The proof of the bounds $A_3, A_4, A_5, A_6 \ll 1$, are the same as in the proofs in part (i).

Part (iii) Finally we prove the third part of proposition 10.7. We have

$$\begin{aligned} &\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}_A^*(z,f;s)\, \frac{dA}{A} \\ &\quad = E_0(z,f;s)\int_0^\infty g(A)\frac{dA}{A} - \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\beta^{-\frac{n}{2}w}}{w(w+1)} E\left(z,f;\ s+\frac{w}{2}\right)\widetilde{g}\left(-\frac{n}{2}w\right)\, dw \\ &\quad - \frac{\Lambda(2ns-2n,\ f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f\times\widetilde{f}\,)}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\beta^{-\frac{n}{2}w}}{w(w+1)} E\left(z,f;\ 2-s+\frac{w}{2}\right)\widetilde{g}\left(-\frac{n}{2}w\right)\, dw, \end{aligned}$$

where

$$E_0(z,f;s) := E(z,f;s) - A^{\frac{n}{2}} E\left(z,f;s-\frac{1}{2}\right) + \frac{\Lambda(2ns-2n, f\times\widetilde{f}\,)}{\Lambda(1+2ns-2n, f\times\widetilde{f}\,)} E(z,f;2-s).$$

Now, we are assuming that $\max\limits_{1\le i<2n} y_i \ge \left(|t|^{1+\epsilon}\right)^{n(2n-1)}$. By proposition 10.5, for $\Re(s')$ fixed with $s' \in \mathbb{C}$, $|\Im(s')| \le |t|^{1+\epsilon}$, we have $ND\left(E(z,f;s')\right) \ll |t|^{-N}$. The conclusion follows.

□

Corollary 10.8 *Let $g \in \mathbb{C}_c^\infty\left([1,2]\right)$.*

(i) *If $h(z) \ge \beta$ and $\min\limits_{1\le i<2n} y_i \ge \frac{\sqrt{3}}{2}$, then*

$$\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}_A^*(z,f;s)\,\frac{dA}{A} \ll \beta^{\frac{n}{2}} h(z)^{\frac{n}{2}} \left(\frac{h(z}{\beta}\right)^{-\frac{1}{2}}.$$

(ii) *If $h(z) \le \beta$ and $\min\limits_{1\le i<2n} y_i \ge \frac{\sqrt{3}}{2}$, and $\max\limits_{1\le i<2n} y_i \ge \left(t^{1+\epsilon}\right)^{n(2n-1)}$, then*

$$\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}_A^*(z,f;s)\,\frac{dA}{A} \;\ll\; \beta' = \beta^{\frac{n}{2}} \left(y_n^n y_{n+1}^{n-1} \cdots y_{2n-1}\right)^{\frac{n}{2}}.$$

Proof: This follows directly from proposition 10.7. □

§11 Upper bound for the integral $\mathcal{I}$:

The key idea for proving our main theorem 1.3 is in obtaining suitable upper and lower bounds for the integral $\mathcal{I}$ which we now define.

Definition 11.1 (The integral $\mathcal{I}$) *Let $t \in \mathbb{R}$ with $|t| \gg 1$ and set*

$$\beta = |t|^{n^{10}}, \qquad \delta = \beta^{-1}.$$

Let f be a cusp form on $GL(n)$ as in (4.1). Set

$$\mathfrak{h}^{2n} := GL(2n,\mathbb{R})/(O(2n,\mathbb{R})\cdot\mathbb{R}^\times).$$

We now define the integral $\mathcal{I} = \mathcal{I}_{g,\psi}(t)$ for test functions $g \in \mathbb{C}_0^\infty([1,2])$ (with $\widetilde{g}(n/2) = 1$) and $\psi : [0,\infty] \to \mathbb{C}$, where g, ψ are non-negative. In addition, we require that for some fixed $a > 0$, the Mellin transform $\widetilde{\psi}(\omega)$ is holomorphic in $-a \le \Re(w) \le a$, and satisfies $\widetilde{\psi}(\omega) \ll e^{-n|w|}$ in this strip. Let

$$\mathcal{I} \;:=\; \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \int\limits_{P_{2n-1,1}(\mathbb{Z})\backslash\mathfrak{h}^{2n}} \left| \int_0^\infty \widehat{E}_A^*(z,f;1+it)\, g\left(\frac{A}{\beta}\right) \frac{dA}{A}\right|^2 \cdot \psi\left(\frac{\det(z)}{\delta}\right) d^*z.$$

Theorem 11.2 (Upper bound for $\mathcal{I}$) *For $|t| \gg 1$ we have the upper bound*

$$\boxed{\mathcal{I}_{g,\psi}(t) \;\ll\; \left|L(1+2int, f\times\widetilde{f}\,)\right| \cdot \delta^{-\frac{1}{2}}\beta^{\frac{1}{2}+n}\left(\log|t|\right)^2 \cdot \left[1 + \left|L(1+2int, f\times\widetilde{f}\,)\right|\right],}$$

where the $\ll$-constant depends at most on n and f.

Proof of theorem 11.2: We begin with some standard computations involving Mellin inversion and Rankin-Selberg unfolding.

$$\mathcal{I}_{g,\psi}(t) = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \int\limits_{P_{2n-1,1}(\mathbb{Z})\backslash\mathfrak{h}^{2n}} \left|\int\limits_0^\infty \widehat{E}^*_A(z,f;1+it)\, g\left(\frac{A}{\beta}\right)\frac{dA}{A}\right|^2 \cdot\left(\int\limits_{2-i\infty}^{2+i\infty}\widetilde{\psi}(-w)\left(\frac{\det(z)}{\delta}\right)^w dw\right) d^*z$$

$$= \frac{\left|L(1+2int, f\times\widetilde{f}\,)\right|^2}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\widetilde{\psi}(-w)\,\delta^{-w} \cdot\int\limits_{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}}\left|\int_0^\infty g\left(\frac{A}{\beta}\right)\widehat{E}^*_A(z,f;1+it)\,\frac{dA}{A}\right|^2\cdot E_{P_{2n-1,1}}(z,w)\,d^*z\,dw.$$

Next, we shift the line of integration in the w-integral to $\Re(w)=\frac{1}{2}$. In doing so, we cross a pole of the maximal parabolic Eisenstein series $E_{P_{2n-1,1}}(z,w)$ at $w=1$. It follows that

$$\mathcal{I}_{g,\psi}(t) = c\cdot\frac{\left|L(1+2int, f\times\widetilde{f}\,)\right|^2\,\widetilde{\psi}(-1)}{\delta}\int\limits_{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}}\left|\int_0^\infty g\left(\frac{A}{\beta}\right)\widehat{E}^*_A(z,f;1+it)\,\frac{dA}{A}\right|^2 d^*z$$

$$+\frac{\left|L(1+2int, f\times\widetilde{f}\,)\right|^2}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\widetilde{\psi}(-w)\,\delta^{-w} \cdot\int\limits_{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}}\left|\int_0^\infty g\left(\frac{A}{\beta}\right)\widehat{E}^*_A(z,f;1+it)\,\frac{dA}{A}\right|^2\cdot E_{P_{2n-1,1}}(z,w)\,d^*z\,dw$$

$$:= \mathcal{I}^{(1)}_{g,\psi}(t)\ +\ \mathcal{I}^{(2)}_{g,\psi}(t).$$

Here $c\neq 0$ is a constant.

Next, we will obtain upper bounds for $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$. To bound $\mathcal{I}^{(1)}$ we break it into 3 pieces:

$$c^{-1}\cdot\mathcal{I}^{(1)} = \mathcal{I}^{(1)}_1+\mathcal{I}^{(1)}_2+\mathcal{I}^{(1)}_3,$$

where

$$\mathcal{I}^{(1)}_1 = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2\,\frac{\widetilde{\psi}(-1)}{\delta}\int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}\\ h(z)\geq\beta}}\left|\int_0^\infty g\left(\frac{A}{\beta}\right)\widehat{E}^*_A(z,f;1+it)\,\frac{dA}{A}\right|^2 d^*z,$$

$$\mathcal{I}_2^{(1)} = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \, \frac{\widetilde{\psi}(-1)}{\delta} \int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}\\ \max\limits_{1\le i<2n} y_i \,\ge\, (t^{1+\epsilon})^{n(2n-1)}\\ h(z)\le\beta}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}^*_A(z,f;1+it)\, \frac{dA}{A}\right|^2 d^*z,$$

$$\mathcal{I}_3^{(1)} = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \, \frac{\widetilde{\psi}(-1)}{\delta} \int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}\\ \max\limits_{1\le i<2n} y_i \,\le\, (t^{1+\epsilon})^{n(2n-1)}\\ h(z)\le\beta}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}^*_A(z,f;1+it)\, \frac{dA}{A}\right|^2 d^*z.$$

By corollary 10.8 (i) and the fact that $h(z)^n \prod\limits_{\ell=1}^{2n-1} y_\ell^{-\ell(2n-\ell)} \ll 1$, it follows that

$$\mathcal{I}_1^{(1)} \ll \delta^{-1}\beta^n \cdot \left|L(1+2int,\ f\times\widetilde{f}\,)\right|^2.$$

Similarly, by corollary 10.8 (ii)

$$\mathcal{I}_2^{(1)} \ll \delta^{-1}\beta^n(\log|t|)^2 \cdot \left|L(1+2int,\ f\times\widetilde{f}\,)\right|^2,$$

and by proposition 6.4, it follows that

$$\mathcal{I}_3^{(1)} \ll \delta^{-1}\beta^n(\log|t|)^2 \cdot \left|L(1+2int,\ f\times\widetilde{f}\,)\right|.$$

It remains to bound $\mathcal{I}^{(2)}$ which we also split into 3 parts:

$$2\pi i\cdot\mathcal{I}^{(2)} = \mathcal{I}_1^{(2)} + \mathcal{I}_2^{(2)} + \mathcal{I}_3^{(2)}.$$

Here

$$\mathcal{I}_1^{(2)} = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \int\limits_{\frac12-i\infty}^{\frac12+i\infty} \widetilde{\psi}(-w)\,\delta^{-w} \int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}\\ h(z)\ge\beta}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}^*_A(z,f;1+it)\, \frac{dA}{A}\right|^2 \cdot E_{P_{2n-1,1}}(z,w)\, d^*z\, dw,$$

$$\mathcal{I}_2^{(2)} = \left|L(1+2int, f\times\widetilde{f}\,)\right|^2 \int\limits_{\frac12-i\infty}^{\frac12+i\infty} \widetilde{\psi}(-w)\,\delta^{-w} \int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n}\\ \max\limits_{1\le i<2n} y_i \,\ge\, (t^{1+\epsilon})^{n(2n-1)}\\ h(z)\le\beta}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}^*_A(z,f;1+it)\, \frac{dA}{A}\right|^2 \cdot E_{P_{2n-1,1}}(z,w)\, d^*z\, dw,$$

$$\mathcal{I}_3^{(2)} = \big|L(1+2int, f \times \widetilde{f}\,)\big|^2 \int\limits_{\frac12 - i\infty}^{\frac12 + i\infty} \widetilde{\psi}(-w)\,\delta^{-w} \int\limits_{\substack{SL(2n,\mathbb{Z})\backslash\mathfrak{h}^{2n} \\ \max\limits_{1\le i<2n} y_i \,\le\, (t^{1+\epsilon})^{n(2n-1)} \\ h(z)\le\beta}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right) \widehat{E}^*_A(z,f;1+it)\,\frac{dA}{A}\right|^2$$

$$\cdot\, E_{P_{2n-1,1}}(z,w)\,d^*z\,dw.$$

By corollary 10.8 (i), proposition 3.4, and the fact that

$$h(z)^n \sum_{1\le k\le 2n} \left[\left(y_1y_2^2\ \cdots\ y_{2n-k}^{2n-k}\right)^{\frac12} \left(y_{2n-k+1}^{k-1}y_{2n-k+2}^{k-2}\ \ \cdots\ \ y_{2n-1}\right)^{\frac12} \right.$$

$$\left. +\left(y_1y_2^2\ \cdots\ y_{2n-k}^{2n-k}\right)^{\frac{k-1}{2n}} \left(y_{2n-k+1}^{k-1}y_{2n-k+2}^{k-2}\ \ \cdots\ \ y_{2n-1}\right)^{\frac{2n-k+1}{2n}} \right] \cdot \prod_{\ell=1}^{2n-1} y_\ell^{-\ell(2n-\ell)} \ \ll\ h(z)^{\frac12},$$

it follows that

$$\mathcal{I}_1^{(2)} \ll \delta^{-\frac12}\beta^{\frac12+n}(\log|t|)\cdot\big|L(1+2int, f\times\widetilde{f}\,)\big|^2.$$

By corollary 10.8 (ii) and proposition 3.4 it follows that

$$\mathcal{I}_2^{(2)} \ll \delta^{-\frac12}\beta^{\frac12+n}(\log|t|)\cdot\big|L(1+2int, f\times\widetilde{f}\,)\big|^2.$$

Next, by proposition 3.4, for $\frac{\sqrt{3}}{2}\le y_i\le |t|^{(1+\epsilon)n(2n-1)}$ $(1\le i<2n)$, we obtain the bound

$$E_{P_{2n-1,1}}(z,w) \ll \sum_{1\le k\le 2n} \left[\left(|t|^{1+\epsilon}\right)^{n(2n-1)\left(\frac{(2n-k)(2n-k+1)}{2}\ +\ \frac{(k-1)k}{2}\right)} \right.$$

$$\left. +\ \left(|t|^{1+\epsilon}\right)^{n(2n-1)\left(\frac{(2n-k)(2n-k+1)(k-1)}{4n}\ +\ \frac{(k-1)k(2n-k+1)}{4n}\right)} \right]$$

$$\ll\ |t|^{100n^4}(\log|t|)\ \ll\ \beta^{\frac12}.$$

Consequently, by proposition 6.4 it follows that

$$\mathcal{I}_3^{(2)} \ll \delta^{-\frac12}\beta^{n+\frac12}(\log|t|)^2\cdot\big|L(1+2int, f\times\widetilde{f}\,)\big|.$$

Finally, combining all the above bounds, we get

$$\mathcal{I}_{g,\psi}(t) \ll \big|L(1+2int, f\times\widetilde{f}\,)\big|\cdot\delta^{-\frac12}\beta^{n+\frac12}(\log|t|)^2\cdot\Big[1+\big|L(1+2int, f\times\widetilde{f}\,)\big|\Big].$$

□

§12 Lower bound for the integral $\mathcal{I}$:

The main aim of this section is to prove the following lower bound for the integral $\mathcal{I}$.

Theorem 12.1(Lower bound for $\mathcal{I}$) *Assume that the cusp form f for $SL(n,\mathbb{Z})$ has Langlands parameters $(i\alpha_1,\ldots,i\alpha_n)$. Assume further that $\widetilde{\psi}(w)$ vanishes (to order n^4) at $w=i(\alpha_j-\alpha_k)\neq 0$ for $1\le j,\ k\le n$ and that $\widetilde{\psi}(0)=1$. Then, under the same assumptions as in definition 11.1 and theorem 11.2, we have*

$$\boxed{\mathcal{I}_{g,\psi}(t) \;\gg\; \beta^n\delta^{-1}/(\log|t|),}$$

where the $\gg$-constant depends at most on n and f.

Remark: To show that it is possible for ψ to vanish to high order at finitely many pure imaginary points and also satisfy the conditions specified in definition 11.1 (i.e., positivity and exponential decay of the Mellin transform in a strip) can be seen as follows. Let Ψ be a function satisfying the conditions in definition 11.1. For $\lambda>0$, define $\psi(y):=\Psi(\lambda y)+\Psi(y)$. Clearly ψ satisfies the conditions specified in definition 11.1. Then

$$\widetilde{\psi}(w)=\big(\lambda^{-w}+1\big)\widetilde{\Psi}(w).$$

If we choose $\lambda=e^{\pi/\alpha}$ then it is clear that $\widetilde{\psi}(i\alpha)=0$. By iterating this procedure we can construct a test function ψ having the properties required in theorem 12.1. For example, we may initially choose a test function of type $\Psi(y)=y^Re^{-y^{1/2n}}$ (for some large positive R) and then apply the above procedures. We may, therefore, take

$$\widetilde{\psi}(w):=\frac{\Gamma\big(2n(R+w)\big)}{2^{4(n^2-L)}\,\Gamma(2nR)}\prod_{\substack{1\le j,k\le n\\ \alpha_j\neq\alpha_k}}\Big(e^{\frac{\pi w}{\alpha_j-\alpha_k}}+1\Big)^4.$$

where $L=\#\{j,k\mid \alpha_j=\alpha_k\}$. From now on we shall assume that $\widetilde{\psi}$ is of this form with $R\gg 1$ sufficiently large and independent of n and f.

We defer the proof of theorem 12.1 until later. A key ingredient of the proof is the following orthogonality condition stating that the degenerate part of the Fourier expansion of F is orthogonal to the non-degenerate part.

Proposition 12.2 *Suppose F is an automorphic function for $SL(k,\mathbb{Z})$ with $k\ge 2$ as in theorem 8.3. Define $D(F)$ and $ND(F)$ as in definition 8.4. Then*

$$\int\limits_{P_{k-1,1}(\mathbb{Z})\backslash\mathfrak{h}^k}\overline{D}(F)\cdot ND(F)\psi\left(\frac{\det(z)}{\delta}\right)\,d^\star z=0.$$

Proof of proposition 12.2: We have

$$\overline{D}(F)\cdot ND(F) = \sum_{m_1\neq 0}\cdots\sum_{m_{k-1}\neq 0}\ \sum_{\gamma\in U_{k-1}(\mathbb{Z})\backslash SL(k-1,\mathbb{Z})}$$

$$\cdot\, \overline{D}(F)\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z\right)\int_0^1\cdots\int_0^1 F\left(\begin{pmatrix}1 & u_{1,2} & & \cdots & u_{1,k}\\ & 1 & u_{2,3} & \cdots & u_{2k}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & u_{k-1,k}\\ & & & & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z\right)$$

$$\cdot\, e^{-2\pi i(m_1u_{1,2}+\cdots+m_{k-1}u_{k-1,k})}\ d^\times u.$$

Since

$$\bigcup_{\gamma\in U_{k-1}(\mathbb{Z})\backslash SL(k-1,\mathbb{Z})}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\ P_{k-1,1}\backslash\mathfrak{h}^k\ \equiv\ U_k(\mathbb{Z})\backslash\mathfrak{h}^k,$$

we have

$$\int\limits_{P_{k-1,1}\backslash\mathfrak{h}^k}\overline{D}(F)\cdot ND(F)\psi\left(\frac{\det(z)}{\delta}\right)d^* z = \sum_{m_1\neq 0}\cdots\sum_{m_{k-1}\neq 0}\ \int\limits_{U_k(\mathbb{Z})\backslash\mathfrak{h}^k}\overline{D}(F)\int_0^1\cdots\int_0^1$$

$$\cdot\, F\left(\begin{pmatrix}1 & u_{1,2} & & \cdots & u_{1,k}\\ & 1 & u_{2,3} & \cdots & u_{2,k}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & u_{k-1,k}\\ & & & & 1\end{pmatrix} z\right)\cdot e^{-2\pi i\left(m_1u_{1,2}+\cdots+m_{k-1}u_{k-1,k}\right)}$$

$$\cdot\, \psi\left(\frac{\det(z)}{\delta}\right)d^\times u\cdot d^\star z$$

$$= \int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{k-1}=0}^{\infty}\ \int\limits_{x_{1,2}=0}^{1}\cdots\int\limits_{x_{k-1,k}=0}^{1}\overline{D}(F)\cdot\int\limits_{u_{1,2}=0}^{1}\cdots\int\limits_{u_{k-1,k}=0}^{1}$$

$$\cdot\, F\left(\begin{pmatrix}1 & u_{1,2} & & \cdots & u_{1,k}\\ & 1 & u_{2,3} & \cdots & u_{2k}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & u_{k-1,k}\\ & & & & 1\end{pmatrix} z\right)\cdot e^{-2\pi i\left(m_1u_{1,2}+\cdots+m_{k-1}u_{k-1,k}\right)}$$

$$\cdot\, \psi\left(\frac{\det(z)}{\delta}\right)d^\times u\ d^\star z.$$

Consequently, it is enough to prove that

$$\mathcal{J} := \int\limits_{x_{1,2}=0}^{1}\cdots\int\limits_{x_{k-1,k}=0}^{1}\overline{D}(F)\ e^{2\pi i\left(m_1x_{1,2}+\cdots+m_{k-1}x_{k-1,k}\right)}\ d^\times x\ = 0.$$

We write

$$D(F) := D_1(F) + \cdots + D_{k-1}(F)$$

and

$$\mathcal{J} := \mathcal{J}_1 + \cdots + \mathcal{J}_{k-1},$$

where

$$\begin{aligned}\mathcal{J}_1 = \int\limits_{x_{1,2}=0}^{1} \cdots \int\limits_{x_{k-1,k}=0}^{1} \int\limits_{u_{1,k}=0}^{1} \cdots \int\limits_{u_{k-1,k}=0}^{1} F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k} \\ & 1 & \cdots & 0 & u_{2,k} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,k-1} & 0 \\ & 1 & \cdots & x_{2,k-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} y\right) \\ \cdot\, e^{2\pi i\left(m_1 x_{1,2}+\cdots+m_{k-1}x_{k-1,k}\right)}\, d^\times x\, d^\times u \\ = 0,\end{aligned}$$

after computing the $x_{k-1,k}$-integral.

For $1 \le \ell \le k-2$, we define

$$\begin{aligned}\mathcal{J}_{\ell+1} := \int\limits_{x_{1,2}=0}^{1} \cdots \int\limits_{x_{k-1,k}=0}^{1} F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k-\ell} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & 0 & u_{2,k-\ell} & \cdots & u_{2,k-1} & u_{2,k} \\ & & \ddots & \vdots & & \cdots & \vdots & \vdots \\ & & & & & & 1 & u_{k-1,k} \\ & & & & & & & 1 \end{pmatrix}\begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z\right) \\ \cdot \int\limits_{u_{1,k-l}=0}^{1} \cdots \int\limits_{u_{k-1,k}=0}^{1} e^{2\pi i\left(-m'_{k-1}u_{k-1,k}-\cdots-m'_{k-\ell}u_{k-\ell,k-\ell+1}\right)}\, d^\times u \\ \cdot\; e^{2\pi i\left(m_1 x_{1,2}+\cdots+m_{k-1}x_{k-1,k}\right)}\, d^\times x.\end{aligned}$$

Here

$$\begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z \;\equiv\; \begin{pmatrix} 1 & x_{1,2}(\gamma_{k-1}) & x_{1,3}(\gamma_{k-1}) & \cdots & x_{1,k}(\gamma_{k-1}) \\ & 1 & x_{2,3}(\gamma_{k-1}) & \cdots & x_{2,k}(\gamma_{k-1}) \\ & & \ddots & \cdots & \vdots \\ & & & 1 & x_{k-1,k}(\gamma_{k-1}) \\ & & & & 1 \end{pmatrix} y(\gamma_{k-1}) \quad \Big(\text{mod } \mathbb{Z}_k\, O(k,\mathbb{R})\Big).$$

Now $x_{1,k}, x_{2,k}, \ldots, x_{k-1,k}$ do not appear in $y(\gamma_{k-1})$. Note also that

$$x_{k-1,k}(\gamma_{k-1}) = a_{k-1,1}x_{1,k} + a_{k-1,2}x_{2,k} + \;\cdots\; + a_{k-1,k-1}x_{k-1,k},$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k-\ell} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & 0 & u_{2,k-\ell} & \cdots & u_{2,k-1} & u_{2,k} \\ & & \ddots & \vdots & & \cdots & \vdots & \vdots \\ & & & & & & 1 & u_{k-1,k} \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{1,2}(\gamma_{k-1}) & \cdots & x_{1,k-\ell}(\gamma_{k-1}) & \cdots & x_{1,k}(\gamma_{k-1}) \\ & 1 & \cdots & x_{2,k-\ell}(\gamma_{k-1}) & \cdots & x_{2,k}(\gamma_{k-1}) \\ & & \ddots & & \vdots & \vdots \\ & & & & 1 & x_{k-1,k}(\gamma_{k-1}) \\ & & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_{1,2}(\gamma_{k-1}) & \cdots & x_{1,k-\ell}(\gamma_{k-1})+u_{1,k-\ell} & \cdots & x_{1,k}(\gamma_{k-1})+u_{1,k-\ell}x_{k-\ell,k}(\gamma_{k-1})+\cdots+u_{1,k} \\ & 1 & \cdots & x_{2,k-\ell}(\gamma_{k-1})+u_{2,k-\ell} & \cdots & x_{2,k}(\gamma_{k-1})+u_{2,k-\ell}x_{k-\ell,k}(\gamma_{k-1})+\cdots+u_{2,k} \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & & 1 & x_{k-1,k}(\gamma_{k-1})+u_{k-1,k} \\ & & & & & 1 \end{pmatrix}.$$

Now $x_{1,k}, x_{2,k}, \ldots, x_{k-1,k}$ only appear in $x_{1,k}(\gamma_{k-1}), \ldots, x_{k-1,k}(\gamma_{k-1})$. If we make the change variables

$$\begin{aligned} x_{1,k}(\gamma_{k-1}) \ + \ u_{1,k-\ell}x_{k-\ell,k}(\gamma_{k-1}) \ + \ \cdots \ + \ u_{1,k} \ &\longrightarrow \ u_{1,k} \\ x_{2,k}(\gamma_{k-1}) \ + \ u_{2,k-\ell}x_{k-\ell,k}(\gamma_{k-1}) \ + \ \cdots \ + \ u_{2,k} \ &\longrightarrow \ u_{2,k} \\ \vdots \qquad & \\ x_{k-1,k}(\gamma_{k-1}) \ + \ u_{k-1,k} \ \longrightarrow \ u_{k-1,k}, & \end{aligned}$$

and compute the $x_{1,k}, x_{2,k}, \ldots, x_{k-1,k}$ integral, it follows that

$$a_{k-1,1} = \ \cdots \ = a_{k-1,k-2} = 0, \qquad a_{k-1,k-1} = 1,$$

which implies that

$$\gamma_{k-1} \ \in \ \widetilde{P}_{k-1,1}.$$

To complete the proof, we use induction. Assume

$$\gamma_{k-1} \in \widetilde{P}_{k-1,\ell} \backslash \widetilde{P}_{k-1,i-1}$$

for $2 \leq i \leq \ell$. We will show that $\gamma_{k-1} \in \widetilde{P}_{k-1,i}$.

Since

$$\widetilde{P}_{k-1,i-1} = \left\{ \begin{pmatrix} \gamma_{k-i} & * \\ 0 & U_{i-1} \end{pmatrix} \ \middle| \ \gamma_{k-i} \in SL(k-i, \mathbb{Z}) \right\},$$

we can assume γ_{k-1} is of the form $\begin{pmatrix} \gamma_{k-i} & 0 \\ 0 & U_{i-1} \end{pmatrix}$, by multiplying a suitable matrix in $\widetilde{P}_{k-1,\ell}$ on the left.

Let

$$\gamma_{k-i} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k-i} \\ \vdots & \cdots & \vdots \\ a_{k-i,1} & \cdots & a_{k-i,k-i} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \gamma_{k-i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,k-i+1} \\ & 1 & x_{2,3} & \cdots & x_{2,k-i+1} \\ & & \ddots & \dots & \vdots \\ & & & 1 & x_{k-i,k-i+1} \\ & & & & 1 \end{pmatrix} y$$

$$\equiv \begin{pmatrix} 1 & x_{1,2}(\gamma_{k-i}) & x_{1,3}(\gamma_{k-i}) & \cdots & x_{1,k-i+1}(\gamma_{k-i}) \\ & 1 & x_{2,3}(\gamma_{k-i}) & \cdots & x_{2,k-i+1}(\gamma_{k-i}) \\ & & \ddots & \dots & \vdots \\ & & & 1 & x_{k-i,k-i+1}(\gamma_{k-i}) \\ & & & & 1 \end{pmatrix} y(\gamma_{k-i}) \quad \Big(\mathrm{mod} Z_k\, O(k,\mathbb{R})\Big).$$

Here

$$x_{k-i,k-i+1}(\gamma_{k-i}) = a_{k-i,1}x_{1,k-i+1} \ + \ a_{k-i,2}x_{1,k-i+1} \ + \ \cdots \ + a_{k-i,k-i}x_{k-i,k-i+1}$$

does not appear in $y(\gamma_{k-i})$.

Next, change variables

$$\begin{aligned} & u_{1,k-i+1} + \ \cdots \ + x_{1,k-i+1}(\gamma_{k-i}) \ \longrightarrow \ u_{1,k-i+1} \\ & \quad \vdots \\ & u_{k-i,k-i+1} \ + \ x_{k-i,k-i+1}(\gamma_{k-i}) \ \longrightarrow \ u_{k-i,k-i+1}. \end{aligned}$$

Computing the $x_{1,k-i+1}, \quad \dots \quad x_{k-i,k-i+1}$ integrals, it follows that

$$a_{k-i,1} = \cdots = a_{k-i,k-i-1} = 0, \qquad a_{k-i,k-i} = 1,$$

which implies that

$$\gamma_{k-i} \in \widetilde{P}_{k-1,i}.$$

Hence

$$\mathcal{J}_{\ell+1} := \int\limits_{x_{1,2}=0}^{1} \cdots \int\limits_{x_{k-1,k}=0}^{1} F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k-\ell} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 & \cdots & 0 & u_{2,k-\ell} & \cdots & u_{2,k-1} & u_{2,k} \\ & & & \ddots & \vdots & \dots & \vdots & \vdots \\ & & & & & & 1 & u_{k-1,k} \\ & & & & & & & 1 \end{pmatrix} z \right)$$

$$\cdot \int\limits_{u_{1,k-l}=0}^{1} \cdots \int\limits_{u_{k-1,k}=0}^{1} e^{2\pi i\left(-m'_{k-1}u_{k-1,k}-\cdots-m'_{k-\ell}u_{k-\ell,k-\ell+1}\right)} \, d^{\times}u$$

$$\cdot \ e^{2\pi i\left(m_1x_{1,2}+\cdots+m_{k-1}x_{k-1,k}\right)} \, d^{\times}x.$$

Here $1 \le \ell \le k-2$ and

$$\begin{pmatrix} 1 & \overbrace{0 \ \cdots \ 0}^{k-\ell-1 \text{ column}} & u_{1,k-\ell} & \cdots & u_{1,k-1} & u_{1,k} \\ & 1 \ \cdots \ 0 & u_{2,k-\ell} & \cdots & u_{2,k-1} & u_{2,k} \\ & \ddots \ \vdots & \vdots & \cdots & \vdots & \vdots \\ & 1 & u_{k-\ell-1,k-\ell} & \cdots & u_{k-\ell-1,k-1} & u_{k-\ell-1,k} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & u_{k-1,k} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,k-\ell} & \cdots & x_{1,k} \\ & 1 & \cdots & x_{2,k-\ell} & \cdots & x_{2,k} \\ & & \ddots & \vdots & & \vdots \\ & & 1 & x_{k-\ell-1,k-\ell} & \cdots & x_{k-\ell-1,k} \\ & & & & \ddots & \vdots \\ & & & & 1 & x_{k-1,k} \\ & & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,k-\ell-1} & x_{1,k-\ell}+u_{1,k-\ell} & \cdots & u_{1,k}+\cdots+x_{1,k} \\ & \ddots & & \vdots & \vdots & & \vdots \\ & & & 1 & x_{k-\ell,k-\ell}+u_{k-\ell-1,k-\ell} & \cdots & x_{k-\ell-1,k}+u_{k-\ell-1,k-\ell}x_{k-\ell,k}+\cdots+u_{k-\ell-1,k} \\ & & & & \ddots & & \vdots \\ & & & & & & 1 \end{pmatrix}.$$

The change of variables

$$\begin{aligned} & x_{k-\ell-1,k-\ell} + u_{k-\ell-1,k-\ell} \longrightarrow u_{k-\ell-1,k-\ell} \\ & x_{k-\ell-1,k-\ell+1} + u_{k-\ell-1,k-\ell}\, x_{k-\ell,k-\ell+1} + u_{k-\ell-1,k-\ell+1} \longrightarrow u_{k-\ell-1,k-\ell+1} \\ & \qquad \vdots \\ & x_{k-\ell-1,k} + u_{k-\ell-1,k-\ell}\, x_{k-\ell,k} + \ \cdots \ + u_{k-\ell-1,k} \longrightarrow u_{k-\ell-1,k} \end{aligned}$$

and the computation of the $x_{k-\ell-1,k-\ell}$-integral shows that $\mathcal{J}_{\ell+1} = 0$.
$\square$

Corollary 12.3 *We have for any $N > 1$ that*

$$\mathcal{I}_{g,\psi}(t) \gg \left|L(1+2int,\ f \times \widetilde{f}\,)\right|^2 \int_0^\infty \cdots \int_0^\infty \sum_{N \le p \le 2N}$$

$$\cdot \left| \int_0^1 \cdots \int_0^1 \int_0^\infty \widehat{E}_A^* \left(\begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,2n} \\ & 1 & \cdots & u_{2,2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} y,\ f;\ 1+it \right) g\left(\frac{A}{\beta}\right) \frac{dA}{A} \right.$$

$$\left. \cdot\, e^{2\pi i \left(-p u_{2n-1,2n} \,-\, u_{2n-2,2n-1} - \ \cdots \ - u_{1,2}\right)} \, d^*u \right|^2 \psi\left(\frac{\mathrm{Det}(y)}{\delta}\right) \, d^*y,$$

where the sum above goes over primes $N \le p \le 2N$.

Proof: This follows from proposition 12.2 and definition 11.1 after noting that only the absolute value squared of the non-degenarate terms (see definition 8.4) contribute. The lower rank average in the Fourier expansion is gone because of the unfolding in the beginning of the proof of proposition 12.2. For the lower bound we only consider the contributions of the $(1, \ldots, 1, p)$ Fourier coefficients with $N \le p \le 2N$. Another key step in the proof of theorem 12.1 is the choice of N. $\square$

The following lemma is due to [Stade, 2002].

Lemma 12.4 *Fix an integer* $m \ge 2$*, a vector* $\nu \in \mathbb{C}^{m-1}$*, and the associated Whittaker function* $W_{m,\nu}$*. Let* $(\beta_1, \ldots, \beta_m)$ *be the Langlands parameters associated to* (m, ν) *as defined in (2.3). Then*

$$\int_0^\infty \cdots \int_0^\infty \Big| W_{m,\nu}(y) \Big|^2 (\operatorname{Det} y))^w \; d^*y \;=\; \frac{\pi^{-\frac{(m-1)m}{2}w} \cdot 2^{\frac{(m-1)m(m+1)}{6}}}{\Gamma\left(\frac{mw}{2}\right)}$$

$$\cdot \left| \prod_{j=1}^{m-1} \prod_{j \le k \le m-1} \cdot \pi^{-\frac{1}{2} - \frac{1}{2}(\beta_{m-k} - \beta_{m-k+j})} \, \Gamma\left(\frac{1 + \beta_{m-k} - \beta_{m-k+j}}{2} \right) \right|^{-2}$$

$$\cdot \prod_{j=1}^{m} \prod_{k=1}^{m} \Gamma\left(\frac{w + \beta_j + \overline{\beta_k}}{2} \right).$$

Next, we use corollary 12.3, lemma 12.4 and results from sieve theory to get a lower bound for $\mathcal{I}_{g,\psi}$.

Proof of theorem 12.1: By corollary 12.3, we need to consider the contribution from different parts of (8.9). The main contribution to the lower bound will come from the sum (over primes p) of the $(1, \ldots, 1, p)$ Fourier coefficients of $E(z, f; \frac{1}{2} + it)$. The Fourier coefficient for an individual prime p (denoted $a_1(1, \ldots, 1, p)$) is given by (see [Go], Proposition 10.9.3, [Shahidi], 2010)

$$a_1(1, \ldots, 1, p) \;:=\; \int_0^1 \cdots \int_0^1 E\left(\begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,2n} \\ & 1 & \cdots & u_{2,2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} y, \; f; \; \frac{1}{2} + it \right)$$

$$\cdot \, e^{2\pi i \left(-p u_{2n-1,2n} \, - \, u_{2n-2,2n-1} - \; \cdots \; - u_{1,2} \right)} \, d^*u$$

$$= \; \frac{c_n \, \overline{\lambda(p) \, \eta_{nit}(p)}}{p^{\frac{2n-1}{2}}} \, W_{2n,\nu}(My) \cdot \frac{1}{L(1 + 2int, \; f \times \widetilde{f}\,)},$$

where c_n is an absolute constant depending only on n,

$$\eta_s(n) := \sum_{\substack{ab=n\\ a,b\geq 1}} \left(\frac{a}{b}\right)^s, \qquad M = \begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

and $\lambda(p)$ is the p-th Hecke eigenvalue of f. The Langlands parameters of $E(z, f; \frac{1}{2} + it)$ are

$$\Big(i(nt+\alpha_1),\ \ldots\ , i(nt+\alpha_n),\ i(-nt+\alpha_1),\ \ldots\ , i(-nt+\alpha_n)\Big),$$

where $(i\alpha_1,\ \ldots\ , i\alpha_n)$ are the Langlands parameters of f. For simplicity, we assume $\Re(\alpha_j) = 0$ for $1 \leq j \leq n$.

Let

$$I_1^*(p) := \int_0^\infty \cdots \int_0^\infty \left| \int_0^\infty A^{\frac{n}{2}} \cdot a_1(1, \ldots, 1, p)\ g\left(\frac{A}{\beta}\right)\ \frac{dA}{A} \right|^2 \psi\left(\frac{\text{Det (z)}}{\delta}\right)\ d^*y$$

$$= \beta^n |\lambda(p)|^2\ |\eta_{nit}(p)|^2 \cdot \frac{|c_n|^2}{\left|L(1+2int,\ f \times \widetilde{f}\,)\right|^2} \cdot I_{1,1}^*(p),$$

where

(12.5)

$$I_{1,1}^*(p) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \widetilde{\psi}(-w)\ (p\delta)^{-w} \int_0^\infty \cdots \int_0^\infty |W_{2n,\nu}(y)|^2\ \ (\text{Det}\, z)^w\ \ d^*y\, dw$$

$$= \frac{\pi^{2n^2} 2^{\frac{(2n-1)2n(2n+1)}{6}}}{2\pi i} \prod_{j=1}^n \prod_{k=1}^n \left| \Gamma\left(\frac{1+i(\alpha_k-\alpha_j)}{2}\right) \Gamma\left(\frac{1+2int+i(\alpha_k-\alpha_j}{2}\right)\right|^{-2}$$

$$\cdot \int_{1-i\infty}^{1+i\infty} \frac{\widetilde{\psi}(-w)(p\delta)^{-w}}{\pi^{\frac{2n(2n-1)w}{2}}}\ \frac{\prod\limits_{j=1}^n \prod\limits_{k=1}^n \Gamma\left(\frac{w+i(\alpha_k-\alpha_j)}{2}\right)^2 \Gamma\left(\frac{w+2int+i(\alpha_k-\alpha_j}{2}\right) \Gamma\left(\frac{w-2int+i(\alpha_k-\alpha_j}{2}\right)}{\Gamma(nw)} dw.$$

Under the assumptions on $\widetilde{\psi}$, we can assume the above integrand is analytic at the points $w = i(\alpha_j - \alpha_k) \neq 0$ for $1 \leq j,\ k \leq n$. Shifting the line of integration in the w-integral above to $\Re(w) = -\frac{1}{\log\log R}$, we pick up residues from poles at

$$w = 0, \quad 2int + i(\alpha_j - \alpha_k), \quad -2int + i(\alpha_j - \alpha_k), \quad (1 \leq j, k \leq n).$$

At $w = 0$ there will be a pole of order $2L - 1$ where

$$L = \#\{(j,k) \mid \alpha_j = \alpha_k,\ 1 \leq j, k \leq n\}.$$

The residue at $w=0$ is

(12.6)

$$\begin{aligned}\operatorname*{Res}_{w=0}\Bigg[\frac{\widetilde{\psi}(-w)(p\delta)^{-w}}{\pi^{\frac{2n(2n-1)w}{2}}}\;\frac{\Gamma\left(\frac{w}{2}\right)^{2L}}{\Gamma(nw)}\prod_{\substack{1\le j<k\le n\\ \alpha_k\ne\alpha_j}}\Gamma\left(\frac{w+i(\alpha_k-\alpha_j)}{2}\right)^2\Gamma\left(\frac{w-i(\alpha_k-\alpha_j)}{2}\right)^2\\ \cdot\prod_{j=1}^{n}\prod_{k=1}^{N}\Gamma\left(\frac{w+i(2nt+\alpha_k-\alpha_j)}{2}\right)\Gamma\left(\frac{w-i(2nt-\alpha_k+\alpha_j)}{2}\right)\Bigg].\end{aligned}$$

Let $z=x+iy$. Now, by Stirling's asymptotic expansion [Whittaker-Watson, §12.33, 1927] we have for $|z|\to\infty$ and $|\arg z|<\pi$ the asymptotic expansion

$$\Gamma(z)=\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^{z}\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}\;+\;\mathcal{O}\left(\frac{1}{|z|^4}\right)\right).$$

It follows that for $-\frac{1}{2}\le x\le\frac{1}{2}$ and $|y|\to\infty$ that

$$\begin{aligned}\Gamma\left(\frac{x+iy}{2}\right)\Gamma\left(\frac{x-iy}{2}\right)=2\pi e^{-x}\left(\frac{x^2+y^2}{4}\right)^{\frac{x-1}{2}}\,e^{-|y|\left(\arctan\left(\frac{|y|}{x}\right)+\delta(x)\right)}\\ \cdot\left|1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}\;+\;\mathcal{O}\left(\frac{1}{|z|^4}\right)\right|^2,\end{aligned}$$

where

$$\delta(x)=\begin{cases}0 & \text{if } x>0,\\ \pi & \text{if } x<0.\end{cases}$$

Such asymptotic expansions can also be derived for derivatives of the Gamma function. Consequently, for any fixed integer $\ell\ge 0$, if we take the ℓ^{th} derivative in x, we get for $-\frac{1}{2}\le x\le\frac{1}{2}$ and $y\to\infty$, the asymptotic formula:

(12.7)

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\ell}\Gamma\left(\frac{x+iy}{2}\right)\Gamma\left(\frac{x-iy}{2}\right)=\left[\left(\frac{\partial}{\partial x}\right)^{\ell}2\pi e^{-x}\left(\frac{x^2+y^2}{4}\right)^{\frac{x-1}{2}}\,e^{-|y|\left(\arctan\left(\frac{|y|}{x}\right)+\delta(x)\right)}\right]\\ \cdot\left(1+\mathcal{O}\left(\frac{1}{|y|}\right)\right),\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\ell}\Gamma\left(\frac{x+iy}{2}\right)\Gamma\left(\frac{x-iy}{2}\right)\Bigg|_{x=0}&=\;\big(\log|y|/2\big)^{\ell}\,\frac{4\pi e^{-\frac{\pi|y|}{2}}}{|y|}\left(1+\mathcal{O}\left(\frac{1}{|y|}\right)\right)\\ &=\big(\log|y|/2\big)^{\ell}\,\Gamma\left(\frac{iy}{2}\right)\Gamma\left(\frac{-iy}{2}\right)\left(1+\mathcal{O}\left(\frac{1}{|y|}\right)\right).\end{aligned}$$

It follows from (12.7) that if we choose

$$N\delta = \prod_{j=1}^{n}\prod_{k=1}^{n} \exp\left(\log\left(\frac{|2nt+\alpha_k-\alpha_j|}{2}\right)\right) \sim (n|t|)^{n^2},$$

then, for any integer $\ell \geq 1$, we have

$$\left(\frac{\partial}{\partial w}\right)^{\ell}\left[(N\delta)^{-w}\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{w+i(2nt+\alpha_k-\alpha_j)}{2}\right)\Gamma\left(\frac{w-i(2nt-\alpha_k+\alpha_j)}{2}\right)\right]_{w=0}$$

$$= \sum_{m=0}^{\ell}\binom{\ell}{m}\big(-\log(N\delta)\big)^{m}$$
$$\cdot\left(\frac{\partial}{\partial w}\right)^{\ell-m}\left[\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{w+i(2nt+\alpha_k-\alpha_j)}{2}\right)\Gamma\left(\frac{w-i(2nt-\alpha_k+\alpha_j)}{2}\right)\right]_{w=0}$$

$$= \sum_{m=0}^{\ell}\binom{\ell}{m}\big(-\log(N\delta)\big)^{m}\cdot\big(\log(N\delta)^{\ell-m} \;+\; \mathcal{O}\left(|t|^{-1}\right)\big)$$
$$\cdot\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{w+i(2nt+\alpha_k-\alpha_j)}{2}\right)\Gamma\left(\frac{w-i(2nt-\alpha_k+\alpha_j)}{2}\right)\Bigg|_{w=0}$$

$$\ll\; \frac{(\log|t|)^{\ell}}{|t|}\cdot\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\Gamma\left(\frac{i(2nt+\alpha_k-\alpha_j)}{2}\right)\right|^{2}.$$

It now follows from the above computation that for $p\delta = N\delta\lambda_p$ with $1 \leq \lambda_p \leq 2$ and any integer $\ell \geq 0$

(12.8)
$$\left(\frac{\partial}{\partial w}\right)^{\ell}\left[(p\delta)^{-w}\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{w+i(2nt+\alpha_k-\alpha_j)}{2}\right)\Gamma\left(\frac{w-i(2nt-\alpha_k+\alpha_j)}{2}\right)\right]_{w=0}$$

$$= \left((-1)^{\ell}|\log\lambda_p|^{\ell} + \mathcal{O}\left(\frac{(\log|t|)^{\ell}}{|t|}\right)\right)\cdot\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\Gamma\left(\frac{i(2nt+\alpha_k-\alpha_j)}{2}\right)\right|^{2}$$

$$\leq \left(|\log 2|^{\ell} + \mathcal{O}\left(\frac{(\log|t|)^{\ell}}{|t|}\right)\right)\cdot\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\Gamma\left(\frac{i(2nt+\alpha_k-\alpha_j)}{2}\right)\right|^{2}.$$

To complete the determination of the residue at $w = 0$ in (12.6), we need to compute

$$\left(\frac{\partial}{\partial w}\right)^{\ell}\left[\frac{\widetilde{\psi}(-w)}{\pi^{\frac{2n(2n-1)w}{2}}}\prod_{\substack{1\le j<k\le n\\ \alpha_k\ne\alpha_j}}\Gamma\left(\frac{w+i(\alpha_k-\alpha_j)}{2}\right)^2\Gamma\left(\frac{w-i(\alpha_k-\alpha_j)}{2}\right)^2\right]$$

for all $0 \le \ell \le 2L-1$.

Recall that

$$\widetilde{\psi}(-w) = \frac{\Gamma\big(2n(R-w)\big)}{2^{4(n^2-L)}\,\Gamma(2nR)}\prod_{\substack{1\le j,k\le n\\ \alpha_j\ne\alpha_k}}\left(e^{\frac{\pi w}{\alpha_j-\alpha_k}}+1\right)^4.$$

If we assume that n and α_j $(j = 1, 2\ldots, n)$ are fixed and R is sufficiently large then by Stirling's asymptotic formula we see that

$$\left(\frac{\partial}{\partial w}\right)^{\ell}\widetilde{\psi}(-w)\bigg|_{w=0} \sim \big(-2n\log(2nR)\big)^{\ell}$$

as $R \to \infty$. Consequently

(12.9)

$$\left(\frac{\partial}{\partial w}\right)^{\ell}\left[\frac{\widetilde{\psi}(-w)}{\pi^{\frac{2n(2n-1)w}{2}}}\prod_{\substack{1\le j<k\le n\\ \alpha_k\ne\alpha_j}}\Gamma\left(\frac{w+i(\alpha_k-\alpha_j)}{2}\right)^2\Gamma\left(\frac{w-i(\alpha_k-\alpha_j)}{2}\right)^2\right]_{w=0}$$

$$\sim \big(-2n\log(2nR)\big)^{\ell}\cdot\prod_{\substack{1\le j<k\le n\\ \alpha_k\ne\alpha_j}}\left|\Gamma\left(\frac{i(\alpha_k-\alpha_j)}{2}\right)\right|^4$$

as $R \to \infty$, for all $0 \le \ell \le 2L-1$.

The estimations (12.8) and (12.9) show that the main contribution to the residue at $w = 0$ in (12.6) comes from the $(2L-2)^{th}$ derivative of $\widetilde{\psi}(w)$ at $w = 0$. It now follows from (12.8) and(12.9) that the residue at $w = 0$ in (12.6) is asymptotic to

$$\sim\ n\cdot 2^{2L}\frac{\big(2n\log(2nR)\big)^{2L-2}}{(2L-2)!}\prod_{\substack{1\le j<k\le n\\ \alpha_k\ne\alpha_j}}\left|\Gamma\left(\frac{i(\alpha_k-\alpha_j)}{2}\right)\right|^4\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\Gamma\left(\frac{i(2nt+\alpha_k-\alpha_j}{2}\right)\right|^2,$$

for R sufficiently large and $|t| \to \infty$.

The other residues at $\pm 2int + i(\alpha_j - \alpha_k)$ will be bounded by

$$\begin{aligned} &\ll \left|\widetilde{\psi}^{(j)}\Big(- \big(\pm 2int + i(\alpha_j - \alpha_k)\big)\Big)\right| \cdot |t|^{-n^2} \\ &\ll |t|^{-B} \end{aligned}$$

for arbitrary $B > 1$, because of the rapid decay of $\widetilde{\psi}$ and its derivatives.

On the line $\Re(w) = -\frac{1}{\log\log R}$ we estimate (with Stirling's asymptotic formula) the integral

(12.10)

$$\int\limits_{-\frac{1}{\log\log R}-i\infty}^{-\frac{1}{\log\log R}+i\infty} \frac{\widetilde{\psi}(-w)(p\delta)^{-w}}{\pi^{\frac{2n(2n-1)w}{2}}} \frac{\Gamma\left(\frac{w}{2}\right)^{2L}}{\Gamma(nw)} \prod_{\substack{j=1 \\ \alpha_j \neq \alpha_k}}^{n} \prod_{k=1}^{n} \Gamma\left(\frac{w + i(\alpha_k - \alpha_j)}{2}\right)^2$$

$$\cdot \prod_{j=1}^{n}\prod_{k=1}^{n} \Gamma\left(\frac{w + 2int + i(\alpha_k - \alpha_j}{2}\right) \Gamma\left(\frac{w - 2int + i(\alpha_k - \alpha_j}{2}\right) dw$$

$$\ll (\log\log R)^{2L-1} \cdot \prod_{j=1}^{n}\prod_{k=1}^{n} \left|\Gamma\left(\frac{2int + i(\alpha_k - \alpha_j}{2}\right)\right|^2.$$

Here we have used the fact that if we shift the line of integration in the above integral then the growth in $|t|$ does not change because of $(p\delta)^{-w} \approx (t^{n^2})^{-w}$ which cancels the polynomial term (coming from Stirling's formula) in the product of Gamma functions.

It follows from (12.5), (12.9), (12.10) that

$$I^*_{1,1}(p) \gg (\log R)^{2L-2} \cdot |t|^{-n^2}, \qquad (|t| \to \infty), \tag{12.11}$$

where the $\gg$-constant depends at most on n and the α_i $(i = 1, 2, \ldots, n)$.

Lemma 12.12 *Let f be a Hecke Maass cusp form for $SL(n, \mathbb{Z})$ which is tempered at every rational prime. Then there exist at least $\frac{1}{10n^2}\frac{N}{\log N}$ primes p in the interval $[N, 2N]$ such that*

$$|\lambda(p)| \geq \frac{1}{100},$$

where $\lambda(p)$ is the p-th Hecke eigenvalue of f.

Proof: By [Liu, Wang, Ye, 2005] it is known that

$$\sum_{N \leq m \leq 2N} \Lambda(m) \cdot |\lambda(m)|^2 \sim N. \tag{12.13}$$

Define a prime to be good if $|\lambda(p)| \geq \frac{1}{100}$, and otherwise define the prime to be bad. Now, suppose the conclusion of lemma 12.12 is wrong. We will show that this leads to a contradiction. If lemma 12.12 is false, the left side of (12.13) can be asymptotically estimated as follows:

$$\begin{aligned} \sum_{N \leq m \leq 2N} \Lambda(m) \cdot |\lambda(m)|^2 &\sim \sum_{\substack{N \leq p \leq 2N \\ p \text{ good}}} \Lambda(p) \cdot |\lambda(p)|^2 \; + \sum_{\substack{N \leq p \leq 2N \\ p \text{ bad}}} \Lambda(p) \cdot |\lambda(p)|^2 \\ &\leq n^2 \frac{1}{10n^2} N \; + \sum_{N \leq p \leq 2N} \Lambda(p) \frac{1}{100^2} \qquad (12.14) \\ &\leq \frac{N}{10} \; + \; \frac{N}{100^2}. \end{aligned}$$

Here we used the Ramanujan bound $|\lambda(p)| \leq n$, i.e., the fact that f is tempered at p. Since (12.14) contradicts (12.13) this proves the lemma. □

Lemma 12.15 *Let $N \geq t^2 \geq 1$. Then there exist at least $\left(1 - \frac{1}{20n^2}\right) \frac{N}{\log N}$ primes p in the interval $[N, 2N]$ such that*

$$\big|\eta_{nit}(p)\big| \; \geq \; \frac{1}{2000^2 n^4}.$$

Proof: We have

$$\big|\eta_{nit}(p)\big| \; = \; \big|p^{2nit} + 1\big| \; \geq \; \big|\cos(2nt)\,(\log p) + 1\big|.$$

Let

$$\Delta = \frac{1}{2000^2 n^4}.$$

Assume for some integer m and a prime $N \leq p \leq 2N$ that

$$\big|2nt(\log p) - (2m+1)\pi\big| \; \geq 10\sqrt{\Delta}. \qquad (12.16)$$

Then it follows that

$$\big|\cos(2nt)(\log p) + 1\big| \; \geq \; \Delta.$$

Define S_m to be the set of primes p in the interval $[N, 2N]$ which don't satisfy (12.16). For $p \in S_m$ we have the inequalities:

$$e^{\frac{(2m+1)\pi}{2n|t|}} \left(1 - \frac{\sqrt{\Delta}}{2nt}\right) \; \leq \; p \; \leq \; e^{\frac{(2m+1)\pi}{2nt}} \left(1 + \frac{\sqrt{\Delta}}{2n|t|}\right).$$

Sieve theory tells us [Bombieri-Davenport, 1969] that for

$$M = e^{\frac{(2m+1)\pi}{2nt}} \frac{\sqrt{\Delta}}{n|t|}$$

we have the following bound for the cardinality of S_m:

$$\#S_m \le \frac{3M}{\log M}$$

Now

$$\frac{N}{2} \le e^{\frac{(2m+1)\pi}{2n|t|}} \le 3N.$$

This implies that

$$\frac{2n|t|(\log N)}{2\pi} - \frac{1}{2} \;\le\; m \;\le\; \frac{2n|t|(\log N)}{2\pi} - \frac{1}{2} + \frac{n|t|(\log 2)}{\pi}.$$

Hence

$$\#\left(\bigcup_m S_m\right) \;\le\; \frac{2N\sqrt{\Delta}}{n|t|}\cdot\frac{n|t|(\log 9)}{\log N - \log|t|} \;\le\; \frac{100N\sqrt{\Delta}}{\log N} \;=\; \frac{1}{20n^2}\frac{N}{\log N}.$$

□

Lemma 12.17 *Let f be a Hecke Maass cusp form for $SL(n,\mathbb{Z})$ which is tempered at every rational prime. Then there exist at least $\frac{1}{20n^2}\frac{N}{logN}$ primes p in the interval $[N,2N]$ such that*

$$\big|\lambda(p)\eta_{nit}\big| \gg_{n,f} 1.$$

Proof: This follows immediately from lemmas 12.12 and 12.15 since the densities $\frac{1}{10n^2}\frac{N}{\log N}$ and $\left(1-\frac{1}{20n^2}\right)\frac{N}{\log N}$ imply an overlap of positive density. □

Now, it follows from (12.5), (12.11), lemma 12.17, and the previous choice we made, by setting $N\delta \sim (n|t|)^{n^2}$, that

$$\boxed{\big|L(1+2nit,\ f\times\widetilde{f}\,)\big|^2\sum_{N\le p\le 2N} I_1^*(p) \;\gg\; \beta^n\frac{N}{\log N}|t|^{-n^2} \;\gg\; \frac{\beta^n\delta^{-1}}{\log|t|}} \tag{12.18}$$

where the $\gg$-constant depends at most on R,n,f and is independent of $|t|\to\infty$. From now on we assume

$$L(1+2int,\ f\times\widetilde{f}\,) \;\ll\; \frac{1}{(\log|t|)^3}.$$

Otherwise, we already proved our main theorem.

Consider the contribution of the $(1,1,\ldots,1,p)$ coefficient of $E(z,f;1+it)$ which we denote as

$$a(p,1+it) := \frac{c_n\,\overline{\lambda(p)\eta_{\frac{n}{2}+nit}(p)}}{p^{\frac{2n-1}{2}}}\,W_{2n,\nu'}\,(My)\cdot\frac{1}{L(n+1+2nit,\ f\times\widetilde{f})}.$$

Let $s = 1 + it$. The Langlands parameters of $W_{2n,\nu'}$ are

$$\Big(ns - \frac{n}{2} + i\alpha_1, \quad \cdots \quad , ns - \frac{n}{2} + i\alpha_n,\ -ns + \frac{n}{2} + i\alpha_1, \quad \cdots \quad , -ns + \frac{n}{2} + i\alpha_n\Big).$$

Define

$$\begin{aligned} I_2^*(p) &:= \int_0^\infty \cdots \int_0^\infty \left| \int_0^\infty a(p, 1+it) g\left(\frac{A}{\beta}\right) \frac{dA}{A} \right|^2 \psi\left(\frac{\text{Det } z}{\delta}\right) d^\times y \\ &= |\lambda(p)|^2 \left|\eta_{\frac{n}{2}+nit}(p)\right|^2 \cdot \frac{|c_n|^2}{|L(n+1+2nit,\ f \times \widetilde{f}\,)|^2}\ I_{2,1}^*(p) \end{aligned}$$

where for $c > n$ and $\alpha_{k,j} := \alpha_k - \alpha_j$, we set

$$I_{2,1}^*(p) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{\psi}(-s')\big(p\delta\big)^{-s'} \int_0^\infty \cdots \int_0^\infty |W_{2n,\nu'}(y)|^2 \,(\text{Det} z)^{s'}\ d^\times y\, ds'$$

$$\begin{aligned} &:= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{\psi}(-s')\big(p\delta\big)^{-s'} \frac{\pi^{n^3+2n^2-n(2n-1)s'} 2^{\frac{(2n-1)2n(2n+1)}{6}}}{\Gamma(ns')} \prod_{k=1}^{n}\prod_{j=1}^{n} \\ &\cdot \frac{\Gamma\left(\frac{s'+n(s+\bar{s})-n+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{s'+n(s-\bar{s})+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{s'+n(\bar{s}-s)+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{s'-n(s+\bar{s})+n+i\alpha_{k,j}}{2}\right)}{\left|\Gamma\left(\frac{1+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{1+2ns-n+i\alpha_{k,j}}{2}\right)\right|^2}\, ds' \\ &\ll\ |t|^{(-n+c-1)n^2}\big(p\delta\big)^{-c}. \end{aligned}$$

It follows that

$$\left|L(1+2nit,\ f \times \widetilde{f}\,)\right|^2 \sum_{N \le p \le 2N} I_2^*(p)\ \ll \frac{\delta^{-1-n}}{(\log|t|)^2}\ \ll\ \frac{\beta^n \delta^{-1}}{(\log|t|)^2},$$

upon recalling that $\beta = \delta^{-1}$.

Similarly one shows that the contribution from

$$\frac{\Lambda(2ns-2n,\ f \times \widetilde{f}\,)}{\Lambda(1+2ns-2n,\ f \times \widetilde{f}\,)}\ E(z, f; 2-s)$$

is at most $\beta^n\delta^{-1}/(\log|t|)^2$.

For $c > 0$, let

$$E_4(z,s) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} E\left(z, f;\ s + \frac{w}{2}\right)\ dw.$$

Define $a^*(p,1+it)$ to be the $(1,1,\ldots,1,p)$ coefficient of $E_4(z,s)$, which is given by

$$a^*(p,1+it) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} \frac{c_n\,\overline{\lambda(p)\,\eta_{ns+\frac{nw}{2}-\frac{n}{2}}(p)}}{p^{\frac{2n-1}{2}}} \frac{W_{2n,\nu''}(My)\,dw}{L\left(1+2n(s+\frac{w}{2})-n,\ f\times\widetilde{f}\right)}$$

and the associated Langlands parameters are

$$\Bigg(n\left(s+\frac{w}{2}\right)-\frac{n}{2}+i\alpha_1,\quad \ldots\quad ,n\left(s+\frac{w}{2}\right)-\frac{n}{2}+i\alpha_n,\ -n\left(s+\frac{w}{2}\right)+\frac{n}{2}+i\alpha_1,$$
$$\ldots\quad ,-n\left(s+\frac{w}{2}\right)+\frac{n}{2}+i\alpha_n\Bigg).$$

For $c>1$, let

$$I_4^*(p) := \int\limits_0^\infty\cdots\int\limits_0^\infty\left|\int\limits_0^\infty a^*(p,1+it)g\left(\frac{A}{\beta}\right)\,\frac{dA}{A}\right|^2\ \psi\left(\frac{\text{Det } z}{\delta}\right)\,d^\times y$$

$$\ll \int\limits_0^\infty\cdots\int\limits_0^\infty\left|\int\limits_{c-i\infty}^{c+i\infty}\frac{\beta^{-\frac{n}{2}w}}{w(w+1)}\widetilde{g}(-w)\frac{\overline{\lambda(p)\,\eta_{n(s+\frac{w}{2})-\frac{n}{2}}(p)}}{p^{\frac{2n-1}{2}}}\frac{W_{2n,\nu"}(My)\,dw}{L\left(1+2n\left(s+\frac{w}{2}\right)-n,\ f\times\widetilde{f}\right)}\right|^2$$
$$\psi\left(\frac{\text{Det } z}{\delta}\right)\,d^\times y$$

$$\ll \int\limits_0^\infty\cdots\int\limits_0^\infty\int\limits_{c-i\infty}^{c+i\infty}\left|\ \beta^{-\frac{nc}{2}}\widetilde{g}(-w)\ \frac{\overline{\lambda(p)\,\eta_{n(s+\frac{w}{2})-\frac{n}{2}}(p)}}{p^{\frac{2n-1}{2}}}\,W_{2n,\nu"}(My)\ \right|^2 dw\cdot\psi\left(\frac{\text{Det } z}{\delta}\right)\,d^\times y$$

$$\ll \int\limits_{c-i\infty}^{c+i\infty}\beta^{-nc}\,\left|\widetilde{g}(-w)\right|^2\ \left|\lambda(p)\right|^2\ \left|\eta_{n(s+\frac{w}{2})-\frac{n}{2}}(p)\right|^2\ I_{4,1}^*(p)\,dw,$$

where, for $c' > n + nc$ and $\alpha_{k,j} = \alpha_k - \alpha_j$, we have

$$\begin{aligned}
I^*_{4,1}(p) &:= \int\limits_0^\infty \cdots \int\limits_0^\infty \Big| W_{2n,\nu\text{"}}(y) \Big|^2 \, \psi\left(\frac{\text{Det } z}{p\,\delta}\right) \, d^\times y \\
&= \frac{1}{2\pi i} \int\limits_{c'-i\infty}^{c'+i\infty} \widetilde{\psi}(-s') \; (p\delta)^{-s'} \int\limits_0^\infty \cdots \int\limits_0^\infty \Big| W_{2n,\nu\text{"}}(y) \Big|^2 \, (\text{Det } z)^{s'} \;\, d^\times y \, ds' \\
&= \frac{1}{2\pi i} \int\limits_{c'-i\infty}^{c'+i\infty} \widetilde{\psi}(-s') \; (p\delta)^{-s'} \; \frac{\pi^{n^3+2n^2+n^3c-n(2n-1)s'} 2^{\frac{(2n-1)2n(2n+1)}{6}}}{\Gamma(ns')}
\end{aligned}$$

$$\cdot \prod_{k=1}^n \prod_{j=1}^n \Gamma\left(\frac{s' + n\left(s + \frac{w}{2} + \bar{s} + \frac{\bar{w}}{2} - 1\right) + i\alpha_{k,j}}{2}\right) \Gamma\left(\frac{s' + n\left(s + \frac{w}{2} - \bar{s} - \frac{\bar{w}}{2}\right) + i\alpha_{k,j}}{2}\right)$$

$$\cdot \frac{\Gamma\left(\frac{s'+n\left(\bar{s}+\frac{\bar{w}}{2}-s-\frac{w}{2}\right)+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{s'-n\left(s+\frac{w}{2}+\bar{s}+\frac{\bar{w}}{2}-1\right)+i\alpha_{k,j}}{2}\right)}{\left|\Gamma\left(\frac{1+i\alpha_{k,j}}{2}\right)\Gamma\left(\frac{1+2n(s+\frac{w}{2})-n+i\alpha_{k,j}}{2}\right)\right|^2} \; ds'$$

$$\ll |t|^{-n^3-n^3c+n^2c'-n^2} (p\delta)^{-c'}.$$

It follows that

$$\left|L(1+2nit, \; f \times \widetilde{f}\,)\right|^2 \sum_{N \le p \le 2N} I^*_4(p) \; \ll \; \frac{\delta^{-n-nc}\,\beta^{-nc}\,\delta^{-1}}{(\log|t|)^2} \; \ll \; \frac{\beta^n \delta^{-1}}{(\log|t|)^2},$$

upon recalling that $\beta = \delta^{-1}$. Similarly one shows that the contribution from

$$\frac{\Lambda(2ns - 2n, \; f \times \widetilde{f}\,)}{\Lambda(1 + 2ns - 2n, \; f \times \widetilde{f}\,)} \int\limits_{1-i\infty}^{1+i\infty} \frac{A^{-\frac{n}{2}w}}{w(w+1)} \; E\left(z, f; \; 2 - s + \frac{w}{2}\right) \; dw$$

is at most $\beta^n \delta^{-1}/(\log|t|)^2$.

Combining all of the above, we finish the proof of theorem 12.1, where we see that the main contribution to the lower bound comes from (12.18).
□

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