SPECIAL VALUES OF DERIVATIVES OF L–FUNCTIONS

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§1. Generalities on Modular Forms and 1–Cocycles

Consider an unramified covering

$$\begin{array}{c} Y \\ \big\downarrow \pi \\ X \end{array}$$

of complex manifolds X, Y. Let $\Gamma(O_Y^*)$ denote the group of invertible holomorphic functions on Y, $G = \mathrm{Gal}(Y/X)$, and let $H^1\left(G, \Gamma(O_Y^*)\right)$ be the group of one–cocycles of G with values in $\Gamma(O_Y^*)$.

A map

$$\sigma : G \times Y \longrightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

is an element of H^1 if and only if it satisfies

$$\sigma(g_1 g_2, y) = \sigma(g_1, g_2 y) \cdot \sigma(g_2, y)$$

for all $g_1, g_2 \in G$, $y \in Y$. Given such a σ, we may also define an action of G on $Y \times \mathbb{C}$ via

$$g \cdot (y, w) = (gy, \sigma(g, y) w)$$

where $\sigma(g, y)w$ is ordinary multiplication of complex numbers. Factoring by this action defines a map

$$\phi : H^1 \longrightarrow \mathrm{Pic}(X)$$

given by

$$\phi(\sigma) = G \backslash (Y \times \mathbb{C})$$

for all $\sigma \in H^1$. This leads to the sequence

$$H^1(G, \Gamma(O_Y^*)) \xrightarrow{\ \phi\ } \mathrm{Pic}(X) \xrightarrow{\ \pi^*\ } \mathrm{Pic}(Y),$$

which we claim is exact. To see this, note that for $\mathcal{L} \in \mathrm{Pic}(X)$, we have $\pi^* \mathcal{L} = Y \times \mathbb{C}$ is trivial. Further, for $y \in Y$, $g \in G$,

$$\{y\} \times \mathbb{C} \ \simeq \ \{gy\} \times \mathbb{C}$$

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is an isomorphism of $\mathbb{C}$–vector spaces. Such an isomorphism must be given by $\sigma(g,y) \in \mathbb{C}$,

$$\{y\} \times \{w\} \;\mapsto\; \{gy\} \times \{\sigma(g,y)w\}$$

which implies that $\sigma(g,y) \in H^1$ and $\mathcal{L} \simeq G\backslash(Y \times \mathbb{C})$. Further, $G\backslash(Y \times \mathbb{C})$ is trivial if and only if there exists a section $s \in G\backslash(Y \times \mathbb{C})$ which has no zeros or poles. Thus

$$\mathrm{Ker}(\phi) \;=\; \left\{ \sigma(g,y) = \frac{f(gy)}{f(y)} \;\middle|\; f \in \Gamma(O_Y^*) \right\}.$$

Example: If Y is contractable then $G \cong \pi_1(X)$. If we define

$$K(G,Y) = \left\{ \sigma \in H^1 \;\middle|\; \sigma(g,y) = \frac{f(gy)}{f(y)},\ f \in \Gamma(O_Y^*) \right\}$$

then we have the exact sequence

$$0 \longrightarrow K(G,Y) \longrightarrow H^1(G, \Gamma(O_Y^*)) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

Consider now a covering $\pi : Y \longrightarrow X$ which has ramified points, and a cocycle $\sigma \in H^1$ which may not be invertible. In this case the action of G on $Y \times \mathbb{C}$ may not be well defined. We can circumvent this problem by requiring that

$$\sigma(g,y) = 1$$

for all $g \in G, y \in Y$ such that $gy = y$. Under this assumption, the quotient $\mathcal{L}_\sigma = G\backslash(Y \times \mathbb{C})$ under the action of σ will be a line bundle on X.

We now focus on another example of the general construction outlined above which is of primary interest in number theory. Let

$$\mathfrak{h} = \{z \mid \mathrm{Im}(z) > 0\}$$

denote the upper half–plane, and let

$$\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{i\infty\}$$

denote the extended upper half–plane. Consider a congruence subgroup

$$G \subset \mathrm{SL}(2,\mathbb{Z}),$$

which is of finite index in $\mathrm{SL}(2,\mathbb{Z})$. Then G acts discontinuously on $\mathfrak{h}^*$ by linear fractional transformations. In this special situation we choose $Y = \mathfrak{h}^*$ and $X = G\backslash\mathfrak{h}^*$ in the general construction outlined above. For this example, a one cocycle $\sigma \in H^1$, is a map

$$\sigma : G \times \mathfrak{h}^* \longrightarrow \mathbb{C}$$

which satisfies the cocycle relation

$$\sigma(g_1 g_2, z) = \sigma(g_1, g_2 z) \cdot \sigma(g_2, z),$$

for all $g_1, g_2 \in G$ and $z \in \mathfrak{h}^*$. This leads to an action of G on $\mathfrak{h}^* \times \mathbb{C}$ given by

$$g \cdot (z, w) = (gz, \sigma(g, z)w)$$

for all $g \in G, z \in \mathfrak{h}^*$, and $w \in \mathbb{C}$. We may thus consider the diagram below.

$$\begin{array}{ccc} \mathfrak{h}^* \times \mathbb{C} & \longrightarrow & \mathfrak{h}^* \\ \Big\downarrow & & \Big\downarrow \\ G\backslash(\mathfrak{h}^* \times \mathbb{C}) & \longrightarrow & G\backslash\mathfrak{h}^* \end{array}$$

Fix a cocycle $\sigma \in H^1$. A modular form for G (with cocyle σ) is a holomorphic function $f : \mathfrak{h}^* \longrightarrow \mathbb{C}$ which satisfies

$$f(gz) = \sigma(g, z)f(z)$$

for all $g \in G$ and $z \in \mathfrak{h}^*$. Following Borel **[2]**, a modular form is a section of the line bundle $\mathcal{L} = G\backslash(\mathfrak{h}^* \times \mathbb{C})$ lifted to $\mathfrak{h}^*$ via the natural projection.

Example: We may take $G = \mathrm{SL}(2, \mathbb{Z})$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathfrak{h}^*$ let

$$\sigma(g, z) = (cz + d)^{12}.$$

Then the Ramanujan Delta function,

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

is a modular form with cocycle σ.

§2. Action of the Hecke Operators on Line Bundles

Let G be a congruence subgroup which is of finite index in $\mathrm{SL}(2, \mathbb{Z})$. Consider the commensurator subgroup, denoted $\mathrm{Com}(G)$, which is defined by

$$\mathrm{Com}(G) = \Big\{\rho \in \mathrm{GL}(2, \mathbb{R}) \;\Big|\; [G : (\rho^{-1}G\rho) \cap G] < \infty, \\ [\rho^{-1}G\rho : (\rho^{-1}G\rho) \cap G] < \infty\Big\}.$$

Clearly, $G \leq \mathrm{Com}(G) \leq \mathrm{GL}(2, \mathbb{R})$. For every $\rho \in \mathrm{Com}(G)$, we may write

$$G = \bigcup_k \left((\rho^{-1}G\rho) \cap G\right) \delta_k$$

as a finite union of right cosets. Each such $\rho \in \mathrm{Com}(G)$ defines a Hecke operator, denoted T_ρ, which is defined as a formal sum

$$T_\rho = \sum_k \alpha_k$$

where we have set $\alpha_k = \rho\delta_k$.

Let $\mathcal{L} = G\backslash(\mathfrak{h}^* \times \mathbb{C})$ be a line bundle in $\mathrm{Pic}(G\backslash\mathfrak{h}^*)$ associated to a one cocycle $\sigma \in H^1$. For $\rho \in \mathrm{Com}(G)$, we define the action of the Hecke operator T_ρ on σ by

$$T_\rho\,\sigma(g,z) = \prod_k \sigma(\alpha_k g \alpha_k^{-1}, \alpha_k z)$$

for all $g \in G, z \in \mathfrak{h}^*$.

To check that this is well defined, we observe that

$$\begin{aligned} T_\rho\,\sigma(g_1,g_2,z) &= \prod_k \sigma(\alpha_k g_1 g_2 \alpha_k^{-1}, \alpha_k z) \\ &= \prod_k \sigma(\alpha_k g_1 \alpha_k^{-1} \alpha_k g_2 \alpha_k^{-1}, \alpha_k z) \\ &= \prod_k \sigma(\alpha_k g_1 \alpha_k^{-1}, \alpha_k g_2 z) \cdot \prod_k \sigma(\alpha_k g_2 \alpha_k^{-1}, \alpha_k z) \\ &= T_\rho\,\sigma(g_1, g_2 z) \cdot T_\rho\,\sigma(g_2, z) \end{aligned}$$

Since each one–cocycle $\sigma \in H^1$ defines a line bundle

$$\mathcal{L} = g\backslash(\mathfrak{h}^* \times \mathbb{C})$$

the above action on one–cocycles determines an action of the Hecke operators on line bundles.

§3. A Theorem of Manin

Let $N \geq 1$ be a fixed integer. For the remainder of this paper we shall be working exclusively with the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

For this group we have the following Hecke operators. Let $m > 1$ be an integer which is coprime to N. Then

$$\rho = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Com}(\Gamma_0(N)).$$

A calculation shows that the Hecke operator T_ρ (denoted, henceforth, as T_m) is simply

$$T_m = \sum_{r|m} \sum_{b=0}^{r-1} \begin{pmatrix} mr^{-1} & b \\ 0 & r \end{pmatrix}.$$

Let $M > 1$ be a divisor of N. Define the matrix

$$W_M = \begin{pmatrix} Mx & y \\ Nz & Mw \end{pmatrix}$$

for integers x, y, z, w which satisfy $M^2xw - Nyz = M$. The matrices W_M normalize $\Gamma_0(N)$ and satisfy

$$W_{M'}W_{M''} = W_{M'M''}, \qquad \forall\, M', M'' \mid N,$$
$$\prod_{q^e||N} W_{q^e} = W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

where the last product goes over all prime powers q^e exactly dividing N. For $M|N$, we define the Hecke operator T_M to simply be $T_M = W_M$.

Consider a holomorphic cusp form

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

of weight two for the congruence group $\Gamma_0(N)$. Then

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and

$$f(z)dz$$

is a holomorphic differential one–form for the Riemann surface

$$X_0(N) = \Gamma_0(N)\backslash\mathfrak{h}^*.$$

Furthermore, every differential one–form arises in this manner from a weight two holomorphic cusp form for $\Gamma_0(N)$. The Hecke operators T_m act on differential one–forms as follows. For $(m, N) = 1$, we define the action

$$T_m\, f(z)dz = \sum_{r|m} \sum_{b=0}^{r-1} f\left(\frac{mr^{-1}z+b}{r}\right) d\left(\frac{mr^{-1}z+b}{r}\right),$$

and for $M|N$,

$$T_M\, f(z)dz = f(W_M z)d(W_M z).$$

Following Atkin and Lehner [**1**], we say $f(z)$ is a newform if

$$a(1) = 1$$
$$T_p\, f(z)dz = a(p)f(z)dz \qquad \forall \text{ primes } p,\ p \nmid N$$
$$T_q\, f(z)dz = -a(q)f(z)dz \qquad \forall \text{ primes } q|N.$$

Now, suppose $\gamma \in \Gamma_0(N)$, and $\tau \in \mathfrak{h}^*$. The integral

$$I(\gamma) = -2\pi i \int_{\tau}^{\gamma\tau} f(z)\, dz$$

is independent of τ (this can easily be seen by differentiating with respect to τ, the result is 0) and must be a period of $X_0(N)$. Manin studied the action of the Hecke operators on homology by defining

$$T_m\, I(\gamma) = -2\pi i \int_\tau^{\gamma\tau} T_m\, f(z)\, dz.$$

A change of variable gives the action of T_m on the closed loop joining τ and $\gamma\tau$.

Let $f(z) = \sum_{n=1}^\infty a(n)e^{2\pi i n z}$ be a newform for $\Gamma_0(N)$ with associated L–function

$$L_f(s) = \sum_{n=1}^\infty a(n) n^{-s}.$$

Let $m > 1$ be an integer coprime to N. Manin **[3]** proved the beautiful identity

$$L_f(1) = \frac{2\pi i}{A} \sum_{r|m} \sum_{b=0}^{r-1} \int_0^{\frac{b}{r}} f(z)\, dz$$

where

$$A = \left(\sum_{r|m} r \right) \; - \; a(m).$$

The integrals

$$-2\pi i \int_0^{\frac{b}{r}} f(z)\, dz$$

are period integrals since 0 is equivalent to $\frac{b}{r}$ under the action of $\Gamma_0(N)$ when $r|m$ and $(m, N) = 1$.

§4. A Formula for $\mathbf{L}'_f(1)$

Let $N \geq 1$ be a fixed integer. For the remainder of this paper we shall be working exclusively with the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod N \right\},$$

and

$$\Gamma_0^*(N) = \big\langle \Gamma_0(N), W_N \big\rangle$$

which is the group generated by $\Gamma_0(N)$ and the involution W_N.

Fix a prime p with $p \not| N$ and set

$$\alpha_k = \begin{pmatrix} p^(- 1) & kp^(- 1) \\ 0 & 1 \end{pmatrix} \qquad (0 \leq k < p)$$
$$\alpha_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $f(z)$ is a newform of weight two for $\Gamma_0(N)$ it follows that $f(z)dz$ is an eigenfunction for all the T_p. We have

$$\begin{aligned} T_p\, f(z)dz &= \sum_{k=0}^{p} f(\alpha_k z) d(\alpha_k z) \\ &= a(p) f(z) dz. \end{aligned}$$

The basic 1–cocycle for the group $\mathrm{SL}(2,\mathbb{R})$ is

$$j(\gamma, z) = cz + d \qquad \text{for } z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

which satisfies the multiplicative cocycle relation

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) \cdot j(\gamma_2, z)$$

for all $\gamma_1, \gamma_2 \in \mathrm{SL}(2,\mathbb{R}), z \in \mathfrak{h}$.

Assume there exists a non–constant function

$$u : \mathfrak{h}^* \longrightarrow \mathbb{C}$$

which is holomorphic for $z \in \mathfrak{h}$, has polynomial growth at infinity, i.e.

$$|u(z)| << |\log z|^\delta \qquad \text{for some } \delta > 0, \text{ as } z \to i\infty,$$

and satisfies

$$u(\gamma z) = u(z) + c \log(j(\gamma, z)) + c' \log(j(\gamma, z_0)) + u(\gamma z_0) - u(z_0)$$

for fixed complex constants c, c' and all $\gamma \in \Gamma_0^*(N),\ z, z_0 \in \mathfrak{h}^*$ (Note that we must have $c' = -c$, which can be seen upon setting $z = z_0$). Let $f(z)$ be a newform of weight 2 for $\Gamma_0(N)$. For $\gamma \in \Gamma_0^*(N)$ and $\tau \in \mathfrak{h}$ we shall define the function

$$\sigma(\gamma, \tau) = \int_\tau^{\gamma\tau} f(z) u(z)\, dz.$$

Clearly this integral is independent of the path of integration for $\tau \in \mathfrak{h}$ since $u(z)$ is holomorphic on $\mathfrak{h}$. It is also independent of the path of integration for $\tau \in \mathbb{Q} \cup \{i\infty\}$ since $f(z)$ vanishes at the cusps and $u(z)$ has at most polynomial growth at the cusps.

We compute

$$\begin{aligned} \sigma(\gamma_1\gamma_2, \tau) &= \int_\tau^{\gamma_1\gamma_2\tau} f(z) u(z)\, dz \\ &= \int_\tau^{\gamma_2\tau} f(z) u(z)\, dz \ + \ \int_{\gamma_2\tau}^{\gamma_1\gamma_2\tau} f(z) u(z)\, dz \\ &= \sigma(\gamma_1, \gamma_2\tau) \ + \ \sigma(\gamma_2, \tau), \end{aligned}$$

so that σ is an additive cocycle for the group $\Gamma_0^*(N)$.

We wish to compute the action of the Hecke operators on this cocycle. To facilitate this matter we introduce the following notation. Let α_k be as in the beginning of this section. For every such α_k and $\gamma \in \Gamma_0(N)$ there exists a permutation $\pi(k)$ (of the integers between 0 and p) and a matrix $\gamma_k^* \in \Gamma_0(N)$ such that

$$\begin{aligned} \alpha_k \gamma &= \gamma_k^* \, \alpha_{\pi(k)} \\ \gamma_k^* &= \alpha_k \gamma \, \alpha_{\pi(k)}^{-1}. \end{aligned}$$

Since the differential one–form $f(z)dz$ is an eigenfunction of all the Hecke operators, it follows that for all primes $p \not| N$,

$$\begin{aligned} T_p f(z)dz &= \sum_{k=0}^{p} f(\alpha_k z) d(\alpha_k z) \\ &= a(p) f(z) dz \end{aligned}$$

where $a(p)$ is the p^{th} coefficient in the Fourier expansion of $f(z)$.

For $\gamma \in \Gamma_0(N)$, we let $z_\gamma \in \mathfrak{h}^*$ denote the fixed point of γ. Thus

$$\gamma z_\gamma = z_\gamma.$$

With this notation in place, we will show that for all $\tau, \tau_0 \in \mathfrak{h}^*$, $\gamma \in \Gamma_0(N)$,

$$T_p \Big(\sigma(\gamma, \tau) - \sigma(\gamma, \tau_0) \Big) = a(p) \cdot \Big(\sigma(\gamma, \tau) - \sigma(\gamma, \tau_0) \Big) \;\; + \;\; \sum_{k=0}^{p} C_k$$

where

$$C_k = \Big(- c' \log(j(\gamma, z_\gamma)) + c' \log(j(\gamma_k^*, z_{\gamma_k^*})) \Big) \cdot \int_{\alpha_{\pi(k)}\tau_0}^{\alpha_{\pi(k)}\tau} f(z) \, dz.$$

To prove this identity note that for any $\tau_1 \in \mathfrak{h}^*$

$$\begin{aligned} \frac{d}{d\tau} \sigma(\gamma, \tau) &= \frac{f(\gamma\tau)}{j(\gamma, \tau)^2} u(\gamma\tau) \; - \; f(\tau) u(\tau) \\ &= f(\tau) \cdot \Big[c \log(j(\gamma, \tau)) + c' \log(j(\gamma, \tau_1)) + u(\gamma\tau_1) - u(\tau_1) \Big]. \end{aligned}$$

Choosing $\tau_1 = z_\gamma$ (the fixed point of γ) yields

$$\frac{d}{d\tau} \sigma(\gamma, \tau) = f(\tau) \cdot \Big[c \log(j(\gamma, \tau)) + c' \log(j(\gamma, z_\gamma)) \Big].$$

It follows that

$$\sigma(\gamma, \tau) - \sigma(\gamma, \tau_0) = \int_{\tau_0}^{\tau} f(z) \cdot \Big[c \log(j(\gamma, z)) + c' \log(j(\gamma, z_\gamma)) \Big] \, dz.$$

Next we compute

$$\begin{aligned} T_p\,\sigma(\gamma,\tau) &= \sum_{k=0}^{p} \int_{\alpha_k\tau}^{\alpha_k\gamma\tau} f(z)u(z)\,dz \\ &= \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau}^{\gamma_k^*\alpha_{\pi(k)}\tau} f(z)u(z)\,dz \;+\; \sum_{k=0}^{p} \int_{\alpha_k\tau}^{\alpha_{\pi(k)}\tau} f(z)u(z)\,dz. \end{aligned}$$

But

$$\sum_{k=0}^{p} \int_{\alpha_k\tau}^{\alpha_{\pi(k)}\tau} g(z)\,dz = 0$$

for any function $g(z)$.

Hence

$$\begin{aligned} T_p\,\sigma(\gamma,\tau) &= \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau}^{\gamma_k^*\alpha_{\pi(k)}\tau} f(z)u(z)\,dz \\ &= \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau_0}^{\alpha_{\pi(k)}\tau} f(z)\Big[c\log j\left(\gamma_k^*,z\right) + c'\log j\left(\gamma_k^*, z_{\gamma_k^*}\right)\Big]\,dz + \\ &\qquad + \sum_{k=0}^{p} \sigma(\gamma_k^*, \alpha_{\pi(k)}\tau_0). \end{aligned}$$

But

$$\begin{aligned} \sum_{k=0}^{p} \sigma\left(\gamma_k^*, \alpha_{\pi(k)}\tau_0\right) &= \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau_0}^{\gamma_k^*\alpha_{\pi(k)}\tau_0} f(z)u(z)\,dz \\ &= T_p\,\sigma(\gamma,\tau_0). \end{aligned}$$

It follows that

$$\begin{aligned} & T_p\left[\sigma(\gamma,\tau)-\sigma(\gamma,\tau_0)\right] = \\ &\quad = \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau_0}^{\alpha_{\pi(k)}\tau} f(z)\Big[c\log j\left(\gamma_k^*,z\right) + c'\log j\left(\gamma_k^*, z_{\gamma_k^*}\right)\Big]\,dz \\ &\quad = \sum_{k=0}^{p} \int_{\tau_0}^{\tau} f\left(\alpha_{\pi(k)}z\right)\Big[c\log j\left(\gamma_k^*,\alpha_{\pi(k)}z\right) + c'\log j\left(\gamma_k^*, z_{\gamma_k^*}\right)\Big]\,d\left(\alpha_{\pi(k)}z\right). \end{aligned}$$

We observe that

$$\begin{aligned} \log\, j\left(\gamma_k^*, \alpha_{\pi(k)}z\right) &= \log\, j\left(\alpha_k\gamma\alpha_{\pi(k)}^{-1}, \alpha_{\pi(k)}z\right) \\ &= \log\, j\left(\alpha_k\gamma, z\right) \;+\; \log\, j\left(\alpha_{\pi(k)}^{-1}, \alpha_{\pi(k)}z\right) \\ &= \log\, j\left(\alpha_k, \gamma z\right) \;+\; \log\, j(\gamma,z) \;+\; \log\, j\left(\alpha_{\pi(k)}^{-1}, \alpha_{\pi(k)}z\right). \end{aligned}$$

Since

$$\log\, j\left(\alpha_k^{\pm 1}, w\right) = 0$$

for any $w \in \mathfrak{h}$ and all α_k, it immediately follows that

$$\log\, j\left(\gamma_k^*, \alpha_{\pi(k)} z\right) = \log\, j(\gamma, z).$$

We obtain

$$\begin{aligned} T_p & \Big[\sigma(\gamma,\tau) - \sigma(\gamma,\tau_0)\Big] = \\ & = \sum_{k=0}^{p} \int_{\tau_0}^{\tau} f(\alpha_{\pi(k)} z)\Big[c \log j\,(\gamma, z) + c' \log j\,\Big(\gamma_k^*, z_{\gamma_k^*}\Big)\,\Big]\, d\left(\alpha_{\pi(k)} z\right) \\ & = a(p) \int_{\tau_0}^{\tau} f\,(z)\, c \log j(\gamma, z)\, dz \ + \sum_{k=0}^{p} \int_{\alpha_{\pi(k)}\tau_0}^{\alpha_{\pi(k)}\tau} f(z) c' \log j\,\Big(\gamma_k^*, z_{\gamma_k^*}\Big)\, dz, \end{aligned}$$

from which the stated result easily follows.

These results will now be applied to obtain a closed formula for $\mathrm{L}_f'(1)$. We begin with the well known formula

$$(2\pi)^{-s}\Gamma(s+1)\mathrm{L}_f(s+1) = -2\pi i \int_0^{i\infty} f(z)\mathrm{Im}(z)^s dz.$$

Henceforth, we assume that $\mathrm{L}_f(s)$ has a zero of odd order at $s = 1$. Upon differentiating with respect to s and setting $s = 0$ it follows that

$$\mathrm{L}_f'(1) = -2\pi i \int_0^{i\infty} f(z) \log z\, dz.$$

Recall the identities

$$\sigma(\gamma, \tau) = \int_{\tau}^{\gamma\tau} f(z) u(z)\, dz,$$

$$\sigma(\gamma,\tau) - \sigma(\gamma,\tau_0) = \int_{\tau_0}^{\tau} f(z)\Big[c \log j(\gamma, z) + c' \log j(\gamma, z_\gamma)\Big]\, dz.$$

Choosing $\gamma = W_N$, $\tau_0 = 0$, and $\tau = i\infty$ in the above identities yields

$$-2 \int_0^{i\infty} f(z) u(z)\, dz = \int_0^{i\infty} f(z) c \log j(W_N, z)\, dz.$$

Here we have used the fact that $\mathrm{L}_f(1) = 0$ which is equivalent to

$$\int_0^{i\infty} f(z)\, dz = 0,$$

in addition to the identity

$$f(W_N z) d(W_N z) = f(z) dz$$

which is equivalent to the fact that $\mathrm{L}_f(s)$ has an odd order zero at $s = 1$. Note that

$$j(W_N, z) = Nz.$$

Hence

$$-2\pi i \int_0^{i\infty} f(z)\log z\, dz = \frac{4\pi i}{c}\int_0^{i\infty} f(z)u(z)\, dz$$

or equivalently

$$\begin{aligned} \mathrm{L}_f'(1) &= \frac{4\pi i}{c}\int_0^{i\infty} f(z)u(z)\, dz \\ &= \frac{2\pi i}{c}\Big[\sigma(W_N, 0) - \sigma(W_N, i\infty)\Big]. \end{aligned}$$

Let p be a prime which does not divide N. Then we define the Hecke action

$$T_p\, \mathrm{L}_f'(1) = \frac{2\pi i}{c} T_p\left[\sigma(W_N, 0) - \sigma(W_N, i\infty)\right].$$

It immediately follows from our previous computations that

$$T_p\, \mathrm{L}_f'(1) = a(p)\mathrm{L}_f'(1) \ + \sum_{k=0}^{p} C_k$$

where

$$C_k = A_k \int_{\alpha_{\pi(k)}0}^{\alpha_{\pi(k)}i\infty} f(z)\, dz$$

and

$$A_k = \frac{2\pi i}{c}\left[-c\log j(\gamma, z_\gamma) + c'\log j\left(\gamma_k^*, z_{\gamma_k^*}\right)\right]$$

with $\gamma = W_N$. By results of Manin **[3]**, and the fact that the fixed point z_γ always lies in a quadratic number field, it follows that

$$\sum_{k=0}^{p} C_k$$

must be a complex number which lies in the field generated by $\mathbb{Q}$, c, $c', \pi i$, the periods of f, and the logarithms of quadratic algebraic numbers.

We shall now prove our main theorem.

Theorem[1] *Let $f(z)$ be a holomorphic newform of weight two for $\Gamma_0(N)$ for which $\mathrm{L}_f(s)$ has an odd order zero at $s = 1$. Let $u : \mathfrak{h}^* \to \mathbb{C}$ be a holomorphic function on $\mathfrak{h}$ having polynomial growth at ∞ which satisfies*

$$u(\gamma z) = u(z) + c\log j(\gamma, z) + c'\log j(\gamma, z_0) + u(\gamma z_0) - u(z_0)$$

for fixed constants $c, c' \in \mathbb{C}$ and all $\gamma \in \Gamma_0^*(N)$, $z, z_0 \in \mathfrak{h}^*$. Then for any integer $m > 1$, coprime to N, we have

$$\mathrm{L}_f'(1) = \frac{1}{A}\sum_{r|m}\sum_{b=0}^{r-1}\int_0^{\frac{b}{r}} f(z)u(z)\, dz \quad + \quad B$$

where

$$A = \frac{c}{4\pi i}\left[\left(\sum_{r|m} r\right) \;-\; a(m)\right],$$

and B lies in the field generated by $\mathbb{Q}, c, c', \pi i, a(m)$, the periods of f, and the logarithms of quadratic algebraic numbers.

Remark: This formula expresses $\mathrm{L}_f'(1)$ as a finite linear combination of additive one cocycles for $\Gamma_0(N)$. It gives the natural generalization of Manin's theorem on $\mathrm{L}_f(1)$ (see section 3) to higher derivatives.

Proof: We give the proof when $m = p$ is a prime number. The general case is similar and left to the reader. We have already shown that

$$T_p\,\mathrm{L}_f'(1) = a(p)\mathrm{L}_f'(1) \;+\; B$$

with

$$B = \sum_{k=0}^{p} C_k.$$

But

$$\begin{aligned}
T_p\,\mathrm{L}_f'(1) &= \frac{2\pi i}{c} T_p\big[\sigma(W_N, 0) - \sigma(W_N, i\infty)\big]\\
&= \frac{4\pi i}{c}\sum_{k=0}^{p}\int_{\alpha_k 0}^{\alpha_k i\infty} f(z)u(z)\,dz\\
&= \frac{4\pi i}{c}\sum_{k=0}^{p}\int_{\alpha_k 0}^{i\infty} f(z)u(z)\,dz\\
&= \frac{4\pi i}{c}\sum_{k=0}^{p}\left[\int_{\alpha_k 0}^{0} \;+\; \int_{0}^{i\infty}\right] f(z)u(z)\,dz\\
&= \frac{4\pi i}{c}\sum_{k=0}^{p}\int_{\alpha_k 0}^{0} f(z)u(z)\,dz \;+\; (p+1)\mathrm{L}_f'(1).
\end{aligned}$$

The stated result follows immediately from this computation.

§5. Construction of Special One–Cocycles

In the previous section we outlined a closed formula for the derivative of an L–function (associated to a newform f of weight two for $\Gamma_0(N)$) at the special value $s = 1$ in terms of one–cocycles of the form

$$\sigma(\gamma, \tau) = \int_{\tau}^{\gamma\tau} f(z)u(z)\,dz.$$

It was required that $u(z)$ have polynomial growth at ∞ and satisfy

$$u(\gamma z) = u(z) + c\log j(\gamma, z) + c'\log j(\gamma, z_\gamma) + u(\gamma z_0) - u(z_0),$$

for all $\gamma \in \Gamma_0(N)$ and $z, z_0 \in \mathfrak{h}^*$. We now explicitly construct such a function $u(z)$.

Define

$$u(z) = \log\Big(\Delta(z)\cdot\Delta(Nz)\Big)$$

where

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}\left(1 - e^{2\pi inz}\right)^{24}$$

is the Ramanujan cusp form of weight twelve for the modular group. Then $u(z)$ satisfies the modular relations

$$u(\gamma z) = u(z) + 24\log j(\gamma, z)$$

for all $\gamma \in \Gamma_0(N)$.

Furthermore, for the involution W_N, we have

$$u(W_N z) = \log\left(\Delta\left(\frac{-1}{Nz}\right)\cdot\Delta\left(\frac{-1}{z}\right)\right)$$

$$= \log\Big((Nz)^{12}\Delta(Nz)\cdot z^{12}\Delta(z)\Big)$$

$$= u(z) + 24\log(Nz) - 12\log(N).$$

It follows that $u(z)$ satisfies

$$u(\gamma z) = u(z) + 24\Big[\log j(\gamma, z) - \log j(\gamma, z_0)\Big] + u(\gamma z_0) - u(z_0)$$

for all $\gamma \in \Gamma_0^*(N),\ z, z_0 \in \mathfrak{h}^*$. Furthermore, $u(z)$ has polynomial growth at infinity and is holomorphic for $z \in \mathfrak{h}$. Thus we may express $\mathrm{L}_f'(1)$ in terms of the special one–cocycles

$$\sigma(\gamma, \tau) = \int_{\tau}^{\gamma\tau} f(z)\log\Big(\Delta(z)\cdot\Delta(Nz)\Big)\,dz$$

for $\gamma \in \Gamma_0(N),\ \tau \in \mathbb{Q}$.

Another explicit one–cycle may be constructed from the (almost holomorphic) Eisenstein series $\mathrm{E}_2(z)$ of weight two for the modular group. We have

$$\mathrm{E}_2(z) = \frac{-2\pi i}{z - \bar{z}} \;+\; \frac{\pi^2}{3} \;-\; 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi inz}$$

where

$$\sigma_s(n) = \sum_{\substack{d|n\\ d\ge 1}} d^s.$$

If we define the holomorphic function

$$\mathrm{E}_2^*(z) = \mathrm{E}_2(z) \;+\; \frac{2\pi i}{z - \bar{z}}$$

then a simple computation shows that $\mathrm{E}_2^*(z)$ satisfies the modular relations

$$\frac{\mathrm{E}_2^*(\gamma z)}{j(\gamma, z)^2} = \mathrm{E}_2^*(z) \; - \; \frac{2\pi i c}{cz+d}$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

If we lift $\mathrm{E}_2^*(z)$ to $\Gamma_0(N)$ by defining

$$\mathrm{E}_2^*(z, N) = \mathrm{E}_2^*(z) \; + \; \mathrm{E}_2^* N(z)$$

then the antiderivative of $\mathrm{E}_2^*(z, N)$ with respect to z can be used to define a function $u(z)$.

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§8. References

[1] A.O.L Atkin, J. Lehner, *Hecke operators on* $\Gamma_o(m)$*,* Math. Ann. **185** (1970), 134-160.

[2] A. Borel, *Introduction to automorphic forms,* Proc. Symp. Pure Math. **9**, Amer. Math. Soc. (1966), 199-210.

[3] Y.T. Manin, *Parabolic points and zeta functions of modular forms,* (Russian) Isv. Acad. Nauk. Vol 6 (1972), No. 1, 19-64.