Analytic Number Theory  
FINAL EXAM  
(due Wednesday, May 4, 2011)

**Problem 1:** Fix $u > 0$. Show that the Fourier transform of $\frac{u}{\pi(x^2 + u^2)}$ is $e^{-2\pi|x|u}$. Plug this into the Poisson summation formula to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^{\infty} e^{-2\pi|n|u}.$$

By letting $u \to 0$ prove that $\zeta(2) = \pi^2/6$.

**Problem 2:** Let $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ (for $s \in \mathbb{C}$ with $\Re(s) > 1$) be the usual Dirichlet L-function for a Dirichlet character $\chi$. Explain what it means for an infinite product of complex numbers to converge. Prove that the infinite product $\prod_{k=1}^{\infty} L(k, \chi)$ converges.

**Problem 3:** Fix a prime $p$. Show that

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{p} \right\}$$

is a subgroup of $SL(2, \mathbb{Z})$.

**Problem 4:** Fix a prime $p$ and a Dirichlet character $\chi \pmod{p}$. For $z$ in the upper half plane and an even integer $k \geq 4$, define the Eisenstein series

$$E_k(z, \chi) := \sum_{c \equiv 0 \pmod{p}} \sum_{d \in \mathbb{Z}}^{(c,d)=1} \chi(d)(cz + d)^{-k}.$$

Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ then

$$E_k \left( \frac{az + b}{cz + d}, \chi \right) = \chi(d)^{-1}(cz + d)^k E_k(z, \chi).$$
Problem 5: For \( s \in \mathbb{C} \) with \( \Re(s) > 8 \), let

\[
L(s) := \sum_{n=1}^{\infty} a(n)n^{-s}, \quad (a(n) \in \mathbb{C} \text{ for } n = 1, 2, \ldots)
\]

where \( a(1) = 1 \) and \( |a(n)| \ll n^\gamma \) (for \( n = 1, 2, \ldots \)). Assume that the function 
\[
\Phi(s) := (2\pi)^{-s} \Gamma(s)L(s)
\]
is an entire function which is bounded in any fixed vertical strip \( \{ s \in \mathbb{C} \mid a < \Re(s) < b \} \) and satisfies the functional equation

\[
\Phi(s) = \Phi(12 - s)
\]

for all \( s \in \mathbb{C} \). Using the inverse Mellin transform, prove that

\[
\sum_{n=1}^{\infty} a(n)e^{2\pi i nz}, \quad (z \in \mathfrak{h})
\]
is the Ramanujan cusp form of weight 12.

Problem 6: Consider the extended upper half-plane \( \mathfrak{h}^* := \mathfrak{h} \cup i\infty \), where \( \mathfrak{h} \) is the upper half-plane. For \( z \in \mathfrak{h}^* \), define

\[
S_z := \{ \gamma \in SL(2, \mathbb{Z}) \mid \gamma z = z \}
\]
to be the stabilizer of \( z \). Show that \( S_z \) is trivial (for \( z \) in the fundamental domain \( \{ z \mid -1/2 \leq \Re(z) \leq 1/2, \ |z| \geq 1 \} \)) unless \( z = i\infty, i, \pm1/2 + i\sqrt{3}/2 \). Explicitly determine \( S_z \) for the four cases \( z = i\infty, i, \pm1/2 + i\sqrt{3}/2 \).