THE FUNCTIONAL EQUATIONS OF LANGLANDS EISENSTEIN SERIES FOR SL($n, \mathbb{Z}$)

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Dedicated to the memory of Chen Jingrun

Abstract. This paper presents a very simple explicit description of Langlands Eisenstein series for SL$(n, \mathbb{Z})$. The functional equations of these Eisenstein series are heuristically derived from the functional equations of certain divisor sums and certain Whittaker functions that appear in the Fourier coefficients of the Eisenstein series. We conjecture that the functional equations are unique assuming they take the form of a real affine transformation of the s variables defining the Eisenstein series. The uniqueness conjecture is proved in certain special cases.

Contents

1. Introduction

1.1. Early history of the analytic theory of Eisenstein series. As remarked by Moeglin and Waldspurger [MW95], the analytic theory of Eisenstein series really began with the work of Maass [Maa49] who formally defined the series

$$E(z,s) := \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c,d\in\mathbb{Z}}} \frac{y^s}{|cz+d|^{2s}}$$

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which converges absolutely for $s \in \mathbb{C}$ and $\Re(s) > 1$ for all $z = x + iy$ in the upper half-plane, i.e., $x \in \mathbb{R}, y > 0$. Let $\zeta(s)$ denote the Riemann zeta function which satisfies the functional equation

$$\zeta^*(s) := \pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \zeta(s) = \zeta^*(1-s). \tag{1.1}$$

Maass obtained the meromorphic continuation and functional equation of the completed Eisenstein series

$$E^*(z,s) := \zeta^*(2s)E(z,s) = E^*(z,1-s)$$

from the Fourier expansion of $E(z,s)$ together with the functional equation (1.1). This approach was generalized by Roelcke [Roe56] to discrete groups commensurable with $\mathrm{SL}(2,\mathbb{Z})$ and completed by Selberg [Sel56], [Sel63] for all Eisenstein series on $\mathrm{GL}(2,\mathbb{R})$. In his talk at the International Congress in Stockholm 1962, Selberg [Sel63] presented a new proof of the functional equation of rank one Eisenstein series which did not make use of the Fourier expansion of Eisenstein series except for the constant term. This approach was generalized by Langlands, (see [Lan66], [Lan76], [MW95]) who defined more general Eisenstein series in the higher rank case and extended Selberg's proof of the meromorphic continuation and functional equations. The basic principle in Selberg's proof is to show the analytic continuation of the Eisenstein series and its constant term simultaneously by using the fact that the resolvent of an operator has analytic continuation to the complement of its spectrum. In 1967 Selberg found another proof of the functional equation of Eisenstein series which was not published but shown to Dennis Hejhal, Paul Cohen, and Peter Sarnak which Selberg suggested would also work in the case of higher rank, but it took at least two decades before Selberg's claim was vindicated. In the 1980's Joseph Bernstein simplified Selberg's second proof. More recently Bernstein and Lapid [BL19] found a *"soft"* uniform proof of the meromorphic continuation and functional equations of Eisenstein series induced from a general automorphic form (not necessarily cuspidal or in the discrete spectrum).

1.2. **Elementary introduction to Langlands Eisenstein series for** $\mathrm{SL}(n,\mathbb{Z})$**.** We now present a very elementary explanation of the notation for Langlands Eisenstein series. For a formal definition see §4.

Let $n \geq 2$. The Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ depends on an integer partition

$$n = n_1 + n_2 + \cdots + n_r, \qquad \big(n_1, n_2, \ldots, n_r \in \mathbb{Z}_{>0} \text{ and } 2 \leq r \leq n\big),$$

which we denote by $\mathcal{P} = \mathcal{P}_{n_1,n_2,\ldots n_r}$. In addition, the Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ also depends on a product of automorphic functions

$$\Phi := \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r,$$

where each $\phi_j : \mathrm{GL}(n_j,\mathbb{R}) \mapsto \mathbb{C}$ is a smooth function invariant under left multiplication by the discrete subgroup $\mathrm{SL}(n_j,\mathbb{Z})$ and the center of $\mathrm{GL}(n_j,\mathbb{R})$, and right invariant by $\mathrm{O}(n_j,\mathbb{R})$. We denote the Langlands Eisenstein series associated to $\mathcal{P}$ and Φ by $E_{\mathcal{P},\Phi}(g,s)$. where $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$. It is required that s satisfies $\sum\limits_{i=1}^{r} n_i s_i = 0$.

1.3. **Motivation and Main Theorems of this paper.** After seeing our paper [GSW23] (where certain Fourier coefficients of Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ are explicitly computed), Peter Sarnak raised the question if it might be possible to prove the meromorphic continuation and functional equation of Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ by the original method of Maass which just uses the explicit form of the Fourier coefficients of Eisenstein series.

The aim of this paper is to show that the non-constant Fourier coefficients of Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ all satisfy the same functional equations and are entire functions of the complex variables defining the Eisenstein series. It was proved in [GSW23] that the first coefficient of $E_{\mathcal{P},\Phi}(g,s)$ is given as a recipricol of a certain product of completed Rankin-Selberg L-functions. If we multiply $E_{\mathcal{P},\Phi}(g,s)$ by this product of completed Rankin-Selberg L-functions we obtain the normalized Eisenstein series $E^*_{\mathcal{P},\Phi}(g,s)$ defined in our main Theorem 6.3. We will show by direct computation of the Fourier coefficients (see Corollary 6.5) that the functional equation of $E^*_{\mathcal{P},\Phi}(g,s)$ proved by Langlands (see [Lan76], [MW95]) is given by

$$\boxed{E^*_{\mathcal{P},\Phi}(g,s) = E^*_{\sigma\mathcal{P},\sigma\Phi}(g,\sigma s)} \tag{1.2}$$

where the permutation $\sigma \in S_r$ satisfies

$$\sigma\mathcal{P} = \mathcal{P}_{n_{\sigma(1)},n_{\sigma(2)},\ldots,n_{\sigma(r)}}, \quad \sigma\Phi = \phi_{\sigma(1)} \otimes \phi_{\sigma(2)} \otimes \cdots \phi_{\sigma(r)}, \quad \sigma s = \big(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(r)}\big).$$

For some simple concrete examples of functional equations of Langlands Eisenstein series see §3.

We also conjecture that the functional equations 1.2 are unique in the sense that if there exists a real valued affine transformation $\mu(s)$ of the variables $s = (s_1, s_2, \ldots, s_k)$ then μ has to be a permutation in S_r. See §7 where this conjecture is stated and proved in the case where $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r$ and $\phi_1 = \phi_2 = \cdots = \phi_r$. As far as we know the uniqueness of functional equations of Langlands Eisenstein series has not been investigated before.

If we knew that every Fourier-Whittaker coefficient of the Langlands Eisenstein series $E^*_{\mathcal{P},\Phi}(g,s)$ had meromorphic continuation in all its complex variables and satisfied the same functional equations, then this would give a new significantly simpler proof of the meromorphic continuation and functional equations of all Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$. We conjecture that it's enough to prove this for the constant Fourier-Whittaker coefficient and the

$$(1,1,\ldots,1,\underbrace{\;p\;}_{j^{\text{th}}\text{ entry}},1,\ldots,1)$$

coefficients, for every prime p and all j with $1 \le j \le n-1$.

2. Basic notation

Definition 2.1. (Generalized upper half plane $\mathfrak{h}^n$) We define the *generalized upper half plane* as

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R}) / \big(\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times\big).$$

By the Iwasawa decomposition of $\mathrm{GL}(n)$ (see [Gol06]) every element of $\mathfrak{h}^n$ has a coset representative of the form $g = xy$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in U_n(\mathbb{R}), \qquad y = \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \tag{2.2}$$

with $y_i > 0$ for each $1 \le i \le n-1$. The group $\mathrm{GL}(n,\mathbb{R})$ acts as a group of transformations on $\mathfrak{h}^n$ by left multiplication.

Remark 2.3. In the case $n = 2$, the above definition gives us the classical upper half-plane

$$\mathfrak{h}^2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ y > 0 \right\}.$$

Note that the $\mathrm{GL}(2,\mathbb{R})$ power function y^s is a function of the y-component of $\mathfrak{h}^2$. It is natural to try to do the same thing on $\mathfrak{h}^n$.

Consider a partition $n = n_1 + n_2 + \cdots + n_r$, where $n_i \in \mathbb{Z}_{>0}$ for $1 \le i \le r$. We can define a power function on matrices $\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & * & \cdots & * \\ & \mathfrak{m}_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \mathfrak{m}_r \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R})$ and $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ satisfying $\sum\limits_{i=1}^{r} n_i s_i = 0$ as follows:

$$\big|\mathfrak{m}\big|_{\mathcal{P}}^{s} \ := \ \prod_{i=1}^{r} \big|\det \mathfrak{m}_i\big|^{s},$$

where $\mathcal{P} = \mathcal{P}_{n_1, n_2, \ldots n_r}$ denotes the partition.

Remark 2.4. The condition $\sum\limits_{i=1}^{r} n_i s_i = 0$ assures that the above power function is invariant under multiplication by elements of the center of $\mathrm{GL}(n,\mathbb{R})$.

Definition 2.5. A *Langlands parameter* for $\mathrm{GL}(n)$ is an n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ satisfying $\sum\limits_{i=1}^{n} \alpha_i = 0$.

Definition 2.6. (Langlands parameter for an automorphic form) Let $F\colon \mathfrak{h}^n \to \mathbb{C}$ be a smooth function invariant under $\mathrm{SL}(n,\mathbb{Z})$, and suppose that F is an eigenfunction of all the $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators on $\mathfrak{h}^n$ (see [Gol06]). We say that F has Langlands parameter $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ if F has the same eigenvalues as the power function

$$\big| * \big|_{\mathcal{B}}^{\alpha + \rho_{\mathcal{B}}},$$

where $\mathcal{B}$ denotes the partition $n = 1 + 1 + \cdots + 1$, and

$$\rho_{\mathcal{B}} = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2} \right).$$

Remark 2.7. The notation $\mathcal{B}$ is used to connote the Borel parabolic subgroup. There is a one-to-one correspondence between partitions and parabolic subgroups, up to conjugacy, cf. Definition 4.1.

Example 2.8. *A Maass form for* $\mathrm{SL}(2,\mathbb{Z})$ *with Laplace eigenvalue* $\frac{1}{4} - \beta^2$ *has Langlands parameter* $(\alpha_1, \alpha_2) = (\beta, -\beta)$.

3. Examples for Langlands Eisenstein series of small rank

The Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$ is constructed by summing all translates of $(\operatorname{Im}\gamma z)^s$ over $\gamma \in \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$. Here, for $z = x+iy$ in the upper half plane, $(\operatorname{Im} z)^s = y^s$ can be thought of as a power function. To generalize Eisenstein series to $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$, it is necessary to construct a generalization of the power function y^s. (Note: we intend to describe everything, including in this case in terms of the *Langlands parameters.*)

Example 3.1. ($\mathbf{SL(2,\mathbb{Z})}$ Eisenstein series) Here we have the partition $2 = 1+1$.

Let $s = (s_1, s_2) \in \mathbb{C}^2$ with $s_1 + s_2 = 0$. Then the power function is

$$\left|\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right|^s_{\mathcal{P}_{1,1}} = y^s$$

and we can define the Eisenstein series

$$E_{\mathcal{P}_{1,1}}(g,s) = \sum_{\gamma\in\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\backslash\,\mathrm{SL}(2,\mathbb{Z})} \left|\gamma g\right|^{s_1+1/2}_{\mathcal{P}_{1,1}}.$$

Remark 3.2. Shifting s by $\frac{1}{2}$ in the power function simplifies the functional equation, whose derivation we now explain.

The Fourier expansion

$$\begin{aligned}(3.3)\qquad & E_{\mathcal{P}_{1,1}}(g,s)\\ &= y^{s_1+\frac12} + \phi(s_1+\tfrac12)y^{\frac12 - s_1} + \frac{1}{\zeta^*(2s_1+1)}\sum_{m\neq 0} \sigma_{2s_1}(m)|m|^{-s_1}\sqrt{y}\,K_{s_1}(|m|y)e^{2\pi i m x},\end{aligned}$$

where

$$\zeta^*(s) = \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s), \quad \phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)}, \quad \sigma_s(n) = \sum_{\substack{d|n\\ d>0}} d^s,$$

and K denotes the classical K-Bessel function, is well-known.

To see the functional equation of $E_{\mathcal{P}_{1,1}}(g,s)$ from the Fourier expansion, it is necessary to define

$$E^*_{\mathcal{P}_{1,1}}(g,s) = \zeta^*(2s+1)E_{\mathcal{P}_{1,1}}(g,s);$$

that is, we are clearing the denominator.

The main object of this paper is to show that the functional equations of all Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ can be easily seen by observing the Fourier coefficients in the Fourier expansion of the Eisenstein series.

In the case of $\mathrm{SL}(2,Z)$,

$$\int_0^1 E^*_{\mathcal{P}_{1,1}}\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}g, s\right)e^{2\pi i m u}\,du = \frac{\sigma_{2s_1}(m)|m|^{-s_1}}{\zeta^*(2s_1+1)}\sqrt{y}\,K_{s_1}(|m|y).$$

This immediately implies that, if $E_{\mathcal{P}_{1,1}}(g,s)$ satisfies a functional equation in s, then each of its Fourier coefficients must also satisfy the same functional equation. Since the m^{th} Fourier coefficient is easily seen to be invariant under $s \to -s$, the functional equation is $E^*_{\mathcal{P}_{1,1}}(g,s) = E^*_{\mathcal{P}_{1,1}}(g,-s)$.

Example 3.4. (The Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g,s)$ for $\mathrm{SL}(3,\mathbb{Z})$)
In the case $n=3$, the above definition of $\mathfrak{h}^n$ yields

$$\mathfrak{h}^3 = \left\{ xy = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R},\ y_1, y_2 > 0 \right\}.$$

Let $s=(s_1,s_2,s_3)\in\mathbb{C}^3$ with $s_1+s_2+s_3=0$. Then the power function is given by

$$\left|dxyk\right|^s_{\mathcal{P}_{1,1,1}} := \left|\begin{pmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|^s_{\mathcal{P}_{1,1,1}} = (y_1y_2)^{s_1} y_2^{s_2},$$

where d is in the center of $\mathrm{GL}(3,\mathbb{R})$ and $k\in\mathrm{O}(n,\mathbb{R})$. Then for $g\in\mathrm{GL}(3,\mathbb{R})$we have

$$E_{\mathcal{P}_{1,1,1}}(g,s) = \sum_{\gamma \in \left(\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right) \backslash \mathrm{SL}(3,\mathbb{Z})} \left|\gamma g\right|^{s+(1,0,-1)}_{\mathcal{P}_{1,1,1}}.$$

The shift by $(1,0,-1)$ makes the form of the functional equations as simple as possible. Note that this shift is a special value of the ρ-function defined in ....

Proposition 3.5. *Let $g\in\mathrm{GL}(3,\mathbb{R})$ and $s=(s_1,s_2,s_3)\in\mathbb{C}^3$ with $s_1+s_2+s_3=0$. Define*

$$E^*_{\mathcal{P}_{1,1,1}}(g,s) = \left(\prod_{1\le j<\ell\le 3} \zeta^*(1+s_j-s_\ell)\right) E_{\mathcal{P}_{1,1,1}}(g,s).$$

Then $E^*_{\mathcal{P}_{1,1,1}}(g,s)$ satisfies the functional equation

$$E^*_{\mathcal{P}_{1,1,1}}(g,s_1,s_2,s_3) = E^*_{\mathcal{P}_{1,1,1}}(g,s_{\sigma(1)},s_{\sigma(2)},s_{\sigma(3)})$$

for any $\sigma\in S_3$.

Proof. It's well-known that $E_{\mathcal{P}_{1,1,1}}(g,s)$ has analytic continuation and satisfies various functional equations (see [Bum84]). As we did for $\mathrm{SL}(2,\mathbb{Z})$, we will determine these functional equations by considering the $(m,1)^{\text{th}}$ Fourier coefficient of $E_{\mathcal{P}_{1,1,1}}(g,s)$, for a generic $m\in\mathbb{Z}_{>0}$.
It is proved in [Bum84] that this $(m,1)^{\text{th}}$ Fourier coefficient is given by

$$\int_0^1\int_0^1\int_0^1 E_{\mathcal{P}_{1,1,1}}\left(\left(\begin{smallmatrix} 1 & u_1 & u_3 \\ & 1 & u_2 \\ & & 1 \end{smallmatrix}\right) g, s\right) e^{-2\pi i(mu_1+u_2)}\, du_1\, du_2\, du_3 \tag{3.6}$$

$$= \frac{1}{m\cdot\prod\limits_{1\le j<\ell\le 3}\zeta^*(1+s_j-s_\ell)} \left(\sum_{\substack{c_1,c_2,c_3\in\mathbb{Z}_{>0}\\ c_1c_2c_3=m}} c_1^{s_1}c_2^{s_2}c_3^{s_3}\right) W^{(3)}_{(s_1,s_2,s_3)}\left(\left(\begin{smallmatrix} m & & \\ & 1 & \\ & & 1 \end{smallmatrix}\right) g\right),$$

where $W^{(3)}_{(s_1,s_2,s_3)}$ denotes the unique rapidly decaying $\mathrm{GL}(3,\mathbb{R})$ Whittaker function with Langlands parameter $(s_1,s_2,s_3)\in\mathbb{C}^3$ (see [Bum84], [Gol15]). It is known ([Gol15]) that $W^{(3)}_{(s_1,s_2,s_3)}$ is invariant under any permutation of s_1,s_2,s_3. It is also immediate that the divisor sum satisfies the same invariances. Therefore, so does $E^*_{\mathcal{P}_{1,1,1}}(g,s)$. □

Remark 3.7. Note that the product $\prod_{1\le j<\ell\le 3} \zeta^*(1+s_j-s_\ell)$ is not invariant under permutations of (s_1, s_2, s_3). It is for this reason that we need to multiply the Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g,s)$ by this product to obtain our functional equations for $E^*_{\mathcal{P}_{1,1,1}}(g,s)$.

Example 3.8. (The Eisenstein series $E_{\mathcal{P}_{1,2},1\otimes\phi}(g,s)$, $E_{\mathcal{P}_{2,1},\phi\otimes 1}(g,s)$ for $\mathrm{SL}(3,\mathbb{Z})$)
Here we consider the partitions $3 = 1+2$ and $3 = 2+1$. In these cases we are twisting the Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g,s)$ by a Maass form for $\mathrm{SL}(2,\mathbb{Z})$. The notation $1\otimes\phi$ refers to the situation where the 1 denotes the constant function 1 on the upper left 1×1 block of our 3×3 matrix, and ϕ denotes a Maass form for $\mathrm{SL}(2,\mathbb{Z})$, which is a function on the lower right 2×2 block. Similarly, the notation $\phi\otimes 1$ refers to the situation where the Maass form ϕ is a function on the upper left 2×2 block of our 3×3 matrix, and the the constant 1 is a function on the lower right 1×1 block.

We first consider the partition $3 = 1+2$ represented by $\mathcal{P}_{2,2}$. The power function in this case takes the following form: let $g = dxyk$, with d a central element of $\mathrm{GL}(3,\mathbb{R})$ and $k\in \mathrm{O}(3,\mathbb{R})$. Then for $s = (s_1,s_2)\in\mathbb{C}^2$ with $2s_1+2s_2 = 0$, we have

$$\big|g\big|^s_{\mathcal{P}_{2,2}} := \left|\begin{pmatrix} y_1y_2 & 0 & 0\\ 0 & y_1 & 0\\ 0 & 0 & 1\end{pmatrix}\right|^s_{\mathcal{P}_{1,2}} = (y_1y_2)^{s_1}\left|\det\begin{pmatrix} y_1 & 0\\ 0 & 1\end{pmatrix}\right|^{s_2} = y_1^{s_1+s_2}y_2^{s_1}.$$

Suppose ϕ is a Maass form for $\mathrm{SL}(2,\mathbb{Z})$. Associated to ϕ is a Langlands parameter $\alpha = (\alpha_1,\alpha_2)\in\mathbb{C}^2$, where $\alpha_1+\alpha_2 = 0$ and $\frac14-\alpha_1^2 = \frac14-\alpha_2^2$ is the Laplace eigenvalue of ϕ.

By the Iwasawa decomposition, every $g\in\mathrm{GL}(3,\mathbb{R})$ can be written in the form $g = \begin{pmatrix}\mathfrak{m}_1(g) & * \\ & \mathfrak{m}_2(g)\end{pmatrix}k$ for some $k\in \mathrm{O}(3,\mathbb{R})$, where $\mathfrak{m}_1(g)\in\mathrm{GL}(1,\mathbb{R})\cong\mathbb{R}^\times$ and $\mathfrak{m}_2(g)\in\mathrm{GL}(2,\mathbb{R})$. Then we define the Eisenstein series

$$E_{\mathcal{P}_{1,2},1\otimes\phi}(g,s) = \sum_{\gamma\in\left(\begin{smallmatrix} * & * & *\\ & * & *\\ & * & *\end{smallmatrix}\right)\backslash \mathrm{SL}(3,\mathbb{Z})} \phi\big(\mathfrak{m}_2(\gamma g)\big)\,\big|\mathfrak{m}_1(\gamma g)\big|^{s_1+1}\,\big|\det\mathfrak{m}_2(\gamma g)\big|^{s_2-1/2}. \tag{3.9}$$

Next we consider the partition $3 = 2+1$ represented by $\mathcal{P}_{2,1}$. The power function in this case takes the following form: let $g = dxyk$, with d a central element of $\mathrm{GL}(3,\mathbb{R})$ and $k\in\mathrm{O}(3,\mathbb{R})$. Then for $s = (s_1,s_2)\in\mathbb{C}^2$ with $2s_1+s_2 = 0$, we have

$$\big|g\big|^s_{\mathcal{P}_{2,1}} := \left|\begin{pmatrix} y_1y_2 & 0 & 0\\ 0 & y_1 & 0\\ 0 & 0 & 1\end{pmatrix}\right|^s_{\mathcal{P}_{2,1}} = \left|\det\begin{pmatrix} y_1y_2 & 0\\ 0 & y_1\end{pmatrix}\right|^{s_1} = y_1^{2s_1}y_2^{s_1}.$$

Suppose ϕ is a Maass form for $\mathrm{SL}(2,\mathbb{Z})$ with Langlands parameter $\alpha = (\alpha_1,\alpha_2)\in\mathbb{C}^2$, where $\alpha_1+\alpha_2 = 0$. By the Iwasawa decomposition, every $g\in\mathrm{GL}(3,\mathbb{R})$ can be written in the form $g = \begin{pmatrix}\mathfrak{m}_1(g) & * \\ & \mathfrak{m}_2(g)\end{pmatrix}k$ for some $k\in\mathrm{O}(3,\mathbb{R})$, where $\mathfrak{m}_1(g)\in\mathrm{GL}(2,\mathbb{R})$ and $\mathfrak{m}_2(g)\in\mathrm{GL}(1,\mathbb{R})\cong\mathbb{R}^\times$. Then we define the Eisenstein series

$$E_{\mathcal{P}_{2,1},\phi\otimes 1}(g,s) := \sum_{\gamma\in\left(\begin{smallmatrix} * & * & *\\ * & * & *\\ & & *\end{smallmatrix}\right)\backslash \mathrm{SL}(3,\mathbb{Z})} \phi\big(\mathfrak{m}_1(\gamma g)\big)\,\big|\mathfrak{m}_1(\gamma g)\big|^{s_1+1/2}\,\big|\det\mathfrak{m}_2(\gamma g)\big|^{s_2-1}. \tag{3.10}$$

Recall the L-function $L(s,\phi)$ as defined in [Gol15]. The completed L-function for this is given by

$$L^*(s,\phi) := \pi^{-s}\Gamma(\tfrac{s+\alpha_1}{2})\Gamma(\tfrac{s+\alpha_2}{2})L(s,\phi) = L^*(1-s,\phi).$$

Proposition 3.11. *Let $E_{\mathcal{P}_{1,2},1\otimes\phi}$ and $E_{\mathcal{P}_{2,1},\phi\otimes 1}$ be as in equations* (3.9) *and* (3.10)*. Define*

$$\begin{aligned} E^*_{\mathcal{P}_{1,2},1\otimes\phi}(g,s) &:= L^*(1+s_2-s_1,\phi)E_{\mathcal{P}_{1,2},1\otimes\phi}(g,s), \qquad \big(s=(s_1,s_2)\in\mathbb{C}^2,\ s_1+2s_2=0\big),\\ E^*_{\mathcal{P}_{2,1},\phi\otimes 1}(g,s) &:= L^*(1+s_2-s_1,\phi)E_{\mathcal{P}_{2,1},\phi\otimes 1}(g,s), \qquad \big(s=(s_1,s_2)\in\mathbb{C}^2,\ 2s_1+s_2=0\big).\end{aligned}$$

Then the functional equation takes the form

$$E^*_{\mathcal{P}_{1,2},1\otimes\phi}\big(g,(s_1,s_2)\big) = E^*_{\mathcal{P}_{2,1},\phi\otimes 1}\big(g,(s_2,s_1)\big)$$

for all $s_1,s_2\in\mathbb{C}$.

Remark 3.12. In $E^*_{\mathcal{P}_{1,2},1\otimes\phi}\big(g,(s_1,s_2)\big)$, as required we have $s_1+2s_2=0$. Similarly, in $E^*_{\mathcal{P}_{2,1},\phi\otimes 1}\big(g,(s_2,s_1)\big)$, as required we have $2s_2+s_1=0$, which is, of course, the same condition.

Proof. Recall the definition of the adjoint L-function: $L(s,\operatorname{Ad}\phi) := L(s,\phi\times\overline{\phi})/\zeta(s)$ where $L(s,\phi\times\overline{\phi})$ is the Rankin-Selberg convolution L-function as in §12.1 of [Gol15]. The completed adjoint L-function at $s=1$ is given by

$$L^*(1,\operatorname{Ad}\phi) := \Gamma(\tfrac{1}{2}+\alpha_1)\Gamma(\tfrac{1}{2}+\alpha_2)L(1,\operatorname{Ad}\phi).$$

It is proved in [GMW21] that, if $s_1+2s_2=0$, then the $(m,1)^{\text{th}}$ Fourier coefficient of $E_{\mathcal{P}_{1,2},1\otimes\phi}$ is given by

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 E_{\mathcal{P}_{1,2},1\otimes\phi}\left(\begin{pmatrix}1&u_1&u_3\\&1&u_2\\&&1\end{pmatrix}g,(s_1,s_2)\right)e^{-2\pi i(mu_1+u_2)}\,du_1\,du_2\,du_3\\ &= \frac{1}{mL^*(1,\operatorname{Ad}\phi)^{1/2}L^*(1+s_2-s_1,\phi)}\left(\sum_{\substack{c_1,c_2\in\mathbb{Z}_{>0}\\ c_1c_2=m}}\lambda_\phi(c_1)c_1^{s_1}c_2^{s_2}\right)W^{(3)}_{(s_1,s_2+\alpha_1,s_2+\alpha_2)}\left(\begin{pmatrix}m&&\\&1&\\&&1\end{pmatrix}g\right).\end{aligned}$$

We therefore have

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 E^*_{\mathcal{P}_{1,2},1\otimes\phi}\left(\begin{pmatrix}1&u_1&u_3\\&1&u_2\\&&1\end{pmatrix}g,(s_1,s_2)\right)e^{-2\pi i(mu_1+u_2)}\,du_1\,du_2\,du_3\\ &\qquad= \frac{1}{mL^*(1,\operatorname{Ad}\phi)^{1/2}}\left(\sum_{\substack{c_1,c_2\in\mathbb{Z}_{>0}\\ c_1c_2=m}}\lambda_\phi(c_1)c_1^{s_1}c_2^{s_2}\right)W^{(3)}_{(s_1,s_2+\alpha_1,s_2+\alpha_2)}\left(\begin{pmatrix}m&&\\&1&\\&&1\end{pmatrix}g\right).\end{aligned}\tag{3.13}$$

Similarly, it is proved in [GMW21] that, if $2s_1+s_2=0$, then the $(m,1)^{\text{th}}$ Fourier coefficient of $E^*_{\mathcal{P}_{2,1},\phi\otimes 1}$ is given by

$$\int_0^1\int_0^1\int_0^1 E^*_{\mathcal{P}_{2,1},\phi\otimes 1}\left(\begin{pmatrix}1&u_1&u_3\\&1&u_2\\&&1\end{pmatrix}g,(s_1,s_2)\right)e^{-2\pi i(mu_1+u_2)}\,du_1\,du_2\,du_3 \tag{3.14}$$
$$=\frac{1}{mL^*(1,\operatorname{Ad}\phi)^{1/2}}\left(\sum_{\substack{c_1,c_2\in\mathbb{Z}_{>0}\\ c_1c_2=m}}\lambda_\phi(c_2)c_1^{s_1}c_2^{s_2}\right)W^{(3)}_{(s_1+\alpha_1,s_1+\alpha_2,s_2)}\left(\begin{pmatrix}m&&\\&1&\\&&1\end{pmatrix}g\right).$$

Note that, if we interchange s_1 and s_2 in the divisor sum in (3.13), we get the divisor sum appearing in (3.14). Also, this interchange sends $W^{(3)}_{(s_1,s_2+\alpha_1,s_2+\alpha_2)}$ to $W^{(3)}_{(s_2,s_1+\alpha_1,s_1+\alpha_2)}$, which equals $W^{(3)}_{(s_1+\alpha_1,s_1+\alpha_2,s_2)}$, since $W^{(3)}_{(a,b,c)}$ is invariant under any permutation of (a,b,c). Finally, this switch transforms the condition $s_1+2s_2=0$ to the condition $2s_1+s_2=0$. ☐

Example 3.15. (The Eisenstein series $E_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}(g,s)$ for $\mathrm{SL}(4,\mathbb{Z})$)

Here we consider the partition $4=2+2$. In this case our construction involves a twist by two Maass forms for $\mathrm{SL}(2,\mathbb{Z})$: a Maass form ϕ_1 with Langlands parameter $(\alpha_{1,1},\alpha_{1,2})\in\mathbb{C}^2$ with $\alpha_{1,1}+\alpha_{1,2}=0$, and a Maass form ϕ_2 with Langlands parameter $(\alpha_{2,1},\alpha_{2,2})\in\mathbb{C}^2$ with $\alpha_{2,1}+\alpha_{2,2}=0$. Here, $\frac14-\alpha_{j,1}^2$ is the Laplace eigenvalue of ϕ_j, for $j=1,2$.

The power function in this case takes the following form: let $g=dxyk$, with d a central element of $\mathrm{GL}(4,\mathbb{R})$ and $k\in\mathrm{O}(4,\mathbb{R})$. Then for $s=(s_1,s_2)\in\mathbb{C}^2$ with $2s_1+2s_2=0$, we have

$$|g|^s_{\mathcal{P}_{2,2}}:=\left|\begin{pmatrix}y_1y_2y_3&0&0&0\\0&y_1y_2&0&0\\0&0&y_1&0\\0&0&0&1\end{pmatrix}\right|^s_{\mathcal{P}_{2,2}}=\left|\det\begin{pmatrix}y_1y_2y_3&0\\1&y_1y_2\end{pmatrix}\right|^{s_1}\cdot\left|\det\begin{pmatrix}y_1&0\\0&1\end{pmatrix}\right|^{s_2}$$
$$=y_1^{2s_1+s_2}y_2^{2s_1}y_3^{s_1}.$$

By the Iwasawa decomposition, every $g\in\mathrm{GL}(4,\mathbb{R})$ can be written in the form $g=\begin{pmatrix}\mathfrak{m}_1(g)&*\\&\mathfrak{m}_2(g)\end{pmatrix}k$ for some $k\in\mathrm{O}(4,\mathbb{R})$, where $\mathfrak{m}_1(g),\mathfrak{m}_2(g)\in\mathrm{GL}(2,\mathbb{R})$. Then we define the Eisenstein series

$$E_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}(g,s)=\sum_{\gamma\in\begin{pmatrix}*&*&*&*\\ *&*&*&*\\&&*&*\\&&*&*\end{pmatrix}\backslash\mathrm{SL}(4,\mathbb{Z})}\phi_1\big(\mathfrak{m}_1(\gamma g)\big)\,\phi_2\big(\mathfrak{m}_2(\gamma g)\big)\cdot\big|\gamma g\big|^{s+(1,-1)}_{\mathcal{P}_{2,2}}. \tag{3.16}$$

Recall the Rankin-Selberg L-function $L^*(s,\phi_1\times\phi_2)$ as defined in [Gol15]. The completed L-function for this is given by

$$L^*(s,\phi_1\times\phi_2):=\pi^{-2s}\left(\prod_{j,k=1}^{2}\Gamma\big(\tfrac{s+\alpha_{1,j}+\alpha_{2,k}}{2}\big)\right)\cdot L(s,\phi_1\times\phi_2).$$

Proposition 3.17. *Let $E_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}$ be as in equation* (3.16)*. Define*

$$E^*_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}(g,s) := L^*(1+s_2-s_1,\phi_1\times\phi_2)E_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}(g,s),$$

where $s=(s_1,s_2)\in\mathbb{C}^2$ satisfies $2s_1+2s_2=0$. Then the functional equation takes the form

$$E^*_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}\big(g,(s_1,s_2)\big) = E^*_{\mathcal{P}_{2,2},\phi_2\otimes\phi_1}\big(g,(s_2,s_1)\big)$$

for all $s_1,s_2\in\mathbb{C}$.

Remark 3.18. In $E^*_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}\big(g,(s_1,s_2)\big)$, as required we have $2s_1+2s_2=0$. Similarly, in $E^*_{\mathcal{P}_{2,2},\phi_2\otimes\phi_1}\big(g,(s_2,s_1)\big)$, as required we have $2s_2+2s_1=0$, which is, of course, the same condition.

Proof. It is proved in [GMW21] that, if $2s_1+2s_2=0$, then the $(m,1,1)^{\text{th}}$ Fourier coefficient of $E^*_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}$ is given by

$$\int_0^1\cdots\int_0^1 E^*_{\mathcal{P}_{2,2},\phi_1\otimes\phi_2}\left(\begin{pmatrix}1&u_1&u_4&u_6\\&1&u_2&u_5\\&&1&u_3\\&&&1\end{pmatrix}g,(s_1,s_2)\right)e^{-2\pi i(mu_1+u_2+u_3)}\prod_{i=1}^6 du_i \tag{3.19}$$
$$= \frac{\left(\sum\limits_{\substack{c_1,c_2\in\mathbb{Z}_{>0}\\ c_1c_2=m}}\lambda_{\phi_1}(c_1)\lambda_{\phi_2}(c_2)c_1^{s_1}c_2^{s_2}\right)W^{(4)}_{(s_1+\alpha_{1,1},s_1+\alpha_{1,2},s_2+\alpha_{2,1},s_2+\alpha_{2,2})}\left(\begin{pmatrix}m&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}g\right)}{m^{3/2}L^*(1,\operatorname{Ad}\phi_1)^{1/2}L^*(1,\operatorname{Ad}\phi_2)^{1/2}}.$$

Interchanging ϕ_1 and ϕ_2 we find that, if $2s_1+2s_2=0$, then the $(m,1,1)^{\text{th}}$ Fourier coefficient of $E_{\mathcal{P}_{2,2},\phi_2\otimes\phi_1}$ is given by

$$\int_0^1\cdots\int_0^1 E^*_{\mathcal{P}_{2,2},\phi_2\otimes\phi_1}\left(\begin{pmatrix}1&u_1&u_4&u_6\\&1&u_2&u_5\\&&1&u_3\\&&&1\end{pmatrix}g,(s_1,s_2)\right)e^{-2\pi i(mu_1+u_2+u_3)}\prod_{i=1}^6 du_i \tag{3.20}$$
$$= \frac{\left(\sum\limits_{\substack{c_1,c_2\in\mathbb{Z}_{>0}\\ c_1c_2=m}}\lambda_{\phi_2}(c_1)\lambda_{\phi_1}(c_2)c_1^{s_1}c_2^{s_2}\right)W^{(4)}_{(s_1+\alpha_{2,1},s_1+\alpha_{2,2},s_2+\alpha_{1,1},s_2+\alpha_{1,2})}\left(\begin{pmatrix}m&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}g\right)}{m^{3/2}L^*(1,\operatorname{Ad}\phi_1)^{1/2}L^*(1,\operatorname{Ad}\phi_2)^{1/2}}.$$

Note that, if we interchange s_1 and s_2 in the divisor sum in (3.19), we get the divisor sum appearing in (3.20). Also, this interchange sends the Whittaker function with Langlands parameter

$$(s_1+\alpha_{1,1},s_1+\alpha_{1,2},s_2+\alpha_{2,1},s_2+\alpha_{2,2})$$

to the Whittaker function with Langlands parameter

$$(s_2+\alpha_{1,1},s_2+\alpha_{1,2},s_1+\alpha_{2,1},s_1+\alpha_{2,2}). \tag{3.21}$$

The Whittaker function with Langlands parameter given by (3.21) equals the Whittaker function in equation (3.20) because the Whittaker function is invariant under permutations of the Langlands parameters. □

4. Eisenstein series for a parabolic subgroup of $\mathrm{GL}(n,\mathbb{R})$

Definition 4.1. (Parabolic Subgroup) *For $n \geq 2$ and $1 \leq r \leq n$, consider a partition of n given by $n = n_1 + \cdots + n_r$ with positive integers $n_1, \cdots, n_r$. We define the standard parabolic subgroup*

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * & \cdots & * \\ 0 & \mathrm{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

Letting I_r denote the $r \times r$ identity matrix, the subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}$$

is the unipotent radical of $\mathcal{P}$. The subgroup

$$M^{\mathcal{P}} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}(n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

is the standard choice of Levi subgroup of $\mathcal{P}$.

Definition 4.2. (Automorphic form Φ associated to a parabolic $\mathcal{P}$) *Let $n \geq 2$. Consider a partition $n = n_1 + \cdots + n_r$ with $1 < r < n$. Let $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} \subseteq \mathrm{GL}(n,\mathbb{R})$. For $i = 1, 2, \ldots, r$, let $\phi_i : \mathrm{GL}(n_i,\mathbb{R}) \to \mathbb{C}$ be either the constant function 1 (if $n_i = 1$) or a Maass cusp form for $\mathrm{SL}(n_i,\mathbb{Z})$ (if $n_i > 1$). The form $\Phi := \phi_1 \otimes \cdots \otimes \phi_r$ is defined on $\mathrm{GL}(n,\mathbb{R}) = \mathcal{P}(\mathbb{R})K$ (where $K = \mathrm{O}(n,\mathbb{R})$) by the formula*

$$\Phi(u\mathfrak{m}k) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_i), \qquad (u \in N^{\mathcal{P}}, \mathfrak{m} \in M^{\mathcal{P}}, k \in K)$$

where $\mathfrak{m} \in M^{\mathcal{P}}$ has the form $\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r \end{pmatrix}$, with $\mathfrak{m}_i \in \mathrm{GL}(n_i,\mathbb{R})$. In fact, this construction works equally well if some or all of the ϕ_i are Eisenstein series.

Definition 4.3. (Power function for a parabolic subgroup) Let $n \geq 2$ and $2 \leq r \leq n$. Fix a partition $n = n_1 + n_2 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r}$. Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ satisfy $\sum\limits_{i=1}^{r} n_i s_i = 0$.

For $g \in \mathcal{P}$, with diagonal block entries $\mathfrak{m}_i \in \mathrm{GL}(n_i,\mathbb{R})$, we define the power function

$$|g|_{\mathcal{P}}^{s} := \prod_{i=1}^{r} |\det(\mathfrak{m}_i)|^{s_i}.$$

Since $\det(\mathfrak{m}_i k_i) = \pm\det(\mathfrak{m}_i)$ for $k_i \in \mathrm{O}(n_i, \mathbb{R})$, we see that $|g|_{\mathcal{P}}^{s}$ is invariant under right multiplication by $\mathrm{O}(n, \mathbb{R})$. Because of this, and the fact that $\sum_{i=1}^{r} n_i s_i = 0$, it follows that $|g|_{\mathcal{P}}^{s}$ extends to a well-defined function on $\mathrm{GL}(n, \mathbb{R})$, invariant by the center and right multiplication by $\mathrm{O}(n, \mathbb{R})$.

Definition 4.4. ($\boldsymbol{\rho}$-function for a parabolic) *Let $n \geq 2$ and $2 \leq r \leq n$. Let $\mathcal{P}$ be the parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1, n_2, \ldots, n_r}$. Then we define*

$$\rho_{\mathcal{P}}(j) := \begin{cases} \frac{n-n_1}{2}, & j = 1, \\ \frac{n-n_j}{2} - n_1 - \cdots - n_{j-1}, & j \geq 2, \end{cases}$$

and

$$\rho_{\mathcal{P}} = (\rho_{\mathcal{P}}(1), \rho_{\mathcal{P}}(2), \ldots, \rho_{\mathcal{P}}(r)).$$

Remark 4.5. The ρ-function is introduced as a normalizing factor (a shift in the s variable) in the definition of Eisenstein series, below, to make later formulae as simple as possible. Note that, in the definition of Eisenstein series for reductive groups, $\rho_{\mathcal{P}}$ equals half the sum of the roots of the parabolic subgroup.

Definition 4.6. (Langlands Eisenstein series) *Let $n \geq 2$ and $2 \leq r \leq n$. Let $\mathcal{P}$ be the parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1, n_2, \ldots, n_r}$. Suppose Φ is a Maass form associated to $\mathcal{P}$, as in Definition 4.2. Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ with $\sum_{i=1}^{r} n_i s_i = 0$ and let $|g|_{\mathcal{P}}^{s}$ be the power function as in Definition 4.3.*

For $\Gamma_n := \mathrm{SL}(n, \mathbb{Z})$, we define the Eisenstein series

$$E_{\mathcal{P}, \Phi}(g, s) := \sum_{\gamma \in (\Gamma_n \cup \mathcal{P}) \backslash \Gamma_n} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s + \rho_{\mathcal{P}}}, \tag{4.7}$$

with $\rho_{\mathcal{P}}$ as in Definition 4.4.

Example 4.8. (Borel Eisenstein series) Consider the partition

$$n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}},$$

associated to the Borel parabolic subgroup

$$\mathcal{B} := \mathcal{P}_{1,1,\ldots,1} := \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \right\}.$$

Then the Borel Eisenstein series is constructed as follows. Let $\Phi = 1$ be the trivial function, and choose $s = (s_1, s_2, \ldots, s_n)$ where $\sum_{i=1}^{n} s_i = 0$. Then $\rho_{\mathcal{B}} = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right)$, and (4.7) becomes

$$E_{\mathcal{B}}(g, s) = \sum_{\gamma \in (\Gamma \cup \mathcal{B}) \backslash \Gamma} |\gamma g|_{\mathcal{B}}^{s + \rho_{\mathcal{B}}}. \tag{4.9}$$

The Borel Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{SL}(3, \mathbb{Z})$ are given in Examples 3.1 and 3.4 respectively.

5. Whittaker function for Langlands Eisenstein series

Definition 5.1. (Jacquet's Whittaker function) Let $g \in \mathrm{GL}(n,\mathbb{R})$ with $n \geq 2$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ with $\sum_{i=1}^{n} \alpha_i = 0$. We define the *completed Whittaker function* $W_\alpha : \mathrm{GL}(n,\mathbb{R}) \big/ \left(\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times\right) \to \mathbb{C}$, *with Langlands parameter* α, by the integral

$$W_\alpha^{(n)}(g) = \prod_{1 \leq j < k \leq n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \cdot \int\limits_{U_n(\mathbb{R})} \left|w_n \cdot ug\right|_{\mathcal{B}}^{s+\rho_{\mathcal{B}}} \overline{\psi_{1,1,\ldots,1}(u)}\, du, \tag{5.2}$$

where w_n is the long element of the Weyl group for $\mathrm{GL}(n,\mathbb{R})$, and $|*|_{\mathcal{B}}^{s}$ is the power function for the Borel $\mathcal{B}$ given in Definition 4.3. This integral converges absolutely if $\mathrm{Re}(\alpha_i - \alpha_{i+1}) > 0$ for $1 \leq i \leq n-1$, has meromorphic continuation to all $\alpha \in \mathbb{C}^n$ satisfying $\sum_{i=1}^{n} \alpha_i = 0$, and is invariant under permutations of $\alpha_1, \alpha_2, \ldots, \alpha_n$ (cf. [GMW21]).

Proposition 5.3. *Let* $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots n_r}$ *be a parabolic subgroup of* $\mathrm{GL}(n,\mathbb{R})$*, and let* $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r$ *be an automorphic form associated to* $\mathcal{P}$*. For* $j = 1, 2, \ldots, r$*, let* $(\alpha_{j,1}, \ldots, \alpha_{j,n_j})$ *denote the Langlands parameter of* ϕ_j*. We adopt the convention that if* $n_j = 1$ *then* $\alpha_{j,1} = 0$*. Then the Langlands parameter of* $E_{\mathcal{P},\Phi}(g,s)$ *(denoted* $\alpha_{\mathcal{P},\Phi}(s)$*) is*

$$\alpha_{\mathcal{P},\Phi}(s) = \Big(\overbrace{\alpha_{1,1}+s_1,\ \ldots\ ,\alpha_{1,n_1}+s_1}^{n_1 \text{ terms}},\ \overbrace{\alpha_{2,1}+s_2,\ \ldots\ ,\alpha_{2,n_2}+s_2}^{n_2 \text{ terms}},\ \ldots\ ,\ \overbrace{\alpha_{r,1}+s_r,\ \ldots\ ,\alpha_{r,n_r}+s_r}^{n_r \text{ terms}}\Big).$$

Proof. Let $\mathcal{B}$ be the Borel parabolic subgroup of $\mathrm{GL}(n)$. We need to show that $E_{\mathcal{P},\Phi}(g,s)$ has the same eigenvalues as the power function $\left|\, * \,\right|_{\mathcal{B}}^{\alpha_{\mathcal{P},\Phi}(s)+\rho_B}$.

By construction, the Eisenstein series $E_{\mathcal{P}}(g,s)$ has the same eigenvalues as the power function $\left|\, * \,\right|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}}$. If we define $s^* + \rho_{\mathcal{P}}^* \in \mathbb{C}^r$ by

$$s^* + \rho_{\mathcal{P}}^* := \Big(\overbrace{s_1+\rho_{\mathcal{P}}(1),\ldots,s_1+\rho_{\mathcal{P}}(1)}^{n_1 \text{ terms}},\ \overbrace{s_2+\rho_{\mathcal{P}}(2),\ldots,s_2+\rho_{\mathcal{P}}(2)}^{n_2 \text{ terms}},\ \ldots\ ,\ \overbrace{s_r+\rho_{\mathcal{P}}(r),\ldots,s_r+\rho_{\mathcal{P}}(r)}^{n_r \text{ terms}}\Big),$$

then $\left|\, * \,\right|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}} = \left|\, * \,\right|_{\mathcal{B}}^{s^*+\rho_{\mathcal{P}}^*}$. To see this, note that it suffices to check this equality on diagonal matrices $\mathrm{diag}(d_1, d_2, \ldots, d_n)$. This can be verified by a simple calculation.

Let $\mathcal{B}_j$ be the Borel parabolic subgroup of $\mathrm{GL}(n_j)$. Let $(\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,n_j})$ be the Langlands parameter of ϕ_j (that is, ϕ_j has the same eigenvalues as $\left|\, * \,\right|_{\mathcal{B}_j}^{\alpha_j+\rho_{\mathcal{B}_j}}$). The Langlands parameter of $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r$ is

$$\alpha_\Phi = \Big(\overbrace{\alpha_{1,1},\ldots,\alpha_{1,n_1}}^{n_1 \text{ terms}},\ \overbrace{\alpha_{2,1},\ldots,\alpha_{2,n_2}}^{n_2 \text{ terms}},\ \ldots\ ,\ \overbrace{\alpha_{r,1},\ldots,\alpha_{r,n_r}}^{n_r \text{ terms}}\Big).$$

Hence Φ has the same eigenvalues as $\big| * \big|_{\mathcal{B}}^{\alpha_\Phi+\rho_\Phi}$, where

$$\rho_\Phi = \Bigg(\overbrace{\tfrac{n_1-1}{2}, \tfrac{n_1-3}{2}, \ldots, \tfrac{1-n_1}{2}}^{\rho_{\mathcal{B}_1}},\ \overbrace{\tfrac{n_2-1}{2}, \tfrac{n_2-3}{2}, \ldots, \tfrac{1-n_2}{2}}^{\rho_{\mathcal{B}_2}},\quad \ldots \quad,\ \overbrace{\tfrac{n_r-1}{2}, \tfrac{n_r-3}{2}, \ldots, \tfrac{1-n_r}{2}}^{\rho_{\mathcal{B}_r}}\Bigg).$$

The eigenvalues of $E_{\mathcal{P},\Phi}(*, s)$ match those of $\big| * \big|_{\mathcal{B}}^{\alpha_\Phi+\rho_\Phi} \cdot \big| * \big|_{\mathcal{B}}^{s^*+\rho^*_{\mathcal{P}}}$. It therefore suffices to show that

$$\alpha_\Phi + \rho_\Phi + s^* + \rho^*_{\mathcal{P}} = \alpha_{\mathcal{P},\Phi}(s) + \rho_{\mathcal{B}}. \tag{5.4}$$

This is equivalent to the identity $\rho_\Phi + \rho^*_{\mathcal{P}} = \rho_{\mathcal{B}}$, which follows immediately from the definitions. □

6. Statement and proof of the Main Theorem of this paper

Proposition 6.1. The M^{th} Fourier coefficient of $E_{\mathcal{P},\Phi}$ *Let*

$$s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r,$$

where $\sum_{i=1}^{r} n_i s_i = 0$. *Consider* $E_{\mathcal{P},\Phi}(*, s)$ *with associated Langlands parameters* $\alpha_{{}_{\mathcal{P},\Phi}}(s)$ *as defined in Proposition 5.3. Let* $M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. *Then the* M^{th} *term in the Fourier-Whittaker expansion of* $E_{\mathcal{P},\Phi}$ *is*

$$\int_0^1 \cdots \int_0^1 E_{\mathcal{P},\Phi}(ug, s) \exp\left(-2\pi i \sum_{i=1}^{n-1} m_i u_{i,i+1}\right) \prod_{1\le i<j\le n} du_{i,j} \ = \ \frac{A_{\mathcal{P},\Phi}(M, s)}{\prod\limits_{k=1}^{n-1} m_k^{k(n-k)/2}}\, W_{\alpha_{{}_{\mathcal{P},\Phi}}(s)}\big(Mg\big),$$

where $A_{\mathcal{P},\Phi}(M, s) = A_{\mathcal{P},\Phi}\big((1, \ldots, 1), s\big) \cdot \lambda_{\mathcal{P},\Phi}(M, s)$, *and*

$$\lambda_{\mathcal{P},\Phi}\big((m, 1, \ldots, 1), s\big) = \sum_{\substack{c_1, c_2, \ldots, c_r \in \mathbb{Z}_{>0} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r}. \tag{6.2}$$

is the $(m, 1, \ldots, 1)^{th}$ *(or more informally the* m^{th}*) Hecke eigenvalue of* $E_{P,\Phi}$.

Moreover, suppose ϕ_j *has Langlands parameter* $(\alpha_{j,1}, \ldots, \alpha_{j,n_j})$, *with the convention that if* $n_j = 1$ *then* $\alpha_{j,1} = 0$. *We also assume that each* ϕ_j *is normalized to have Petersson norm* $\langle \phi_j, \phi_j \rangle = 1$. *Then the first coefficient of* $E_{\mathcal{P},\Phi}$ *is given by*

$$A_{\mathcal{P},\Phi}\big((1, \ldots, 1), s\big) = \prod_{\substack{k=1 \\ n_k \ne 1}}^{r} L^*\big(1, \mathrm{Ad}\ \phi_k\big)^{-\frac{1}{2}} \prod_{1\le j<\ell\le r} L^*\big(1 + s_j - s_\ell,\ \phi_j \times \phi_\ell\big)^{-1}$$

up to a non-zero constant factor with absolute value depending only on n. *Here*

$$L^*(1,\, \mathrm{Ad}\ \phi_k) = L(1,\, \mathrm{Ad}\ \phi_k) \prod_{1\le i\ne j\le n_k} \Gamma\left(\frac{1 + \alpha_{k,i} - \alpha_{k,j}}{2}\right)$$

and

$$L^*(1 + s_j - s_\ell,\ \phi_j \times \phi_\ell) = \begin{cases} L^*(1 + s_j - s_\ell,\, \phi_j) & \text{if } n_\ell = 1 \text{ and } n_j \ne 1, \\ L^*(1 + s_j - s_\ell,\, \phi_\ell) & \text{if } n_j = 1 \text{ and } n_\ell \ne 1, \\ \zeta^*(1 + s_j - s_\ell) & \text{if } n_j = n_\ell = 1. \end{cases}$$

Otherwise, $L^*(1+s_j-s_\ell,\phi_j\times\phi_\ell)$ is the completed Rankin-Selberg L-function.

Proof. The proof of this proposition is given in [GSW23]. $\square$

Theorem 6.3. (Main Theorem) *Let $\mathcal{P}:=\mathcal{P}_{n_1,n_2,\ldots n_r}$ be a parabolic subgroup of $\mathrm{GL}(n,\mathbb{R})$, and let $\Phi=\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_r$ be an automorphic form associated to $\mathcal{P}$. For $j=1,2,\ldots,r$, let $(\alpha_{j,1},\ldots,\alpha_{j,n_j})$ denote the Langlands parameter of ϕ_j. We adopt the convention that if $n_j=1$ then $\alpha_{j,1}=0$.*

Let $E_{\mathcal{P},\Phi}(g,s)$ be as in Definition 4.6. Define

$$E^*_{\mathcal{P},\Phi}(g,s):=\left(\prod_{1\le j<\ell\le r}L^*\big(1+s_j-s_\ell,\ \phi_j\times\phi_\ell\big)\right)E_{\mathcal{P},\Phi}(g,s),$$

where $s=(s_1,s_2,\ldots s_r)\in\mathbb{C}^r$ satisfies $\sum\limits_{j=1}^{r}n_js_j=0$. Then the $M=(m,1,\ldots,1)^{\text{th}}$ Fourier-Whittaker coefficient of $E^*_{\mathcal{P},\Phi}(g,s)$, defined by

$$FW_{\mathcal{P},\Phi}(g,M,s):=\int\limits_0^1\cdots\int\limits_0^1 E^*_{\mathcal{P},\Phi}(ug,s)\exp\left(-2\pi i\sum_{i=1}^{n-1}m_iu_{i,i+1}\right)\prod_{1\le i<j\le n}du_{i,j},$$

satisfies the functional equations

$$FW_{\mathcal{P},\Phi}(g,M,s)=FW_{\sigma\mathcal{P},\sigma\Phi}(g,M,\sigma s)$$

for any $\sigma\in S_r$. The action of σ on $\mathcal{P}$, Φ, and s is given by

$$\begin{aligned}\sigma\mathcal{P}&:=\mathcal{P}_{n_{\sigma(1)},n_{\sigma(2)},\ldots,n_{\sigma(r)}},\\ \sigma\Phi&:=\phi_{\sigma(1)}\otimes\phi_{\sigma(2)}\otimes\cdots\phi_{\sigma(r)},\\ \sigma s&:=\big(s_{\sigma(1)},s_{\sigma(2)},\ldots,s_{\sigma(r)}\big).\end{aligned}$$

Proof. By Proposition 6.1, it suffices to show that each of the following three expressions is invariant under the action of any $\sigma\in S_r$:

$$\prod_{\substack{k=1\\ n_k\neq 1}}^{r}L^*\big(1,\mathrm{Ad}\ \phi_k\big)^{-\frac12},\qquad \sum_{\substack{c_1,c_2,\ldots,c_r\,\in\,\mathbb{Z}_{>0}\\ c_1c_2\cdots c_r=m}}\lambda_{\phi_1}(c_1)\cdots\lambda_{\phi_r}(c_r)\cdot c_1^{s_1}\cdots c_r^{s_r},\qquad W_{\alpha_{\mathcal{P},\Phi}(s)}\big(Mg\big).$$

The first two of these expressions clearly satisfy these invariances. Moreover, by Proposition 5.3, the above action of σ amounts to a certain permutation of the coordinates of the Langlands parameter $\alpha_{\mathcal{P},\Phi}(s)$. It is well-known ([Gol15]) that the Whittaker function $W_{\alpha_{\mathcal{P},\Phi}(s)}$ is invariant under such permutations. $\square$

Corollary 6.4. *Suppose $\sigma\in S_r$ satisfies $\sigma\Phi=\Phi$. Then*

$$FW_{\mathcal{P},\Phi}(g,M,s)=FW_{\mathcal{P},\Phi}(g,M,\sigma s).$$

Proof. This is immediate from our Main Theorem. $\square$

Corollary 6.5. (Functional equations of $E^*_{\mathcal{P},\Phi}(g,s)$) *Suppose $\sigma \in S_r$ satisfies*

$$\sigma\mathcal{P} = \mathcal{P}_{n_{\sigma(1)},n_{\sigma(2)},\ldots,n_{\sigma(r)}}, \quad \sigma\Phi = \phi_{\sigma(1)} \otimes \phi_{\sigma(2)} \otimes \cdots \phi_{\sigma(r)}, \quad \sigma s = \big(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(r)}\big).$$

Then

$$\boxed{E^*_{\mathcal{P},\Phi}(g,s) = E^*_{\sigma\mathcal{P},\sigma\Phi}(g,\sigma s),}$$

for all $g \in \mathrm{GL}(n,\mathbb{R})$.

Proof. This follows from the fact that it is known by Langlands (see [Lan76], [MW95]) that $E^*_{\mathcal{P},\Phi}(g,s)$ satisfies functional equations in the variables s. It is easy to see that every Fourier Whittaker coefficient of $E^*_{\mathcal{P},\Phi}(g,s)$ has to have the same functional equation. Furthermore, one can check that the functional equations found by Langlands match the functional equations of the Fourier Whittaker coefficients given in Theorem. 6.3 □

Example 6.6. The Borel Eisenstein series satisfies $FW_{\mathcal{B}}(g,M,s) = FW_{\mathcal{B}}(g,M,\sigma s)$ for any $\sigma \in S_n$.

Example 6.7. If $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r$ and $\phi_k = \phi_j$ for some $1 \le k \ne j \le r$, and σ is the transposition that interchanges k and j, then $FW_{\mathcal{P},\Phi}(g,M,s) = FW_{\mathcal{P},\Phi}(g,M,\sigma s)$.

7. Uniqueness of functional equations for self-dual Langlands Eisenstein series

Conjecture 7.1. *Let $n \ge 2$. Suppose that $\mathcal{P} = \mathcal{P}_{n_1,n_2,\ldots,n_r}$ with $n = n_1 + n_2 + \cdots + n_r$ with $r \ge 2$ and $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_r$ with each ϕ_j a Maass form for $\mathrm{SL}(n_j,\mathbb{Z})$ if $n_j \ge 2$ and ϕ_j is the constant function one if $n_j = 1$. If $E^*_{\mathcal{P},\Phi}(g,s)$ satisfies a functional equation of the form*

$$E^*_{\mathcal{P},\Phi}(g,s) = E^*_{\mathcal{P},\Phi}(g,\mu(s))$$

for some affine transformation

$$\mu\left(\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & & & \\ a_{11} & a_{12} & \cdots & a_{r1} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}, \tag{7.2}$$

where $a_{ij}, b_i \in \mathbb{R}$ for all $1 \le i,j \le r$. Then, in fact, $\mu = \sigma$ for some $\sigma \in S_r$ for which $\sigma\mathcal{P} = \mathcal{P}$ and $\sigma\Phi = \Phi$.

We give the following two results as evidence for this conjecture.

Proposition 7.3. *Conjecture 7.1 holds in the special case that $\phi_1 = \phi_2 = \cdots = \phi_r$ is the same automorphic form ϕ with $r \ge 2$. (Note: In this case, every permutation $\sigma \in S_r$ has the property that $\sigma\mathcal{P} = \mathcal{P}$ and $\sigma\Phi = \Phi$.)*

Conjecture 7.4. *Fix an integer $\kappa \ge 2$. Assume $\ell_1, \ell_2, \ldots, \ell_\kappa$ are arbitrary integers greater than one. Let $\eta_1, \eta_2, \ldots, \eta_\kappa$ be distinct Maass forms where each η_j is a Maass form for $\mathrm{SL}(\ell_j,\mathbb{Z})$ with associated n^{th} Hecke eigenvalue $\lambda_j(n)$. Then there exists a prime p such that $\lambda_1(p), \lambda_2(p), \ldots, \lambda_\kappa(p)$ are all distinct and non zero.*

Remark 7.5. *Conjecture 7.4 can be proved in the case that $\kappa = 2$ by the methods introduced in* [JS81].

Proposition 7.6. *Assume Conjecture 7.4 and assume that μ given in (7.2) is linear, i.e., $b_i = 0$ for each $i = 1, 2, \ldots, r$. Then Conjecture 7.1 holds.*

Remark 7.7. *The proof uses the fact that if $\phi_j \neq \phi_k$, then we can find p such that $\lambda_j(p)$ and $\lambda_k(p)$ are nonzero and distinct. If μ is not necessarily linear and Φ consists of at least two distinct automorphic forms, our proof breaks down if $\lambda_j(p) = p^b \lambda_k(p)$ for some value of b independent of p.*

Before giving the proofs of these propositions, we give a Lemma which reduces the proof of Conjecture 7.1 to showing that a functional equation for the divisor sum puts strong restrictions on the affine transformation μ.

Lemma 7.8. *Assume that there exist integers $r_1, r_2, \ldots, r_k$ for which $r = r_1 + r_2 + \cdots + r_k$ and*

$$\Phi = \overbrace{\phi_1 \otimes \phi_1 \otimes \cdots \otimes \phi_1}^{r_1 \text{ times}} \otimes \overbrace{\phi_2 \otimes \phi_2 \otimes \cdots \otimes \phi_2}^{r_2 \text{ times}} \otimes \cdots \otimes \overbrace{\phi_k \otimes \phi_k \otimes \cdots \otimes \phi_k}^{r_k \text{ times}}. \tag{7.9}$$

Set $\lambda_j = \lambda_{\phi_j}$, and write

$$\mu_j(s) = a_{j1}s_1 + a_{j2}s_2 + \cdots + a_{jr}s_r + b_j,$$

where $\phi_i = \phi_j$ if and only if $i = j$. Conjecture 7.1 holds if the following is true: If μ is an affine transformation for which, setting $\hat{r}_i = \sum_{j=1}^{i} r_j$,

$$\sum_{j=1}^{k} \lambda_j(p) \left(\sum_{i=\hat{r}_{j-1}+1}^{\hat{r}_j} p^{s_i} \right) = \sum_{j=1}^{k} \lambda_j(p) \left(\sum_{i=\hat{r}_{j-1}+1}^{\hat{r}_j} p^{\mu_i(s)} \right) \tag{7.10}$$

is true for all $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ satisfying $\sum_{j=1}^{r} n_j s_j$ and all primes p, then it must be the case that $\mu = \sigma \in S_r$ is of the form $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_k$ for which $\sigma_j \in S_{r_j}$ permutes the j-th block of r_j forms $\phi_j \otimes \phi_j \otimes \cdots \phi_j$.

Proof. If $E^*_{\mathcal{P},\Phi}(g, s) = E^*_{\mathcal{P},\Phi}(g, \mu(s))$, then it must be true that for every g the product of the divisor sum and the Whittaker function appearing in $FW_{\mathcal{P},\Phi}(g, (p, 1, \ldots, 1), s)$ is invariant under $s \mapsto \mu(s)$. Since we can choose $g = \operatorname{diag}(p^{-1}, 1, \ldots, 1)$, for which the Whittaker function portion of $FW_{\mathcal{P},\Phi}$ is independent of p, it must be the case that the divisor sum by itself is invariant. Hence, it suffices to prove that the only possible affine transformations μ which preserve the divisor sum are in fact permutations σ for which $\sigma\Phi = \Phi$. □

Remark 7.11. *There is no loss in generality in assuming that Φ is in the special form given in (7.9). Indeed, if Φ is not of this form, then there exists some element $\sigma_0 \in S_r$ such that $\sigma_0\Phi$ is of the desired form. Since Corollary 6.5 implies that*

$$E^*_{\mathcal{P},\Phi}(g, s) = E^*_{\sigma_0\mathcal{P},\sigma_0\Phi}(g, \sigma_0(s))$$

we have that

$$E^*_{\mathcal{P},\Phi}(g, s) = E^*_{\mathcal{P},\Phi}(g, \mu(s)),$$

if and only if

$$E^*_{\sigma_0\mathcal{P},\sigma_0\Phi}(g, s) = E^*_{\mathcal{P},\Phi}(g, \sigma_0^{-1}(s)) = E^*_{\mathcal{P},\Phi}(g, (\mu\sigma_0^{-1})(s)) = E^*_{\sigma_0\mathcal{P},\sigma_0\Phi}(g, (\sigma_0\mu\sigma_0^{-1})(s)).$$

Moreover, assuming μ is a permutation of which preserves Φ is equivalent to assuming that $\sigma = (\sigma_0 \mu \sigma_0^{-1})$ is a permutation which preserves $\sigma_0 \Phi$, i.e., of the form $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k$.

7.1. **Proof of Proposition 7.3.** In this case, note that (7.10) simplifies to give

$$\lambda(p)\big(p^{s_1} + p^{s_2} + \cdots + p^{s_r}\big) = \lambda(p)\big(p^{\mu_1(s)} + p^{\mu_2(s)} + \cdots p^{\mu_r(s)}\big).$$

We may assume that p is a prime for which $\lambda(p) \neq 0$, hence

$$p^{s_1} + p^{s_2} + \cdots + p^{s_r} = p^{\mu_1(s)} + p^{\mu_2(s)} + \cdots p^{\mu_r(s)}. \tag{7.12}$$

By Lemma 7.8, we just need to show that the only way this can possibly hold is if each term $\mu_j(s)$ is actually equal to $s_{\sigma(j)}$ for some permutation $\sigma \in S_r$.

To see that this is the case, first fix $s_2, \ldots, s_{r-1}$ and assume that $s_1 \in \mathbb{R}$ with $s_1 \to \infty$. Then in order for the asymptotics of the left hand side of (7.12) to agree with those of the right hand side, it must be the case that $\mu_{j_1}(s) = s_1$ for some $j_1 \in \{1, 2, \ldots, r\}$. This same argument gives, for each $i = 1, 2, \ldots, r-1$, that $s_i = \mu_{j_i}(s)$ for some $j_i \in \{1, 2, \ldots, r\}$. Similarly, we see that $\mu_{j_r}(s) = s_r$ by looking at the case that $s_1 \to -\infty$ with $s_2, \ldots, s_{r-1}$ fixed. Therefore, μ is given by the map $i \mapsto j_i$, which, by the pigeonhole principle, is a permutation. □

7.2. **Proof of Proposition 7.6.** We assume that Φ is as in Lemma 7.8 and that μ is linear, i.e., (7.2) holds with $b_1 = b_2 = \cdots = b_r = 0$.

In order to simplify the proof, we set some notation. Recall, first, that $\hat{r}_i = \sum_{j=1}^{i} r_j$. Then let

$$I_j := \big\{ i \in \mathbb{Z} \mid \hat{r}_{j-1} < i \leq \hat{r}_j \big\}.$$

For each $j = 1, 2, \ldots, k$, we set $\lambda_j := \lambda_{\phi_j}$. Then (7.9) is equivalent to

$$\sum_{j=1}^{k} \lambda_j(p) \left(\sum_{i \in I_j} p^{s_i} \right) = \sum_{j=1}^{k} \lambda_j(p) \left(\sum_{i \in I_j} p^{\mu_i(s)} \right). \tag{7.13}$$

As in the proof of Proposition 7.3, we choose p such that $\lambda_j(p) = \lambda_k(p)$ if and only if $j = k$. Then, for a particular $i \in I_1$, consider the limit $s_i \to +\infty$ (where s_j is fixed for $j \neq i, r$). Then the left hand side of (7.13) is asymptotic to $\lambda_1(p) p^{s_i}$. To agree with the right hand side, it must be the case that $\mu_{j_i}(s) = s_i$ for some choice of $j_i \in I_1$. This shows, again via the pigeonhole principle, that μ permutes the variables $\big\{ s_1, \ldots, s_{r_1} \big\}$.

The same argument holds for $i \in I_k$ for $k = 2, \ldots, r$ by considering $s_i \to +\infty$ with s_j fixed for $j \neq i, 1$. Comparing the asymptotics of both sides of (7.13), we conclude that μ permutes the set $\big\{ s_i \mid i \in I_k \big\}$. Combined with Lemma 7.8, this completes the proof. □

Remark 7.14. *We observe that if one is in a case that μ is linear, i.e., that μ is as in* (7.2) *with the constants $b_i = 0$ for each $i = 1, 2, \ldots, r$, then the representation of μ as a matrix is not unique. Indeed, we can think of μ as an element in the image of the natural map $\psi : M_{r \times r}(\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}^r)$, where*

$$V = \mathbb{C}^r / \{ (s_1, s_2, \ldots, s_r) \mid s_1 + s_2 + \cdots + s_r = 0 \}.$$

Note that ψ is clearly surjective, but its kernel contains the subspace

$$W = \left\{ \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_r & a_r & \cdots & a_r \end{bmatrix} \middle| \, a_1, a_2, \ldots, a_r \in \mathbb{C} \right\}.$$

By a simple dimension counting argument, in fact, we see that $\ker(\psi) = W$.

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