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Inventiones mathematicae

Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series

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§ 0. Introduction

In Sects. 302 and 304 of the Disquisitiones [5, 6], Gauss gives conjectures for the average number of properly primitive classes of a given determinant. We state these conjectures in the more modern form

$$\text{Average } h(D) \approx \frac{2\pi}{7\zeta(3)} |D|^{\frac{1}{2}} \qquad (D<0),$$

$$\text{Average } h(D) \log \varepsilon_D \approx \frac{\pi^2}{12\zeta(3)} D^{\frac{1}{2}} \qquad (D>0)$$

where $h(D)$ is the narrow class number of the order of discriminant D contained in the quadratic field $Q(\sqrt{D})$; and for $D>0$, ε_D is the unit defined by $\varepsilon_D = \frac{1}{2}(t+u\sqrt{D})$, where t, u are the smallest positive integral solutions of $t^2 - Du^2 = 4$. Here, the average is taken over all $D \equiv 0, 1 \pmod 4$ which are not perfect squares. It is well known that D can always be expressed as $D = D_0 m^2$ where D_0 is the discriminant of the field $Q(\sqrt{D})$ and in this case

$$h(D) = h(D_0) m \prod_{p|m} (1 - \chi_{D_0}(p)/p), \qquad \text{if } D<0,$$

$$h(D) \log \varepsilon_D = h(D_0)(\log \varepsilon_{D_0}) m \prod_{p|m} (1 - \chi_{D_0}(p)/p), \qquad \text{if } D>0.$$

Gauss' conjecture for imaginary quadratic fields was first proved by Lipschitz [14] and subsequently improved upon by Mertens [15]. Vinogradov [25–28] has the best result at present, namely

$$\sum_{0<-D<x} h(D) = \frac{4\pi}{21\zeta(3)} x^{\frac{3}{2}} - \frac{2}{\pi^2} x + O(x^{\frac{2}{3}+\varepsilon}). \tag{0.1}$$

Other interesting results have also been obtained by [2] and [23].

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Siegel [22] was the first to prove the Gauss conjecture for real quadratic fields. He obtained the result

$$\sum_{0<D<x} h(D)\log\varepsilon_D=\frac{\pi^2}{18\zeta(3)}x^{\frac{3}{2}}+O(x\log(x)).$$

This was further improved upon by Shintani [21] who showed

$$\sum_{0<D<x} h(D)\log\varepsilon_D=\frac{\pi^2}{18\zeta(3)}x^{\frac{3}{2}}-\frac{1}{2\zeta(2)}x\log x + \frac{\left(1-\log 2\pi+\frac{\zeta'}{\zeta}(2)\right)}{2\zeta(2)}x+O(x^{\frac{3}{4}+\varepsilon}). \tag{0.2}$$

Let $D\equiv 0,1 \pmod 4$ and not a perfect square. We consider the nonprincipal real Dirichlet character ψ_D where

$$\psi_D(n)=\left(\frac{D}{n}\right) \quad \text{(Kronecker's symbol mod}\,|D|), \tag{0.3}$$

and the Dirichlet L-function

$$L(s,\psi_D)=\sum_{n=1}^{\infty}\psi_D(n)n^{-s}.$$

In view of Dirichlet's class number formula [13] the asymptotic formulae (0.1) and (0.2) are equivalent to asymptotics for sums of the type

$$\sum_{0<\pm D<x} L(1,\psi_D). \tag{0.4}$$

It is curious that until very recently, no one has ever considered sums of type (0.4) going over primitive characters or equivalently over fundamental discriminants.

The main purpose of this work is to consider the more general related problem of obtaining asymptotic for sums of the type

$$\sum_{0<\pm m<x} L(\rho,\chi_m) \quad (\mathrm{Re}(\rho)\geqq\tfrac{1}{2}) \tag{0.5}$$

where m is squarefree and χ_m is the real primitive Dirichlet character defined by

$$\chi_m(n)=\begin{cases}\left(\dfrac{m}{n}\right) & m\equiv 1 \pmod 4\\[2ex] \left(\dfrac{4m}{n}\right) & m\equiv 2,3 \pmod 4\end{cases} \tag{0.6}$$

where the symbols on the right are Kronecker symbols (mod m), (mod $4m$) respectively. Note that χ_m is slightly different from the character defined in (0.3). Moreover, $\zeta(\rho)L(\rho,\chi_m)$ is precisely the zeta function associated to the quadratic field $Q(\sqrt{m})$.

In Goldfeld-Viola [8] conjectures for the asymptotics for sums of type (0.5) (going over primitive characters) were formulated. Jutila [12] was the first to prove the conjecture for (both real and imaginary) quadratic Dirichlet L-series. He obtained the result

$$\sum_{0<\pm m<x} L(\tfrac{1}{2},\chi_m)=c_1 x\log x+c_2 x+O(x^{\frac{3}{4}+\varepsilon})$$

for certain constants c_1 and c_2. It is clear that his method also works at a general complex point ρ. Takhtadzjan and Vinogradov [24] gave asymptotics for sums of the type

$$\sum_{0<-D<x} L(\rho,\psi_D) \qquad (\mathrm{Re}(\rho)\geqq\tfrac{1}{2})$$

where the sum goes over all $D\equiv 0,1 (\mathrm{mod}\, 4)$ and not a perfect square, and with ψ_D as in (0.3). They assert that their methods also apply to (0.5) with primitive characters, but mention that in this case the error terms become less manageable.

We now state our main theorems:

Theorem (1). *Let* $\varepsilon>0$ *be fixed. Let* χ_m *be defined as in* (0.6). *Then there exist analytic functions* $c(\rho)$ *and* $c_\pm^*(\rho)$ *with Laurent expansions* $c(\rho)=c_{\frac{1}{2}}/(\rho-\frac{1}{2})+c'_{\frac{1}{2}}+O(\rho-\frac{1}{2})$, $c_\pm^*(\rho)=-c_{\frac{1}{2}}/(\rho-\frac{1}{2})+c^{*\prime}_{\pm\frac{1}{2}}+O(\rho-\frac{1}{2})$ *such that*

$$\sum_{\substack{1<\pm m<x\\ m\ \text{squarefree}}} L(\rho,\chi_m)$$

$$=\begin{cases} c(\rho)x+O(x^{\frac{1}{2}+\varepsilon}), & \text{if } \mathrm{Re}(\rho)\geqq 1\\ c(\rho)x+c_\pm^*(\rho)x^{\frac{3}{2}-\rho}+O(x^{\theta+\varepsilon}), & \text{if } \rho\neq\frac{1}{2},\ \frac{1}{2}\leqq\mathrm{Re}(\rho)<1\\ c_{\frac{1}{2}}x\log x+(c'_{\frac{1}{2}}+c^{*\prime}_{\pm\frac{1}{2}}-c_{\frac{1}{2}})x+O(x^{\frac{19}{32}+\varepsilon}) & \text{if } \rho=\frac{1}{2}. \end{cases}$$

Here

$$c(\rho)=\tfrac{3}{4}(1-2^{-2\rho})\zeta(2\rho)\prod_{p\neq 2}(1-p^{-2}-p^{-2\rho-1}+p^{-2\rho-2}),$$

$$c_{\frac{1}{2}}=\tfrac{3}{16}\prod_{p\neq 2}(1-2p^{-2}+p^{-3})$$

and

$$\theta=\begin{cases} \dfrac{1}{2} & \text{if } \mathrm{Re}(\rho)>\dfrac{-5+\sqrt{193}}{12}\\ \dfrac{19+3\,\mathrm{Re}(\rho)-6\,\mathrm{Re}(\rho)^2}{24+16\,\mathrm{Re}(\rho)} & \text{if } \dfrac{1}{2}\leqq Re(\rho)\leqq\dfrac{-5+\sqrt{193}}{12} \end{cases}$$

and all 0*-constants depend at most on* ρ,ε.

Theorem (2). *Let* $\mathrm{Re}(\rho)\geqq\frac{1}{2}$. *The Dirichlet series*

$$Z_\pm(\rho,w)=\sum_{\substack{\pm m>1\\ m\ \text{squarefree}}} L(\rho,\chi_m)|m|^{-w}$$

converges absolutely for $\mathrm{Re}(w)>1$. *It has a meromorphic continuation to the half-plane* $\mathrm{Re}(w)>\frac{1}{2}$ *with simple poles at* $w=1$, $\frac{3}{2}-\rho$ *unless* $\rho=\frac{1}{2}$ *when it has a double pole at* $w=1$. *For* $\rho\neq\frac{1}{2}$, *the residues at* $w=1$, $\frac{3}{2}-\rho$ *are* $c(\rho)$ *and* $c_{\pm}^{*}(\rho)$. *Finally, for* $\frac{1}{2}<\mathrm{Re}(w)\leqq 1$ *and* $\varepsilon>0$, *we have the growth estimate*

$$Z(\rho,w)\ll|\mathrm{Im}(w)|^{1-\mathrm{Re}(w)+\varepsilon}, \qquad |\mathrm{Im}(w)|\to\infty$$

where the $\ll$ *constant depends at most on* ρ,ε.

Remarks. For technical reasons, we have only computed $c_{\pm}^{*}(\rho)$ for the modified sum $\sum\limits_{1<\pm m<x} L_2(\rho,\chi_m)$ where

$$L_2(\rho,\chi_m)=(1-\chi_m(2)2^{-\rho})L(\rho,\chi_m),$$

is the L-series with 2-factor removed. In this case, $c_{\pm}^{*}(\rho)$ is given in (4.15), (4.16).

Theorem (1) is an average over fundamental discriminants whereas the previous results (0.1) and (0.2) go over all discriminants.

The leading terms in Theorem (1), namely $c(\rho)x$ and $c_{\frac{1}{2}}x\log x$ are the same for both positive and negative discriminants. This is in contradistinction to Gauss' original conjecture.

By use of the Rankin-Selberg method [17, 18] it is possible to obtain a version of Theorem (1) for the squares $|L(\rho,\chi_m)|^2$. It does not appear as if these methods can improve Theorem (1).

The first theorem is not a consequence of the second. Although the main terms in the asymptotic formula given in Theorem (1) can be derived from Theorem (2), it is not possible to obtain the same error terms.

In some sense, the exponent of $\frac{1}{2}$ in the error term of Theorem (1), for the range

$$\mathrm{Re}(\rho)>\frac{-5+\sqrt{193}}{12}\approx 0.74$$

is best possible without knowledge approaching a generalized Riemann hypothesis. This is related to the fact that [29, 16]

$$\sum_{\substack{0<m<x\\ m\text{ squarefree}}} 1=\frac{6}{\pi^2}x+O\left(x^{\frac{1}{2}}e^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}\right) \qquad \text{(unconditionally)},$$

$$\sum_{\substack{0<m<x\\ m\text{ squarefree}}} 1=\frac{6}{\pi^2}x+O\left(x^{\frac{9}{28}+\varepsilon}\right) \qquad \text{(assuming R.H.)}.$$

We conjecture that $\frac{1}{4}$ should be the correct exponent (for all ρ, $\mathrm{Re}(\rho)\geqq\frac{1}{2}$) for the error term in Theorem (1).

The proofs of Theorems (1) and (2) make use of the theory of Eisenstein series of metaplectic type. These Eisenstein series have the interesting property that their Fourier coefficients involve quadratic twists of the zeta function. It is clear (see [7]) that our methods can be generalized to obtain mean value theorems for quadratic twists of any fixed Hecke L-function with grössencharakter of any fixed number field. At present, however, we cannot obtain mean value theorems for quadratic twists of an arbitrary L-function associated to an automorphic form. We have also been unable to obtain mean value theorems for twists by cubic and higher order characters as conjectured in [9]. These appear to be difficult problems and their solutions may ultimately involve the analytic number theory of $GL(n)$.

§ 1. Eisenstein series on $\Gamma_0(4)$

Let $\Gamma_0(4)$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer coefficients, determinant one, and $c \equiv 0 \pmod 4$. This group has three inequivalent cusps at 0, $\frac{1}{2}$, $i\infty$. Let

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\}$$

be the stabilizer of $i\infty$.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we define

$$j_\gamma(z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz+d)^{\frac{1}{2}}$$

where

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv 3 \pmod 4, \end{cases} \quad (d > 0)$$

$\left(\frac{c}{d}\right)$ is the Legendre symbol, and $(cz+d)^{\frac{1}{2}}$ is chosen so that $|\arg(cz+d)^{\frac{1}{2}}| < \frac{\pi}{2}$. If $d<0$, then $j_\gamma(z)$ is defined by the relation $j_\gamma(z) = j_{-\gamma}(z)$.

It is well known [19] that $j_\gamma(z)$ is a multiplier system of weight $\frac{1}{2}$ for $\Gamma_0(4)$. This is equivalent to the fact that $j_\gamma(z)$ satisfies the cocycle relation

$$j_{\gamma\gamma'}(z) = j_\gamma(\gamma' z) j_{\gamma'}(z) \qquad \gamma, \gamma' \in \Gamma_0(4)$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma z = \dfrac{az+b}{cz+d}$.

Let k be any odd rational integer. We now construct the Eisenstein series of weight $k/2$ for the group $\Gamma_0(4)$. For $z = x+iy$; $y>0$, $x \in \mathbb{R}$, and $\mathrm{Im}(z) = y$ we define the Eisenstein series at the cusp $i\infty$ to be

$$E_\infty(z,s,k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\mathrm{Im}\, \gamma z)^s j_\gamma(z)^{-k}. \tag{1.1}$$

It is easily shown that (1.1) converges absolutely and uniformly if $\mathrm{Re}(s) > 1 - \frac{k}{4}$. Since k will usually be fixed, we will frequently suppress it in the notation and simply denote (1.1) by $E_\infty(z,s)$.

There are two additional Eisenstein series at the cusps $0, \frac{1}{2}$. These are defined as follows

$$E_0(z,s,k)=z^{-k/2}E_\infty(-1/(4z),s,k) \tag{1.2}$$
$$E_{\frac{1}{2}}(z,s,k)=(2z+1)^{-k/2}E_0(z/(2z+1),s,k)$$

and satisfy the automorphic relations

$$E_c(\gamma z,s)=j_\gamma(z)^k E_c(z,s) \qquad (c=0,\tfrac{1}{2},\infty)$$
$$E_0(-1/(4z),s)=(4z)^{k/2}i^{-k}E_\infty(z,s)$$
$$E_{\frac{1}{2}}(z+\tfrac{1}{2},s)=2^{k/2}(2z+1)^{-k/2}E_\infty(z/(2z+1),s)$$
$$E_{\frac{1}{2}}(z,s)=i^{-k}E_{\frac{1}{2}}(z+1,s).$$

For our future purposes, we require a more explicit description of the Eisenstein series defined in (1.1) and (1.2). Accordingly, let us put

$$g\left(\frac{m}{n}\right)=\sum_{a(\operatorname{mod} n)} e^{\frac{2\pi i a^2 m}{n}}$$

where $m,n\in\mathbb{Z}$, $n\neq 0$. This Gauss sum was studied by Hecke over arbitrary number fields, and he proved [11] a general reciprocity law for it. For our purposes, it is useful to consider a normalized version of $g\left(\frac{m}{n}\right)$; namely

$$G\left(\frac{m}{n}\right)=g\left(\frac{m}{n}\right)\Big/\left|g\left(\frac{m}{n}\right)\right|.$$

The value of $G\left(\frac{m}{n}\right)$ depends only on the congruence class of $m(\operatorname{mod} n)$ and satisfies

$$\left(\frac{m}{n}\right)G\left(\frac{a}{n}\right)=G\left(\frac{am}{n}\right) \qquad (m,n)=1.$$

A deeper result is Hecke's reciprocity law

$$G\left(\frac{m}{n}\right)=e^{\frac{\pi i}{4}\operatorname{sgn}\left(\frac{m}{n}\right)}G\left(\frac{-n}{4m}\right)$$

where

$$\operatorname{sgn}(a)=\begin{cases}+1 & a>0\\ -1 & a<0.\end{cases}$$

Here m,n are any pair of integers, not necessarily relatively prime.

The propositions of this section describe certain properties of the Eisenstein series (1.1) and (1.2). As the techniques by which these results are obtained are well known we omit the proofs and refer the reader to [20].

Proposition (1.1). *For* $\mathrm{Re}(s) > 1 - k/4$, *the following representations hold:*

$$E_\infty(z,s) = y^s + e^{\frac{\pi i k}{4}} y^s \sum_{\substack{(d,2c)=1\\ c>0}} \frac{G\left(\frac{-d}{4c}\right)^k}{|4cz+d|^{2s}(4cz+d)^{k/2}},$$

$$E_0(z,s) = (y/4)^s \sum_{\substack{(u,2v)=1\\ u>0}} \frac{\left(\frac{-v}{u}\right)\varepsilon_u^k}{|v+uz|^{2s}(v+uz)^{k/2}},$$

$$E_{\frac{1}{2}}(z,s) = (y/4)^s e^{-\frac{\pi i k}{4}} \sum_{\substack{(d,2c)=1\\ d>0}} \frac{G\left(\frac{d-2c}{8d}\right)^k}{|dz+c|^{2s}(dz+c)^{k/2}}.$$

We now give the Fourier expansions for $E_\infty(z,s)$, $E_0(z,s)$ and $E_{\frac{1}{2}}(z,s)$. These expansions are given in the following three propositions.

Proposition (1.2). *We have*

$$E_0(z,s,k) = \sum_{m=-\infty}^{\infty} a_m(s,y,k) e^{2\pi i m x}$$

where

$$a_m(s,y,k) = (y/4)^s \prod_{p \neq 2} \left[\sum_{l=0}^{\infty} \frac{(\varepsilon_{p^l})^k g(-m,p^l)}{p^{l(2s+k/2)}}\right] K_m(s,y,k).$$

In the above

$$g(m,n) = \sum_{\substack{a(\mathrm{mod}\, n)\\ (a,n)=1}} \left(\frac{a}{n}\right) e^{\frac{2\pi i a m}{n}}$$

$$K_m(s,y,k) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(x^2+y^2)^s (x+iy)^{k/2}} dx.$$

Corollary (1.3). *We have*

$$a_0(s,y,k) = \frac{y^s}{4^s} \frac{\zeta(4s+k-2)}{\zeta(4s+k-1)} \frac{(1-2^{-4s-k+2})}{(1-2^{-4s-k+1})} K_0(s,y,k)$$

and for m not divisible by an odd square

$$a_m(s,y,k) = \frac{y^s}{4^s} \frac{L\left(2s+\frac{k-1}{2}, \chi_m\right)}{\zeta(4s+k-1)} \frac{(1-\chi_m(2) 2^{-2s-\frac{k-1}{2}})}{(1-2^{-4s-k+1})} K_m(s,y,k)$$

where χ_m *is the real primitive Dirichlet character associated to the quadratic field* $Q(\sqrt{\mu_k m})$, *where* $\mu_k = (-1)^{(k-1)/2}$.

Proposition (1.4). *We have*

$$E_\infty(z,s)=\sum_{m=-\infty}^{\infty} b_m(s,y,k)e^{2\pi imx}$$

where

$$b_0(s,y,k)=y^s+e^{\frac{\pi ik}{4}}4^s c_0(s,k)a_0(s,y,k)$$

$$c_0(s,k)=\psi_8(k)2^{-4s-k+\frac{1}{2}}(1-2^{-4s-k+2})^{-1}$$

and for $m\neq 0$

$$b_m(s,y,k)=(1+i^k)4^s c_m(s,k)a_m(s,y,k)$$

where for $m\not\equiv 0 \pmod 4$

$$c_m(s,k)=\begin{cases}-2^{-4s-k} & (-1)^{(k-1)/2}m\not\equiv 1 \pmod 4\\ 2^{-4s-k}+\psi_8(m)2^{-6s-(3k/2)+(3/2)} & (-1)^{(k-1)/2}m\equiv 1 \pmod 4\end{cases}$$

and for $m=4^t m_0$, $m_0\not\equiv 0 \pmod 4$

$$c_m(s,k)=2^{-4s-k}\frac{(1-2^{-t(4s+k-2)})}{(1-2^{-(4s+k-2)})}+2^{-t(4s+k-2)}c_{m_0}(s,k).$$

Here, ψ_8 *denotes the real primitive Dirichlet character* (mod 8).

Proposition (1.5). *We have*

$$E_{\frac{1}{2}}(4z,s,k)=\sum_{m=-\infty}^{\infty} d_m(s,y,k)e^{2\pi im(x+\frac{1}{8})}$$

where

$$d_m(s,y,k)=\begin{cases}(1-i^k)a_m(s,y,k)2^{-2s-k+\frac{3}{2}} & m(-1)^{(k-1)/2}\equiv 1 \pmod 8\\ -(1-i^k)a_m(s,y,k)2^{-2s-k+\frac{3}{2}} & m(-1)^{(k-1)/2}\equiv 5 \pmod 8\\ 0 & \textit{otherwise.}\end{cases}$$

Before giving the functional equations satisfied by the Mellin transforms of Eisenstein series on $\Gamma_0(4)$ it is first of all necessary to obtain the transformation laws of the Eisenstein series at the cusps.

Proposition (1.6). *Let* $\alpha_{u,r}=\begin{pmatrix}1 & u/r\\ 0 & 1\end{pmatrix}$, $\tau_r=\begin{pmatrix}0 & -1\\ 4r^2 & 0\end{pmatrix}$ *where* $r\in\mathbb{Z}^+$ *and* $(u,r)=1$. *Then for* $r\equiv 1 \pmod 2$ *and a chosen so that* $-4ua\equiv 1 \pmod r$ *we have*

$$E_0(\alpha_{u,r}\tau_r z,s)=(4rz)^{k/2}(i\varepsilon_r)^{-k}\left(\frac{a}{r}\right)E_\infty(\alpha_{a,r}z,s)$$

and

$$E_\infty(\alpha_{u,r}\tau_r z,s)=(rz)^{k/2}\varepsilon_r^{-k}\left(\frac{a}{r}\right)E_0(\alpha_{a,r}z,s).$$

Proposition (1.7). *Let* τ_r *and* $\alpha_{u,r}$ *be as in Proposition* (1.6). *For a chosen so that* $au\equiv -1 \pmod{4r}$

$$E_0(\alpha_{u,4r}\tau_{2r}z,s)=\left(\frac{4rz}{i}\right)^{k/2}\left(G\left(\frac{a}{4r}\right)\right)^{-k}E_0(\alpha_{a,4r}z,s).$$

Proposition (1.8). *Let τ_r, $\alpha_{u,r}$ be as in Proposition* (1.6). *Then for $r\equiv 1 \pmod 2$ and a chosen so that $au\equiv -1 \pmod{2r}$ we have*

$$E_0(\alpha_{u,2r}\tau_r z,s)=\left(\frac{2rz}{i}\right)^{k/2}\left(G\left(\frac{-a}{8r}\right)\right)^k E_{\frac{1}{2}}(\alpha_{a,2r}z,s)$$

and

$$E_{\frac{1}{2}}(\alpha_{a,2r}\tau_r z,s)=\left(\frac{2rz}{i}\right)^{k/2}\left(G\left(\frac{-a}{8r}\right)\right)^{-k} E_0(\alpha_{u,2r}z,s).$$

We now introduce the Mellin transforms of Eisenstein series on $\Gamma_0(4)$. Let

$$\alpha_{a,r}=\begin{pmatrix}1 & a/r\\ 0 & 1\end{pmatrix}$$

as before. Then we define

$$\Phi_\infty(w,a/r;s,k)=\int_0^\infty (E_\infty(\alpha_{a,r}iy,s,k)-b_0(s,y,k))\,y^{w-1}\,dy,$$

$$\Phi_0(w,a/r;s,k)=\int_0^\infty (E_0(\alpha_{a,r}iy,s,k)-a_0(s,y,k))\,y^{w-1}\,dy,$$

$$\Phi_{\frac{1}{2}}(w,a/r;s,k)=\int_0^\infty E_{\frac{1}{2}}(\alpha_{a,r}iy,s,k)\,y^{w-1}\,dy.$$

Since we are subtracting the constant term from the Eisenstein series it is clear that the above three integrals converge absolutely for $\mathrm{Re}(w)$ sufficiently large.

The following propositions give the functional equations and locate the poles of the Mellin transforms of our Eisenstein series.

Proposition (1.9). *Let $r\equiv 1 \pmod 2$ and let a,u be chosen so that $-4au\equiv 1 \pmod r$. Then*

$$\Phi_0\left(w,s;\frac{u}{r},k\right)=\Lambda_\infty\left(w,\frac{a}{r},k\right)\Phi_\infty\left(\frac{k}{2}-w,s;\frac{a}{r},k\right)$$

where

$$\Lambda_\infty\left(w,\frac{a}{r},k\right)=(2r)^{(k/2)-2w}(-2i)^{k/2}\varepsilon_r^{-k}\left(\frac{a}{r}\right).$$

Moreover $\Phi_0\left(w,s;\frac{u}{r},k\right)$ is holomorphic except for simple poles at $w=s+\frac{k}{2}$, $1-s$, $s+\frac{k}{2}-1$, with corresponding residues equal to $\Lambda_\infty\left(s+\frac{k}{2},\frac{a}{r},k\right)$, $c_1(s,k)\Lambda_\infty\left(1-s,\frac{a}{r},k\right)$, $-c_1(s,k)\psi_8(k)(i)^{-k/2}2^{2s+k-\frac{1}{2}}(1-2^{-4s-k+2})$ where

$$c_1(s,k)=\frac{\psi_8(k)i^{k/2}\zeta(4s+k-2)K_0(s,1,k)}{2^{4s+k-\frac{1}{2}}\zeta(4s+k-1)(1-2^{-4s-k+1})}.$$

Proposition (1.10). *Let a,u be chosen so that $au \equiv -1 \pmod{4r}$. Then*

$$\Phi_0\left(w,s;\frac{u}{4r},k\right)=\Lambda_0\left(w,\frac{a}{4r},k\right)\Phi_0\left(\frac{k}{2}-w,s;\frac{a}{4r},k\right)$$

where

$$\Lambda_0\left(w,\frac{a}{4r},k\right)=2^{k-4w}r^{(k/2)-2w}G\left(\frac{a}{4r}\right)^{-k}.$$

Moreover, $\Phi_0\left(w,\frac{u}{4r};s,k\right)$ is holomorphic except for simple poles at $w=1-s$, $s+\frac{k}{2}-1$. The residue at $w=1-s$ is $\Lambda_0\left(1-s,\frac{a}{4r},k\right)c_2(s,k)\zeta(4s+k-1)^{-1}$ where $c_2(s,k)$ is given by

$$c_2(s,k)=\psi_8(k)\,i^{-k/2}\,2^{2a+k-\frac{1}{2}}\zeta(4a+k-1)(1-2^{-4a-k+2})\,c,(a,k).$$

Proposition (1.11). *Let r be odd and $au \equiv -1 \pmod{2r}$. Then*

$$\Phi_0\left(w,s;\frac{u}{2r},k\right)=\Lambda_{\frac{1}{2}}\left(w,\frac{a}{2r},k\right)\Phi_{\frac{1}{2}}\left(\frac{k}{2}-w,s;\frac{a}{2r},k\right)$$

where

$$\Lambda_{\frac{1}{2}}\left(w,\frac{a}{2r},k\right)=2^{(k/2)-2w}r^{(k/2)-2w}G\left(\frac{-a}{8r}\right)^{k}.$$

Moreover, $\Phi_0\left(w,s;\frac{u}{2r},k\right)$ is holomorphic except for a simple pole at

$$w=s+\frac{k}{2}-1.$$

§ 2. Whittaker functions

We now investigate the Whittaker functions $K_m(s,y,k)$ occurring in the Fourier expansions of the $\frac{1}{2}$-integral weight Eisenstein series we have been considering. It will be shown that the Mellin transform of $K_m(s,y,k)$ is essentially a hypergeometric function.

Proposition (2.1). *For* $\mathrm{Re}(w)>\mathrm{Re}(s)+\frac{k}{2}-1$, $m\neq 0$

$$\int_0^\infty K_m(s,y,k)\,y^{w+s-1}\,dy=\frac{e^{-\frac{\pi i k}{4}}F_\pm(w,s,k)}{2^{2w-k/2}\pi^{w-s-k/2}|m|^{w+1-s-k/2}}$$

where

$$F_\pm(w,s,k)=\Gamma\left(w-s+1-\frac{k}{2}\right)\Gamma(w+s)\begin{cases}\dfrac{F\left(w+s,w-s+1-\frac{k}{2},w+1-\frac{k}{2};\frac{1}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(w+1-\frac{k}{2}\right)} & m>0(+)\\[2ex] \dfrac{F\left(w+s,w-s+1-\frac{k}{2},w+1;\frac{1}{2}\right)}{\Gamma(s)\Gamma(w+1)} & m<0(-)\end{cases}$$

and

$$F(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n$$

is the Gaussian hypergeometric function.

Proof. An easy consequence of the relations given in [1, 15].

It will be convenient, at this point, to change notation and let

$$\rho=2s+(k-1)/2,$$
$$w=s+k/2+\delta.$$

We now define

$$\begin{aligned} G_k(\rho,\delta)&=F_+\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta,\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)\\ G_{-k}(\rho,\delta)&=F_-\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta,\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right).\end{aligned}\tag{2.1}$$

It is clear that there is no ambiguity in this definition since it is easily checked that

$$G_k(\rho,\delta)=G_{-(-k)}(\rho,\delta).$$

Proposition (2.2). *Let $G_k(\rho,\delta)$ be defined as in* (2.1). *Then for ρ fixed, $|\delta|$ large*

$$G_k(\rho,\delta)=\frac{2^{\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta}\,\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}\left(1+O\left(\frac{1}{|\delta|}\right)\right)$$

where the O-symbol depends at most on ρ and k.

Proof. Using the transformation ([1], 15, 3.6)

$$\begin{aligned} F(a,b,c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)\\ &+\frac{(1-z)^{c-a-b}\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1;1-z),\end{aligned}$$

valid for $|\arg(1-z)|<\pi$, and applying this to (2.1), we see that

$$\begin{aligned} G_k(\rho,\delta)=&\frac{\Gamma\left(\frac{5}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}\right)}G_{-k}(\rho,\delta)\\ &+\frac{2^{\frac{1}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}}\,\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}F\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4},\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4},\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4};\frac{1}{2}\right).\end{aligned}\tag{2.2}$$

For large $|\delta|$, the function F in (3.11) is $1+O\left(\frac{1}{|\delta|}\right)$ and the proposition follows by substituting (2.2), with k replaced by $-k$, for $G_{-k}(\rho,\delta)$ in (2.2), and solving the resulting expression for $G_k(\rho,\delta)$. Q.E.D.

Proposition (2.3). *Let*

$$D_k(\rho,\delta)=G_k(\rho,\delta)G_{-k-4}(\rho,\delta)-G_{k+4}(\rho,\delta)G_{-k}(\rho,\delta).$$

Then for fixed ρ, k and $|\delta|\to\infty$, we have

$$D_k(\rho,\delta)\sim 2^{2\delta+\frac{3}{2}+\rho}\frac{\Gamma\left(\delta+\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\delta+\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}.$$

Furthermore, if ρ,δ are fixed there exist infinitely many integers $k\equiv 1 \pmod 4$ and also infinitely many $k\equiv 3 \pmod 4$ such that $D_k(\rho,\delta)\neq 0$.

Proof. The first part of the proposition easily follows from the asymptotics given in Proposition (2.2).

For the second part, we combine (2.1) and Proposition (2.1) to get

$$G_{\pm k}(\rho,\delta)=\Gamma(1+\delta)\Gamma\left(\rho+\frac{1}{2}+\delta\right)\frac{F\left(\rho+\frac{1}{2}+\delta, 1+\delta, \frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta; \frac{1}{2}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}\pm\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta\right)}. \tag{2.3}$$

Now, for ρ,δ fixed and $k\to\pm\infty$, $\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta\notin\mathbb{Z}$ it is easily seen that

$$F\left(\rho+\frac{1}{2}+\delta, 1+\delta, \frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta; \frac{1}{2}\right)=1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}+\delta\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta\right)}+O\left(\frac{1}{k^2}\right).$$

Then by Stirling's formula, one checks that for infinitely many k the two terms in the definition of $D_k(\rho,\delta)$ do not cancel. Q.E.D.

Proposition (2.4). *We have*

$$K_0(s,y,k)=y^{1-2s-k/2}2^{k/2}\sqrt{\pi}\,e^{\frac{-\pi i k}{4}}\frac{\Gamma\left(s+\frac{k}{2}-\frac{1}{2}\right)\Gamma\left(2s+\frac{k}{2}-1\right)}{\Gamma(2s+k-1)\Gamma(s)}.$$

Proof. See [20].

§ 3. Growth estimates

Recalling the Fourier expansion given in Proposition (1.2), namely

$$E_0(z,s,k)=\sum_{m=-\infty}^{\infty}a_m(s,y,k)e^{2\pi i m x}$$

we now define $A_m(\rho, k)$ by the relation

$$a_m(s, y, k) = (y/4)^s A_m(\rho, k) K_m(s, y, k) \tag{3.1}$$

where

$$\rho = 2s + \frac{k-1}{2}.$$

In view of Proposition (1.2) we also have

$$A_m(\rho, k) = \prod_{p \neq 2} \left(\sum_{l=0}^{\infty} \frac{(\varepsilon_{p^l})^k g(-m, p^l)}{p^{l(\rho+\frac{1}{2})}} \right).$$

When m is squarefree it follows from Corollary (1.3) that

$$A_m(\rho, k) = \frac{L(\rho, \chi_m)(1 - \chi_m(2) 2^{-\rho})}{\zeta(2\rho)(1 - 2^{-2\rho})} \tag{3.2}$$

where χ_m is the real primitive Dirichlet character associated to $Q(\sqrt{\mu_k m})$, where $\mu_k = (-1)^{(k-1)/2}$. Note that this field is either real or imaginary according to whether $\mu_k m$ is positive or negative.

Now, for $\mathrm{Re}(\rho) \geqq \frac{1}{2}$ and $\mathrm{Re}(\delta) > 0$, we define the zeta function

$$Z_{\pm}(\rho, \delta, r, k) = \sum_{\substack{\pm m > 0 \\ m \equiv 0 (\mathrm{mod}\, r)}} A_m(\rho, k) |m|^{-1-\delta}. \tag{3.3}$$

The series on the right side of (3.3) converges absolutely for $\mathrm{Re}(\delta) > 0$. This is easily verified because the Rankin-Selberg zeta function

$$\sum_{m \neq 0} |A_m(\rho, k)|^2 |m|^{-1-\delta}$$

has its first pole at $\delta = 0$.

The function $A_m(\rho, k)$ satisfies

$$A_m(\rho, k) = A_m(\rho, k \pm 4) = A_{-m}(\rho, k+2).$$

Therefore $Z_{\pm}(\rho, \delta, r, k)$ satisfies

$$Z_{\pm}(\rho, \delta, r, k) = Z_{\pm}(\rho, \delta, r, k \pm 4) = Z_{\mp}(\rho, \delta, r, k+2).$$

For simplicity we shall henceforth assume that $k \equiv (1 \bmod 4)$.

The main purpose of this section will be to shown that for fixed ρ, k the zeta function in (3.3) has a meromorphic continuation to the entire complex δ-plane and satisfies certain growth estimates uniformly in δ and r.

Proposition (3.1). *Let m be squarefree. Then for* $\mathrm{Re}(\rho) \geqq \frac{1}{2}$, $\mathrm{Re}(w)$ *sufficiently large, we have*

$$\sum_{n=1}^{\infty} A_{mn^2}(\rho, k) n^{-w} = A_m(\rho, k) \frac{\zeta(w)\zeta(w + 2\rho - 1)(1 - 2^{-w-2\rho+1})}{L(\rho + w, \chi_m)(1 - \chi_m(2) 2^{-\rho-w})}$$

where χ_m is the real primitive Dirichlet character associated to $Q(\sqrt{\mu_k m})$.

Proof. See [20].

Proposition (3.2). *Let $m=m_0n^2$ where m_0 is squarefree. Then for* $\mathrm{Re}(\rho)\geqq\frac{1}{2}$

$$|A_{m_0n^2}(\rho,k)|\leqq|A_{m_0}(\rho,k)|\,\sigma(n)2^{w(n)}$$

where $w(n)$ is the number of distinct prime divisors of n and $\sigma(n)$ is the number of divisors of n.

Proof. Equating coefficients in the Dirichlet series occurring in Proposition (3.1), we have

$$A_{m_0n^2}(\rho,k)=A_{m_0}(\rho,k)\sum_{\substack{d_1d_2d_3=n\\(2,d_2d_3)=1}} d_2^{1-2\rho}\chi_{m_0}(d_3)\mu(d_3)d_3^{-\rho}$$

where μ is the Mobius function. The proposition then follows immediately.

Proposition (3.3). *The function $\zeta(2p)Z_\pm(\rho,\delta,r,k)$ defined in* (3.3) *has for fixed ρ with* $\mathrm{Re}(\rho)\geqq\frac{1}{2}$ *a meromorphic continuation to the whole complex δ-plane. Moreover, it has simple poles at $\delta=0,\frac{1}{2}-\rho,-1$.*

Proof. For $\mathrm{Re}(\delta)>0$, the series on the right hand side of (3.3) converges absolutely and defines a holomorphic function. It is, therefore, now necessary to consider the case when $\mathrm{Re}(\delta)\leqq 0$.

We recall the Mellin transform

$$\Phi_0\left(w,s;\frac{a}{r},k\right)=\int_0^\infty[E_0(\alpha_{a,r}iy;s,k)-a_0(s,y,k)]\,y^{w-1}\,dy.$$

In order to simplify notation, we set

$$\Phi_c^*\left(\rho,\delta;\frac{a}{r},k\right)=\Phi_c\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta,\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4};\frac{a}{r},k\right)$$

for $c=0,\infty,\frac{1}{2}$.

Now, using Propositions (1.2), (2.1) together with (2.1) and (3.1) it follows that

$$\begin{aligned}\Phi_0^*\left(\rho,\delta;\frac{a}{r},k\right)=c_k(\rho,\delta)\Big[G_k(\rho,\delta)&\sum_{m>0}A_m(\rho,k)e^{2\pi ima/r}|m|^{-1-\delta}\\ +G_{-k}(\rho,\delta)&\sum_{m<0}A_m(\rho,k)e^{2\pi ima/r}|m|^{-1-\delta}\Big]\end{aligned}\tag{3.4}$$

where

$$c_k(\rho,\delta)=i^{-k/2}\pi^{-\delta}2^{k/2-2\rho-2\delta-1}.$$

We now define

$$\phi_0(\rho,\delta;r,k)=r^{-1}\sum_{a(\mathrm{mod}\,r)}\Phi_0^*\left(\rho,\delta;\frac{a}{r},k\right)\tag{3.5}$$

and summing (3.4) we obtain

$$\phi_0(\rho,\delta;r,k)=c_k(\rho,\delta)[G_k(\rho,\delta)Z_+(\rho,\delta,r,k)+G_{-k}(\rho,\delta)Z_-(\rho,\delta,r,k)].\tag{3.6}$$

Substituting k and $k+4$ into (3.6) we can then solve for $Z_\pm(\rho,\delta,r,k)$ to get

$$\begin{aligned}Z_\pm(\rho,\delta,r,k)=\pm D_k(\rho,\delta)^{-1}[&\phi_0(\rho,\delta;r,k)G_{\mp(k+4)}(\rho,\delta)c_k(\rho,\delta)^{-1}\\ &-\phi_0(\rho,\delta;r,k+4)G_{\mp k}(\rho,\delta)c_{k+4}(\rho,\delta)^{-1}],\end{aligned}\tag{3.7}$$

where

$$D_k(\rho,\delta)=G_k(\rho,\delta)G_{-k-4}(\rho,\delta)-G_{k+4}(\rho,\delta)G_{-k}(\rho,\delta)$$

is the discriminant.

The representation (3.7) gives the meromorphic continuation of $Z_\pm(\rho,\delta,r,k)$ to the whole complex δ-plane. The only possible singularities are the zeros of $D_k(\rho,\delta)$ and the poles of $\phi_0(\rho,\delta;r,k)$. By Proposition (1.9) and (3.5) we see that $\phi_0(\rho,\delta;r,k)$ has three simple poles at $\delta=0,\frac{1}{2}-\rho,\ -1$. Now, since

$$Z_\pm(\rho,\delta;r,k)=Z_\pm(\rho,\delta;r,k+4)$$

it follows that for δ fixed and $\delta\neq 0,\ \frac{1}{2}-\rho,\ -1$ and δ a singularity of $Z_\pm(\rho,\delta,r,k_1)$ then $D_k(\rho,\delta)=0$ for $k\equiv k_1 (\mathrm{mod}\, 4)$. By Proposition (2.3) this is impossible. Q.E.D.

Proposition (3.4). *Let $\varepsilon>0$, $\mathrm{Re}(\rho)\geqq\frac{1}{2}$ be fixed. Let $r>0$ be squarefree. Let $\delta=\sigma+it$ satisfy $|\delta|>\varepsilon$, $|\delta-\frac{1}{2}+\rho|>\varepsilon$, $|\delta+1|>\varepsilon$ and $-\mathrm{Re}(\rho)-\frac{1}{2}-\varepsilon\leqq\sigma\leqq\varepsilon$. Then there exists a positive function $h(r)$ (independent of t,σ) such that*

$$|Z_\pm(\rho,\delta,r^2,k)|\ll r^{-2\sigma-2+2\varepsilon}|t|^{-\sigma+\varepsilon}h(r)$$

where the $\ll$ constant depends at most on ρ, σ, k, ε. Moreover, the function $h(r)$ satisfies the condition

$$\sum_{r=1}^{\infty} h(r)r^{-1-\varepsilon}\ll 1$$

where the above $\ll$-symbol depends at most on ε.

Proof. For $\varepsilon>0$, it follows from (3.3) that

$$|Z_\pm(\rho,\varepsilon+it,r^2,k)|\ll r^{-2-2\varepsilon}h_1(r)$$

where

$$h_1(r)=\sum_{m\neq 0}|A_{mr^2}(\rho,k)|\cdot|m|^{-1-\varepsilon}.$$

Let $m=m_0n^2$ where m_0 is squarefree. Then by Proposition (3.2)

$$\begin{aligned}|A_{mr^2}(\rho,k)|&\leqq|A_{m_0}(\rho,k)|\,\sigma(nr)2^{w(nr)}\\&\ll|A_{m_0}(\rho,k)|(nr)^{\varepsilon/2}\end{aligned}$$

since $\sigma(n)\ll n^{\varepsilon/4}$ and $2^{w(n)}\ll n^{\varepsilon/4}$. Consequently

$$\begin{aligned}\sum_{r=1}^{\infty}\frac{h_1(r)}{r^{1+\varepsilon}}&\ll\sum_{r=1}^{\infty}\sum_{m_0}\sum_{n\neq 0}|A_{m_0}(\rho,k)|\cdot|nr|^{\varepsilon/2}\cdot|m_0n^2r|^{-1-\varepsilon}\\&\ll\sum_{m_0\,\mathrm{sq.free}}|A_{m_0}(\rho,k)|\cdot|m_0|^{-1-\varepsilon}\\&\ll 1.\end{aligned}\tag{3.8}$$

This proves the proposition when $\sigma=\varepsilon$.

In order to estimate the growth of $Z_\pm(\rho,\delta,r^2,k)$ when $\mathrm{Re}(\delta)<\varepsilon$ we shall apply the functional equations of section (1) to the Mellin transforms of the appropriate Eisenstein series. This will give growth estimates when $\mathrm{Re}(\delta)=-\mathrm{Re}(\rho)-\frac{1}{2}-\varepsilon$. The standard Phragmen-Lindelof principle will then be applied to give bounds within the critical strip.

Now, choose

$$\delta=-\rho-\tfrac{1}{2}-\varepsilon-it$$

where ρ, ε satisfy the conditions in the statement of the proposition.

For r odd, we rewrite (3.5) as

$$\phi_0(\rho,\delta,r^2,k)=r^{-2}\sum_{d\mid r^2}\sum_{\substack{u(\mathrm{mod}\, d)\\(u,d)=1}}\Phi_0^*\left(\rho,\delta;\frac{u}{d},k\right). \tag{3.9}$$

By the functional equation given in Proposition (1.9) it follows that for $w=\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta$

$$\Phi_0^*\left(\rho,\delta;\frac{u}{d},k\right)=\Lambda_\infty\left(\frac{k}{2}-w,\frac{a}{d},k\right)\Phi_\infty^*\left(\rho,\varepsilon+it;\frac{a}{d},k\right) \tag{3.10}$$

where

$$-4au\equiv 1(\mathrm{mod}\, d)$$

and

$$\Lambda_\infty\left(w,\frac{a}{d},k\right)=(2d)^{k/2-2w}(-2i)^{k/2}\varepsilon_d^{-k}\left(\frac{a}{d}\right).$$

Now, by Proposition (1.4), we have, for $\delta'=\varepsilon+it$

$$\begin{aligned}\Phi_\infty^*\left(\rho,\varepsilon+it;\frac{a}{d},k\right)&=2^{\rho-k/2+1/2}(1+i^k)\,c_k(\rho,\delta')\\&\cdot\left[G_k(\rho,\delta')\sum_{m>0}\frac{A_m(\rho,k)\,c_m\left(\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4},k\right)e^{2\pi ima/d}}{|m|^{1+\delta}}\right.\\&\left.+G_{-k}(\rho,\delta')\sum_{m<0}\frac{A_m(\rho,k)\,c_m\left(\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4},k\right)e^{2\pi ima/d}}{|m|^{1+\delta}}\right].\end{aligned} \tag{3.11}$$

It is easily seen by Proposition (1.4) that

$$c_m\left(\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4},k\right)\ll\log|m|.$$

So if we substitute (3.11) into (3.10) and sum over $u(\mathrm{mod}\, d)$, $(u,d)=1$ on the left (which is the same as summing over $a(\mathrm{mod}\, d)$, $(a,d)=1$ on the right side of (3.11)), we obtain

$$\begin{aligned}\sum_{\substack{u(\mathrm{mod}\, d)\\(u,d)=1}}\Phi_0^*\left(\rho,\delta;\frac{u}{d},k\right)&\ll\Big[G_k(\rho,\varepsilon+it)\sum_{m>0}|A_m(\rho,k)|\,|g(m,d)|\,|m|^{-1-\varepsilon/2}\\&+G_{-k}(\rho,\varepsilon+it)\sum_{m<0}|A_m(\rho,k)|\,|g(m,d)|\,|m|^{-1-\varepsilon/2}\Big]d^{\rho+\frac{1}{2}+2\varepsilon}\end{aligned} \tag{3.12}$$

where the $\ll$-constant is independent of d, t, and

$$g(m,d)=\sum_{a(\bmod d)}\left(\frac{a}{d}\right)e^{2\pi iam/d}$$

is the imprimitive Gauss sum which satisfies the bound

$$|g(m,d)|\leq\begin{cases}0 & r_2\nmid m\\ r_1^{\frac{1}{2}}r_2\left(\frac{m}{r_2},r_2\right) & r_2\mid m\end{cases} \tag{3.13}$$

where $d=r_1r_2^2$ and $(r_1,r_2)=1$.

Now, by Proposition (2.2) it follows that for $|t|$ sufficiently large

$$G_{\pm k}(\rho,\varepsilon+it)\ll\Gamma\left(\frac{\rho}{2}+\frac{1}{4}\pm\frac{k}{4}+\varepsilon+it\right). \tag{3.14}$$

Combining (3.9), (3.12), (3.13) and (3.14) we obtain

$$\phi_0(\rho,\delta,r^2,k)\ll r^{2\rho-1+4\varepsilon}\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{|k|}{4}+\varepsilon+it\right)h_2(r) \tag{3.15}$$

where

$$h_2(r)=\sum_{r_1r_2r_3=r}r_1^{-\rho-2\varepsilon}r_3^{-2\rho-1-4\varepsilon}\sum_{n\neq 0}|A_{nr_2}(\rho,k)|(n,r_2)|n|^{-1-\varepsilon/2}. \tag{3.16}$$

In order to estimate the right side of (3.7) we use (3.14), Proposition (2.3) and Stirling's formula

$$|t|^{\sigma-\frac{1}{2}}e^{-(\pi/2)|t|}\ll\Gamma(\sigma+it)\ll|t|^{\sigma-\frac{1}{2}}e^{-(\pi/2)|t|},\qquad(|t|\to\infty)$$

in conjunction with (3.15) to obtain, for $k>0$, the bounds

$$\frac{\phi_0(\rho,\delta,r^2,k)G_{-k-4}(\rho,\delta)}{D_k(\rho,\delta)}\ll r^{2\rho-1+4\varepsilon}|t|^{\rho-3/2+2\varepsilon}h_2(r),$$

$$\frac{\phi_0(\rho,\delta,r^2,k+4)G_{-k}(\rho,\delta)}{D_k(\rho,\delta)}\ll r^{2\rho-1+4\varepsilon}|t|^{\rho+1/2+2\varepsilon}h_2(r).$$

This can be put into (3.7) to yield

$$Z_+(\rho,-\rho-\tfrac{1}{2}-\varepsilon-it,r^2,k)\ll r^{2\rho-1+4\varepsilon}|t|^{\rho+1/2+2\varepsilon}h_2(r). \tag{3.17}$$

Since Z_+ depends only on the congruence class of $k(\bmod 4)$ the bound (3.17) must hold for all k. In an identical manner, it is also easily shown that

$$Z_-(\rho,-\rho-\tfrac{1}{2}-\varepsilon-it,r^2,k)\ll r^{2\rho-1+4\varepsilon}|t|^{\rho+\frac{1}{2}+2\varepsilon}h_2(r).$$

Finally, to summarize, we have obtained the growth estimates

$$Z_\pm(\rho,\varepsilon+it,r^2,k)\ll r^{-2-2\varepsilon}h(r)$$

$$Z_\pm(\rho,-\rho-\tfrac{1}{2}-\varepsilon-it,r^2,k)\ll r^{2\rho-1+4\varepsilon}|t|^{\rho+\frac{1}{2}+2\varepsilon}h(r) \tag{3.18}$$

where

$$h(r)\ll h_1(r)+h_2(r).$$

By the Phragmen-Lindelof principle [10] if follows from (3.18) that

$$Z_{\pm}(\rho,\sigma+it,r^2,k)\ll r^{-2\sigma-2+2\varepsilon}|t|^{-\sigma+\varepsilon}h(r)$$

which gives the first part of the proposition.

To prove the second part of the proposition it is enough to show (in view of (3.8)) that

$$\sum_{r=1}^{\infty} h_2(r)r^{-1-\varepsilon}\ll 1. \tag{3.19}$$

This is easily proved by substituting (3.16) into the left side of (3.19) and noting that

$$\begin{aligned}\sum_{r=1}^{\infty} h_2(r)r^{-1-\varepsilon} &\leqq \sum_{r_1=1}^{\infty} r_1^{-\rho-1-2\varepsilon}\sum_{r_3=1}^{\infty} r_3^{-2\rho-2-4\varepsilon}\cdot\sum_{n\neq 0}\sum_{r_2=1}^{\infty}\frac{|A_{nr_2}(\rho,k)|(n,r_2)}{r_2^{1+\varepsilon}|n|^{1+\varepsilon/2}}\\ &\ll \sum_{n\neq 0}\sum_{r=1}^{\infty}\frac{|A_{nr}(\rho,k)|(n,r)}{|nr|^{1+\varepsilon/2}}\\ &=\sum_{m\neq 0}\sum_{d_1d_2=m}\frac{|A_m(\rho,k)|(d_1,d_2)}{|m|^{1+\varepsilon/2}}.\end{aligned} \tag{3.20}$$

Writing $m=m_0n^2$ with m_0 *squarefree it is clear that* $(d_1,d_2)|n$. Hence

$$(d_1,d_2)m^{-1-\varepsilon/2}\leqq m_0^{-1-\varepsilon/2}n^{-1-\varepsilon}. \tag{3.21}$$

Since $\sum_{d_1d_2=m} 1\ll m^{\varepsilon/4}$, it follows from (3.20), (3.21) and Proposition (3.2) that (3.19) holds. This completes the proof of Proposition (3.3) in the case that r is odd.

When r is even, the proof is similar to the above, with Propositions (1.10) and (1.11) used in place of (1.9). We omit the details.

§ 4. Proof of main theorems

It was shown in Proposition (3.3) that for $\mathrm{Re}(\rho)\geqq\frac{1}{2}$, the function $\zeta(2\rho)Z_{\pm}(\rho,\delta,r^2,k)$ has a meromorphic continuation to the whole complex δ-plane whose only singularities, for $\rho\neq\frac{1}{2}$, are simple poles at $\delta=0,\frac{1}{2}-\rho,-1$. For $\rho=\frac{1}{2}$, it has a double pole at $\delta=0$ and a simple pole at $\delta=-1$.

We now give the residues at 0 and $\frac{1}{2}-\rho$. The residue at -1 is $\ll 1$ and is independent of r.

Proposition (4.1). *For* $r>0$ *and squarefree,* $\mathrm{Re}(\rho)\geqq\frac{1}{2}$, $\rho\neq\frac{1}{2}$, *let* $R_{\pm}(\rho,a,r^2,k)$ *denote the residue of* $\zeta(2\rho)Z_{\pm}(\rho,\delta,r^2,k)$ *at* $\delta=a$. *We have*

$$R_{\pm}(\rho,0,r^2,k)=\begin{cases}\zeta(2\rho)r^{-2}\sum_{d|r}d^{-2\rho-1}\phi(d^2) & (r,2)=1\\ \zeta(2\rho)r^{-2}\sum_{d|r_0}d^{-2\rho-1}\phi(d^2) & r=2r_0,\end{cases}$$

$$R_{\pm}(\rho,\tfrac{1}{2}-\rho,r^2,k)=\begin{cases}2^{\rho+\frac{1}{2}}\pi^{\frac{1}{2}-\rho}r^{-2}\sum_{d|r}d^{2\rho-3}\phi(d^2)h_{\pm}(\rho,\tfrac{1}{2}-\rho,k) & (r,2)=1\\ 2^{3\rho-\frac{5}{2}}\pi^{\frac{1}{2}-\rho}r_0^{-2}\sum_{d|r_0}d^{2\rho-3}\phi(d^2)h_{\pm}(\rho,\tfrac{1}{2}-\rho,k) & r=2r_0,\end{cases}$$

where

$$h_{\pm}(\rho,\tfrac{1}{2}-\rho,k)=\pm D_k(\rho,\tfrac{1}{2}-\rho)^{-1}[c(\rho,k)G_{\mp(k+4)}(\rho,\tfrac{1}{2}-\rho)-c(\rho,k+4)G_{\mp k}(\rho,\tfrac{1}{2}-\rho)],$$

and

$$c(\rho,k)=\frac{i^{k/2}\psi_8(k)\zeta(2\rho-1)K_0\left(\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4},1,k\right)}{2^{2\rho+\frac{1}{2}}(1-2^{-2\rho})}.$$

Proposition (4.2). *For $r>0$ and squarefree and $\rho=\frac{1}{2}$ let $R_{\pm}(r^2,k)$, $R'_{\pm}(r^2,k)$ denote the coefficients of δ^{-2} and δ^{-1} of the Laurent expansion of $\zeta(2\rho)Z_{\pm}(\rho,\delta,r^2,k)$ about $\delta=0$. We have*

$$R_{\pm}(r^2,k)=\begin{cases}\frac{1}{2}r^{-2}\sum_{d\mid r}d^{-2}\phi(d^2) & (r,2)=1\\ \frac{1}{8}r_0^{-2}\sum_{d\mid r_0}d^{-2}\phi(d^2) & r=2r_0\end{cases}$$

and

$$R'_{\pm}(r^2,k)=\begin{cases}c_{\pm}(k)r^{-2}\sum_{d\mid r}d^{-2}\phi(d^2)-2r^{-2}\sum_{d\mid r}d^{-2}(\log d)\phi(d^2) & (r,2)=1\\ (c_{\pm}(k)-\log 2)r^{-2}\sum_{d\mid r_0}d^{-2}\phi(d^2)-2r^{-2}\sum_{d\mid r_0}d^{-2}(\log d)\phi(d^2), & r=2r_0\end{cases}$$

where $c_{\pm}(k)$ is given by

$$c_{\pm}(k)=\gamma+\frac{1}{2}\log\pi-\frac{1}{2}\log 2-\frac{1}{2}\frac{g^{\pm\prime}_{1,k}}{g^{\pm}_{1,k}}\left(\frac{1}{2}\right)-\frac{1}{2}\frac{g^{\pm}_{2,k}}{g^{\pm}_{1,k}}\left(\frac{1}{2}\right),\tag{4.1}$$

with

$$g^{\pm}_{1,k}(\rho)=\pm D_k(\rho,\tfrac{1}{2}-\rho)^{-1}[G_{\mp(k+4)}(\rho,\tfrac{1}{2}-\rho)-G_{\mp k}(\rho,\tfrac{1}{2}-\rho)]$$

$$g^{\pm}_{2,k}(\rho)=\pm D_k(\rho,\tfrac{1}{2}-\rho)^{-1}[G_{\mp(k+4)}(\rho,\tfrac{1}{2}-\rho)b_k-G_{\mp k}(\rho,\tfrac{1}{2}-\rho)b_{k+4}],$$

$$b_k=2\frac{\Gamma'}{\Gamma}(1)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-\frac{k}{4}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{k}{4}\right)+4\frac{\zeta'}{\zeta}(0)-10\log 2.$$

Proof of Propositions (4.1) *and* (4.2). Recall from (3.10) that the residues of $Z_{\pm}$ can be obtained from the residues of ϕ_0. These in turn are given in terms of the residues of Φ_0, which are described in Propositions (1.9), (1.10) and (1.11). The proposition then follows after a long but routine calculation.

We now give the proofs for Theorems (1) and (2) of the introduction. Firstly, asymptotics are obtained for sums of type

$$\sum_{\substack{0<\pm m<x\\ m\equiv 0(\mathrm{mod}\, r^2)}} A_m(\rho,1)$$

by use of the analytic properties of the function $Z_{\pm}(\rho,\delta,r^2,1)$ as given in section (3). The reduction to sums over squarefree m

$$\sum_{0<\pm m<x} A_m(\rho,1)$$

is a consequence of the simple linear sieve

$$\sum_{\substack{0<\pm m<x \\ m \text{ squarefree}}} A_m(\rho,1) = \sum_{0<\pm m<x} A_m(\rho,1) \sum_{r^2|m} \mu(r)$$

where $\mu(r)$ is the Möbius function. We, of course, use the fact that for m squarefree, $A_m(\rho,1)$ is essentially a Dirichlet L-series.

Proposition (4.3). *Let $\rho=\beta+it\neq\frac{1}{2}$ with $\beta\geqq\frac{1}{2}$. Let $\varepsilon>0$ be fixed. Then for $x\geqq r^2$ with r squarefree*

$$\zeta(2\rho) \sum_{\substack{0<\pm m<x \\ m\equiv 0 (\mathrm{mod}\, r^2)}} A_m(\rho,1) = R_\pm(\rho,0,r^2,1)x + R_\pm(\rho,\tfrac{1}{2}-\rho,r^2,1)x^{\frac{3}{2}-\rho} + O((x/r^2)^{g(\beta)+\varepsilon} x^\varepsilon h(r))$$

where $R_\pm(\rho,a,r^2,1)$ is given in Proposition (4.1) *and $h(r)$ is given in Proposition* (3.4). *Moreover,*

$$g(\beta) = \begin{cases} (\beta+\frac{3}{2})^{-1}(\frac{19}{16}+\frac{3}{16}\beta-\frac{3}{8}\beta^2) & \frac{1}{2}\leqq\beta\leqq 1 \\ (\beta+\frac{3}{2})^{-1} & \beta>1. \end{cases}$$

The function $g(\beta)$ is monotonically decreasing with $g(\frac{1}{2})=\frac{19}{32}$ and $g(\alpha)=\frac{1}{2}$ with $\alpha = \dfrac{-5+\sqrt{193}}{12} \approx 0.74$.

Proposition (4.4). *Let $\varepsilon>0$ be fixed. Then for $x\geqq r^2$ with r squarefree, we have*

$$\lim_{\rho\to\frac{1}{2}} \zeta(2\rho) \sum_{\substack{0<\pm m<x \\ m\equiv 0 (\mathrm{mod}\, r^2)}} A_m(\rho,1) = R_\pm(r^2,1)x\log x + (R'_\pm(r^2,1)-R_\pm(r^2,1))x + O((x/r^2)^{\frac{19}{32}+\varepsilon} x^\varepsilon h(r))$$

where $R_\pm(r^2,1)$ and $R'_\pm(r^2,1)$ are given in Proposition (4.2) *and $h(r)$ is given in Proposition* (3.4).

Proof. We now prove Propositions (4.3) and (4.4). Let, for $\varepsilon>0$ and $1\leqq T<x/r^2$

$$I = \frac{1}{2\pi i}\int_{\mathscr{R}} \zeta(2\rho) Z_\pm(\rho,\delta,r^2,1) \frac{x^{1+\delta}}{1+\delta}\, d\delta$$

where $\mathscr{R}$ is the rectangle with corners $\varepsilon\pm iT$ and $-\beta-\frac{1}{2}-\varepsilon\pm iT$, $\beta=\mathrm{Re}(\rho)$, traversed in a counter clockwise direction. By Cauchy's theorem, I can be computed by summing the residues of $Z_\pm$. It then follows from Propositions (4.1) and (4.2) that

$$I = \begin{cases} R_\pm(\rho,0,r^2,1)x + R_\pm(\rho,\frac{1}{2}-\rho,r^2,1)x^{\frac{3}{2}-\rho}, & \rho\neq\frac{1}{2} \\ R_\pm(r^2,1)x\log x + (R'_\pm(r^2,1)-R_\pm(r^2,1))x, & \rho=\frac{1}{2}. \end{cases} \tag{4.2}$$

On the other hand, the contribution to I from each of the sides of $\mathscr{R}$ can be computed in a manner similar to [4], p. 104. Using the estimates of Propositions (3.4) and (3.2) together with Burgess' estimate [3], $L(\rho,\chi_m) \ll |m|^{\beta(\rho)+\varepsilon}$, $\beta(\rho)=\frac{3}{8}(1-\mathrm{Re}\,\rho)$, and setting

$$T=\left(\frac{x}{r^2}\right)^{(\frac{1}{2}+\beta(\rho)+\mathrm{Re}(\rho))/(\frac{3}{2}+\mathrm{Re}(\rho))}$$

the propositions follow.

Proof of Theorems (1) *and* (2). Recall that

$$\zeta(2\rho)\sum_{\substack{0<\pm m<x\\ m\,\text{squarefree}}} A_m(\rho,1)=\sum_{r<\sqrt{x}}\mu(r)\sum_{\substack{0<\pm m<x\\ m\equiv 0(\mathrm{mod}\,r^2)}}\zeta(2\rho)A_m(\rho,1).$$

By Proposition (4.1) (i.e. $\rho\neq\frac{1}{2}$)

$$\zeta(2\rho)\sum_{\substack{0<\pm m<x\\ m\,\text{squarefree}}} A_m(\rho,1)=\sum_{r<\sqrt{x}}\mu(r)[R_\pm(\rho,0,r^2,1)x+R_\pm(\rho,\tfrac{1}{2}-\rho,r^2,1)x^{\frac{3}{2}-\rho}]$$
$$+O\left(\sum_{r<\sqrt{x}}\left[\left(\frac{x}{r^2}\right)^{g(\beta)+\varepsilon}x^\varepsilon h(r)\right]\right). \tag{4.3}$$

If $g(\beta)\leqq\frac{1}{2}$ then

$$\sum_{r<\sqrt{x}}\left(\frac{x}{r^2}\right)^{g(\beta)+\varepsilon}x^\varepsilon h(r)\ll x^{\frac{1}{2}+2\varepsilon}\sum_{r<\sqrt{x}}\frac{h(r)}{r^{1+2\varepsilon}}\ll x^{\frac{1}{2}+2\varepsilon} \tag{4.4}$$

by Proposition (3.4).

On the other hand, if $g(\beta)>\frac{1}{2}$

$$\sum_{r<\sqrt{x}}\left(\frac{x}{r^2}\right)^{g(\beta)+\varepsilon}x^\varepsilon h(r)\ll x^{g(\beta)+2\varepsilon}\sum_{r<\sqrt{x}}\frac{h(r)}{r^{2g(\beta)+2\varepsilon}}\ll x^{g(\beta)+2\varepsilon}. \tag{4.5}$$

Letting

$$\alpha_\pm(\rho,a)=\sum_{r=1}^{\infty}\mu(r)R_\pm(\rho,a,r^2,1) \tag{4.6}$$

it is easy to verify by Proposition (4.1) that

$$\begin{aligned}\alpha_\pm(\rho,0)&=3\zeta(2\rho)2^{-2}P(2\rho+1)\\ \alpha_\pm(\rho,\tfrac{1}{2}-\rho)&=\pi^{\frac{1}{2}-\rho}(2^{\frac{1}{2}+\rho}-2^{3\rho-\frac{3}{2}})h_\pm(\rho,\tfrac{1}{2}-\rho,1)P(3-2\rho),\qquad(\mathrm{Re}(\rho)<1)\end{aligned} \tag{4.7}$$

where

$$P(w)=\prod_{p\neq 2}(1-p^{-2}-p^{-w}+p^{-w-1}). \tag{4.8}$$

In view of (4.3) it is necessary to estimate

$$\sum_{r\geqq\sqrt{x}}\mu(r)R_\pm(\rho,0,r^2,1)\ll\sum_{r\geqq\sqrt{x}}r^{-2+\varepsilon}\ll x^{-\frac{1}{2}+\varepsilon} \tag{4.9}$$

and

$$\sum_{r\geqq\sqrt{x}}\mu(r)R_\pm(\rho,\tfrac{1}{2}-\rho,r^2,1)\ll\sum_{r\geqq\sqrt{x}}r^{2\beta-3+\varepsilon}\ll x^{\beta-1+\varepsilon/2} \tag{4.10}$$

since $\beta=\mathrm{Re}(\rho)<1-\varepsilon$.

Consequently, it follows from (4.3), (4.7), (4.9) and (4.10) that

$$\sum_{\substack{0<\pm m<x\\ m\,\text{squarefree}}}\zeta(2\rho)A_m(\rho,1)=\alpha_\pm(\rho,0)x+\alpha_\pm(\rho,\tfrac{1}{2}-\rho)x^{\frac{3}{2}-\rho}+\mathscr{E}(x) \tag{4.11}$$

where

$$\mathscr{E}(x) \ll \begin{cases} x^{g(\beta)+\varepsilon}, & \frac{1}{2} \leqq \beta \leqq \alpha \\ x^{\frac{1}{2}+\varepsilon}, & \alpha \leqq \beta \end{cases}$$

and $\alpha = (\sqrt{193}-5)/12 \approx 0.74$, and moreover, the term $\alpha_{\pm}(\rho, \frac{1}{2}-\rho) x^{\frac{3}{2}-\rho}$ must be omitted when $\mathrm{Re}(\rho) = \beta \geqq 1$.

In the case when $\rho = \frac{1}{2}$, we obtain, in a similar manner using Proposition (4.2)

$$\lim_{\rho \to \frac{1}{2}} \sum_{\substack{0 < \pm m < x \\ m \,\text{squarefree}}} \zeta(2\rho) A_m(\rho, 1) = \alpha_{\pm} x \log x + (\alpha'_{\pm} - \alpha_{\pm}) x + O(x^{\frac{19}{32}}) \tag{4.12}$$

where

$$\alpha_{\pm} = \sum_{r=1}^{\infty} \mu(r) R_{\pm}(r^2, 1) = \tfrac{3}{8} P(2)$$

$$\alpha'_{\pm} = \sum_{r=1}^{\infty} \mu(r) R'_{\pm}(r^2, 1)$$

$$= P(2) \left[\frac{3}{4} c_{\pm}(1) + \frac{1}{4} \log 2 + \frac{3}{2} \sum_{p \neq 2} \frac{(\log p)(p-1)}{p^3 - 2p + 1} \right]. \tag{4.13}$$

Recall that (see 3.2) if m is squarefree

$$\zeta(2\rho) A_m(\rho, 1) = (1 - 2^{-2\rho})^{-1} L_2(\rho, \chi_m).$$

Thus to obtain Theorem (1) with $L(\rho, \chi)$ replaced by $L_2(\rho, \chi)$ we simply multiply (4.11) and (4.12) by $1 - 2^{-2\rho}$. In the special case $\rho = 1$ it is also necessary to remove the term corresponding to $m = 1$.

We then obtain

$$\sum_{\substack{1 < \pm m < x \\ m \,\text{squarefree}}} L_2(\rho, \chi_m) = \begin{cases} c(\rho) x + O(x^{\frac{1}{2}+\varepsilon}), & \text{if } \mathrm{Re}(\rho) \geqq 1 \\ c(\rho) x + c^*_{\pm}(\rho) x^{\frac{3}{2}-\rho} + O(x^{\theta+\varepsilon}), & \text{if } \rho \neq \frac{1}{2},\ \frac{1}{2} \leqq \mathrm{Re}(\rho) < 1 \\ c_{\frac{1}{2}} x \log x + (c'_{\frac{1}{2}} + c^{*\prime}_{\pm\frac{1}{2}} - c_{\frac{1}{2}}) x + O(x^{\frac{19}{32}+\varepsilon}), & \end{cases} \tag{4.14}$$

where

$$c^*_{\pm}(\rho) = (1 - 2^{-2\rho}) \alpha_{\pm}(\rho, \tfrac{1}{2} - \rho) \quad (\rho \neq \tfrac{1}{2}) \tag{4.15}$$

$$c'_{\frac{1}{2}} + c^*_{\pm\frac{1}{2}} - c_{\frac{1}{2}} = \tfrac{1}{2}(\alpha'_{\pm} - \alpha_{\pm}), \tag{4.16}$$

and $c(\rho)$, θ, are given in the statement of Theorem (1). Here $\alpha_{\pm}(\rho, \frac{1}{2} - \rho)$ and $\alpha'_{\pm}, \alpha_{\pm}$ are given in (4.7), (4.13).

We now indicate how the proof of (4.14) can be modified to yield Theorem (1) proper; that is to say, to obtain asymptotics for

$$\sum_{\substack{1 < \pm m < x \\ m \,\text{squarefree}}} L(\rho, \chi_m).$$

Firstly,

$$\sum_{\substack{1<\pm m<x\\ m\,\text{squarefree}}} L_2(\rho,\chi_m)=\sum_{\substack{1<\pm m<x\\ m\,\text{squarefree}}}(1-\chi_m(2)2^{-\rho})L(\rho,\chi_m)$$
$$=\sum_{\substack{1<\pm m<x\\ m\,\text{squarefree}}} L(\rho,\chi_m)-2^{-\rho}\left[\sum_{\substack{1<\pm m<x\\ \pm m\equiv 1\,(\mathrm{mod}\,8)\\ m\,\text{squarefree}}} L(\rho,\chi_m)-\sum_{\substack{1<\pm m<x\\ \pm m\equiv 5\,(\mathrm{mod}\,8)\\ m\,\text{squarefree}}} L(\rho,\chi_m)\right] \tag{4.17}$$

since

$$\chi_m(2)=\begin{cases} 0 & \text{if } m\not\equiv 1\,(\mathrm{mod}\,4)\\ +1 & \text{if } m\equiv 1\,(\mathrm{mod}\,8)\\ -1 & \text{if } m\equiv 5\,(\mathrm{mod}\,8).\end{cases}$$

Let S denote the bracketed difference of two sums above (coefficient of $2^{-\rho}$). To complete the proof, it is enough to show that for $\mathrm{Re}\,\rho<1$, θ as above,

$$S=\alpha_{\pm\frac{1}{2}}(\rho)\,x^{\frac{3}{2}-\rho}+O(x^{\theta+\varepsilon}), \tag{4.18}$$

for some constant $\alpha_{\pm\frac{1}{2}}(\rho)$.

In order to show (4.18) we consider

$$E^*_{\frac{1}{2}}(z,s)=(2z+1)^{-k/2}\,E_\infty\left(\frac{z}{2z+1},s\right)=2^{-k/2}\,E_{\frac{1}{2}}(z+\tfrac{1}{2},s).$$

It can then be shown that $E^*_{\frac{1}{2}}(z,s)$ satisfies the transformation formula (for $(r,2)=1$, $-4au\equiv 1\,(\mathrm{mod}\,r)$)

$$E^*_{\frac{1}{2}}(\alpha_{u,r}\tau_r z,s)=i^{-k/2}(rz)^{k/2}\,G\left(\frac{r-2u}{8r}\right)^k E_0(\alpha_{a,r}z-\tfrac{1}{2},s) \tag{4.19}$$

which is analogous to Propositions (1.6), (1.7), (1.8).

Now, taking Mellin transforms and using Proposition (1.5) it can be shown that the same procedures used previously yield (4.18) when applied to $E^*_{\frac{1}{2}}(z,s)$. Also, note that there is no main term corresponding to x in (4.18). This is due to the fact that by (4.19) $E^*_{\frac{1}{2}}$ transforms to E_0 and not E_∞.

Finally, Theorem (2) follows immediately from Proposition (3.4) and (4.11), (4.12) together with the above remarks. Q.E.D.

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