Analytic Number Theory
Homework #3
(due Thursday, March 12, 2015)

Problem 1: By the functional equation for

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s),$$

the function $s(1-s)\xi(s)$ can be regarded as an entire function of $s^2 - s$; what is the order of this function? Use this to obtain the alternative infinite product

$$\xi(s) = \frac{\xi(1/2)}{4(s-s^2)} \prod_{\rho} \left( 1 - \left( \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}} \right)^2 \right)$$

the product extending over zeros $\rho$ of $\xi(s)$ whose imaginary part is positive. [This symmetrical form eliminates the exponential factors $e^{A+Bs}$ and $e^{s/\rho}$ occurring in the usual Hadamard factorization of $\xi(s)$.]

Problem 2: Calculate the mean value

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 \, dt$$

provided $\sigma > 1$.

Problem 3: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying $\chi(n+15) = \chi(n)$ for all $n \in \mathbb{Z}$.

Problem 4: Let $q$ be an integer which has the property that every Dirichlet character $\chi$ (mod $q$) is real valued (takes on only the values 0, $\pm 1$). Show that $q$ must divide 24.