Analytic Number Theory  
Homework #4  
(due Thursday, May 8, 2014)

**Problem 1:** Let $\chi$ be a Dirichlet character (mod $q$) for some integer $q > 1$. Prove that $L(1, \chi) \ll \log q$.

**Problem 2:** Fix a prime $p$. Show that  
\[
\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \right\} \quad \text{where} \quad c \equiv 0 \pmod{p} 
\]
is a subgroup of $SL(2, \mathbb{Z})$.

**Problem 3:** Define  
\[
P(z) := \sum_{c,d \in \mathbb{Z}} \sum_{(c,d)=1} e^{2\pi i \frac{az+b}{cz+d}} (cz+d)^2k.
\]
Here, for every pair of coprime integers $c, d$ we choose integers $a, b$ so that $ad - bc = 1$. Show that the above series is independent of the choice of $a, b$. Also show that the above series converges absolutely for $k > 1$.

**Problem 4:** Rewrite $P(z)$ as a sum involving $j(\gamma, z) = cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $k > 1$ is an integer, show that $P(z)$ is a holomorphic modular form of weight $2k$ for $SL(2, \mathbb{Z})$.

**Problem 5:** For $s \in \mathbb{C}$ with $\Re(s) > 8$, let  
\[
L(s) := \sum_{n=1}^{\infty} a(n)n^{-s}, \quad (a(n) \in \mathbb{C} \text{ for } n = 1, 2, \ldots)
\]
where $a(1) = 1$ and $|a(n)| \ll n^7$ (for $n = 1, 2, \ldots$). Assume that the function $\Phi(s) := (2\pi)^{-s}\Gamma(s)L(s)$ is an entire function which is bounded in any fixed vertical strip $\{ s \in \mathbb{C} \mid a < \Re(s) < b \}$ and satisfies the functional equation  
\[
\Phi(s) = \Phi(12 - s)
\]
for all $s \in \mathbb{C}$. Using the inverse Mellin transform, prove that  
\[
\sum_{n=1}^{\infty} a(n)e^{2\pi inz}, \quad (z \in \mathbb{H})
\]
is the Ramanujan cusp form of weight 12.