THE DISTRIBUTION OF MODULAR SYMBOLS

Dorian Goldfeld

Dedicated to Andrzej Schinzel

§1. Introduction and statement of results:

Let $f(z)$ denote a fixed Hecke newform of weight two for the congruence subgroup $\Gamma_0(N)$. For $\gamma \in \Gamma_0(N)$, we define the modular symbol

$$< \gamma,\, f > = \; -2\pi i \int_\tau^{\gamma\tau} f(z)\, dz,$$

where the integral above is independent of the value of $\tau \in \mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{i\infty\}$ and $\mathfrak{h}$ is the upper–half plane. It is well known (see [G1]) that modular symbols satisfy the following properties:

$$< \alpha \cdot \beta,\, f > = < \alpha,\, f > + < \beta,\, f > \qquad \text{for all } \alpha, \beta \in \Gamma_0(N),$$

$$\left\langle \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},\, f \right\rangle = \overline{\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix},\, f \right\rangle} \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Further, $< \gamma,\, f > = 0$ if γ is in the subgroup of $\Gamma_0(N)$ generated by the parabolic elements, the elliptic elements, and the commutators of $\Gamma_0(N)$ (see Manin [M]).

Shimura [Sh] has shown that if

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

and

$$k = \mathbb{Q}(a(1), a(2), a(3), \dots)$$

has degree d over $\mathbb{Q}$, then

$$< \gamma,\, f > = \sum_{i=1}^{2d} m_i \Omega_i$$

---

* Supported in part by a grant from the NSF

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $\Omega_i \in \mathbb{C}$ are the periods of $J_0(N)$ and $m_i \in \mathbb{Z}$. We have conjectured [G1] that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ with $|c| \le N^2$ then there exists a fixed constant $\kappa > 0$ such that

$$|m_i| \ll N^\kappa \qquad \text{(for } i = 1, 2, \ldots 2d \text{ and } N \to \infty).$$

In the special case that f is associated to an elliptic curve defined over $\mathbb{Q}$ (i.e., $d = 1$), this conjecture (see [G1], [G–L], [G–S]) implies a version of the ABC–conjecture.

Jerzy Browkin has made some preliminary computations of $< \gamma, f >$ for $\gamma \in \mathcal{G}$ where $\mathcal{G}$ is a certain set of generators for $\Gamma_0(N)$. The results are tabulated in §4 and seem to suggest that κ may be very small and possibly even arbitrarily small and positive.

Very little is known at present on the distribution properties of modular symbols. A new type of Eisenstein series (see [G2])

$$E^*(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} < \gamma, f > \operatorname{Im}(\gamma z)^s$$

was introduced which generalizes the classical non–holomorphic Eisenstein series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \operatorname{Im}(\gamma z)^s$$

and satisfies the automorphic identity

$$E^*(\gamma z, s) = E^*(z,s) - < \gamma, f > E(z,s)$$

for all $\gamma \in \Gamma_0(N)$.

It was recently shown by O'Sullivan (see [O'S]) that $E^*(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with at most a finite number of simple poles on the real axis for $\frac{1}{2} \le s \le 1$. The poles on the real axis occur if and only if the classical Eisenstein series $E(z,s)$ has such poles. He further showed that $E^*(z,s)$ has a certain functional equation for $s \to 1-s$. O'Sullivan obtained his results by generalizing Selberg's proof of the analytic continuation of Eisenstein series to the new Eisenstein series E^*. His method is extremely powerful and also allows one to obtain the analytic continuation and functional equation of more general Eisenstein series

$$E_p^*(z,s,\chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi(\gamma)\, p\big(< \gamma, f >, \overline{< \gamma, g >}\big) \operatorname{Im}(\gamma z)^s$$

where $p(x,y)$ is an arbitrary polynomial defined over $\mathbb{C}$, Γ is a discrete subgroup of $SL(2,\mathbb{R})$, Γ_∞ is the stabilizer of ∞, and χ is a character of Γ. It is clear, that ultimately, this approach will yield vast generalizations and considerably more precise versions of the results proved and conjectured in this paper.

Acknowledgment: *I should like to take this opportunity to thank Jerzy Browkin, Cormac O'Sullivan, and Don Zagier for many helpful discussions in the course of writing this paper.*

We shall show that $E^*(z,s)$ has a meromorphic continuation to $\mathrm{Re}(s) > \frac{1}{2}$ with a single simple pole at $s=1$ and residue given by

$$\frac{3}{N\pi}\prod_{p|N}\left(1+\frac{1}{p}\right)^{-1} F(z)$$

where

$$F(z) = 2\pi i \int_z^{i\infty} f(w)\,dw$$

is the antiderivative of $f(z)$. Further, E^* has a polynomial growth in s in this region (uniformly in z). These properties allow one to establish our first theorem, whose proof is given at the end of §2.

Theorem 1: *Fix an integer $N \ge 1$ and a real number $M > 0$. Assume Selberg's conjecture that there are no exceptional real eigenvalues for $\Gamma_0(N)$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and any fixed $\epsilon > 0$, we have*

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} e^{-\frac{c^2M^2+d^2}{X}} < \gamma, f > \; = \; \frac{3}{N\pi}\prod_{p|N}\left(1+\frac{1}{p}\right)^{-1}\frac{F(iM)}{M}\,X \; + \; O\left(X^{\frac{1}{2}+\epsilon}\right)$$

as $X \to \infty$.

Remarks: If there are m exceptional eigenvalues (i.e., $E(z,s)$ has poles at $s = \kappa_j < 1$, with residue r_j, for $j = 1,2,\ldots m$) then there will be an extra sum of terms

$$\sum_{j=1}^{m} r_j \Gamma(\kappa_j) F(iM)\left(\frac{X}{M}\right)^{\kappa_j}$$

on the right hand side of the asymptotic formula in theorem 1. It would be of great interest to determine the precise dependence on N in the O–constant in theorem 1. One would also like to be able to drop the weighting factor $e^{-\frac{c^2M^2+d^2}{X}}$ and replace the sum with a sum over $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ where $c^2M^2 + d^2 \le X$. This requires very fine estimates on the s–growth of E^*.

The problem of obtaining second and higher moments of modular symbols is a very difficult one and the methods introduced in this paper appear inadequate to a solution. The recent work of O'Sullivan, however, [O'S] does provide a very powerful machine for solving this problem, and it is likely that a solution will be obtained in the near future. For the moment, I should like to state a conjecture for the second moments of the distribution of modular symbols.

Let

$$E^{**}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} |< \gamma, f >|^2\, \mathrm{Im}(\gamma z)^s$$

denote the generalized Eisenstein series formed with absolute squares of modular symbols. O'Sullivan shows that this series has a simple pole at $s=1$. The residue is a function $R^*(z)$ which must satisfy the automorphic relations

$$R^*(\gamma z) = R^*(z) - \overline{<\gamma, f>} R(z) - <\gamma, f> \overline{R(z)} + |<\gamma, f>|^2 \, r_N$$

for all $\gamma \in \Gamma_0(N)$. In §3 we sketch an argument which suggests the following conjecture.

Conjecture 2: *Fix an integer $N \geq 1$ and a real number $M > 0$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and any fixed $\epsilon > 0$, we have*

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e^{-\frac{c^2 M^2 + d^2}{X}} \; |<\gamma, f>|^2 \;\; \sim \;\; \frac{R^*(iM)}{M} X$$

as $X \to \infty$.

Remarks: It is likely that the error term in the above asymptotic formula should again be of the form $O\left(X^{\frac{1}{2}+\epsilon}\right)$. The residue function $R^*(z)$ is still unknown. We can prove that it must be of the form $r_N \, |F(z)|^2 + \eta(z)$ where $\eta(z)$ is a non–holomorphic modular form of weight zero for $\Gamma_0(N)$.

§2. Average values of modular symbols:

Let $\mathcal{S}$ denote the set of cusps of $\Gamma_0(N)$. For $\mathfrak{a} \in \mathcal{S}$ let

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma_0(N) \mid \gamma\mathfrak{a} = \mathfrak{a}\}$$

denote the stability subgroup of the cusp $\mathfrak{a}$. We then define $\sigma_{\mathfrak{a}} \in SL(2, \mathbb{R})$ by the conditions

$$\sigma_{\mathfrak{a}} \infty = \mathfrak{a}, \qquad \sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_\infty.$$

For each cusp $\mathfrak{a} \in \mathcal{S}$ there is a non–holomorphic Eisenstein series

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s$$

with Fourier expansion at another cusp $\mathfrak{b} \in \mathcal{S}$ given by (see [I])

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\mathfrak{b}} y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s} + \sum_{n \neq 0} \phi_{\mathfrak{a}\mathfrak{b}}(n, s) \sqrt{|n| y} \, K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x},$$

where

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}\left(u + u^{-1}\right)} \, u^s \, \frac{du}{u},$$

$$\phi_{\mathfrak{a}\mathfrak{b}}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_c S_{\mathfrak{a}\mathfrak{b}}(0, 0; c) c^{-2s},$$

$$\phi_{\mathfrak{a}\mathfrak{b}}(n,s) = \frac{\pi^s}{\Gamma(s)}|n|^{s-1}\sum_c S_{\mathfrak{a}\mathfrak{b}}(0,n;c)c^{-2s},$$

and

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c) = \sum_{\gamma\,\in\,\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty} e^{2\pi i\frac{an+dm}{c}}, \qquad \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

is the Kloosterman sum.

For modular forms of $f(z), g(z)$ (of weight zero for $\Gamma = \Gamma_0(N)$) let

$$< f, g > = \int\limits_{\Gamma\backslash\mathfrak{h}^*} f(z)\overline{g(z)}\,\frac{dxdy}{y^2}$$

denote the Petersson inner product. The Laplace–Beltrami operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

acts on the Hilbert space $\mathcal{L}^2\left(\Gamma_0(N)\backslash\mathfrak{h}^*\right)$ and induces an orthogonal decomposition of the space into Maass cusp forms, Eisenstein series, and the constant function. For $j = 1, 2, 3, \ldots$ let

$$\eta_j(z) = \sum_{n\neq 0} b_j(n)\sqrt{|n|y}\,K_{ir_j}(2\pi|n|y)e^{2\pi inx}$$

denote the Maass cusp form which satisfies

$$\Delta\eta_j(z) = \left(\frac{1}{4} + r_j^2\right)\eta_j(z)$$

where $\frac{1}{4} + r_j^2$ is the corresponding eigenvalue of the Laplacian.

Every $g \in \mathcal{L}^2\left(\Gamma_0(N)\backslash\mathfrak{h}^*\right)$ (which is orthogonal to the constant function) has a Selberg spectral decomposition given by

$$g(z) = \sum_{j=1}^{\infty} < g, \eta_j > \eta_j(z) \;+\; \sum_{\mathfrak{a}\in\mathcal{S}}\frac{1}{4\pi}\int_{-\infty}^{\infty} < g,\, E_{\mathfrak{a}}(*, \frac{1}{2} + ir) > \; E_{\mathfrak{a}}(z, \frac{1}{2} + ir)\,dr.$$

The Selberg spectral expansion can be used to obtain the meromorphic continuation of

$$E^*(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} < \gamma, f > \;\mathrm{Im}(\gamma z)^s.$$

Define

$$G(z,s) = E^*(z,s) - F(z)E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} (< \gamma, f > - F(z))\,\mathrm{Im}(\gamma z)^s$$

with

$$F(z) = 2\pi i \int_{i\infty}^{z} f(w)\, dw = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n z}.$$

Since $F(\gamma z) = F(z) - < \gamma, f >$ for $\gamma \in \Gamma_0 N)$, it follows that

$$G(z,s) \ = -\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} F(\gamma z)\, \text{Im}(\gamma z)^s.$$

Consequently, $G(z,s)$ is a non–holomorphic automorphic form of weight 0 for $\Gamma_0(N)$. We shall show that for $\text{Re}(s)$ sufficiently large, $G(z,s)$ is an $\mathcal{L}^2$ function and we will compute its Selberg spectral expansion.

For $\text{Re}(s) = \sigma$, sufficiently large, we have

$$|G(z,s)| \ \le \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} |F(\gamma z)| \cdot \text{Im}(\gamma z)^\sigma.$$

Here, we have replaced the sum over $\gamma \in \Gamma_\infty\backslash\Gamma_0(N)$ by the larger sum $\gamma \in \Gamma_\infty\backslash\Gamma_0(1)$. Since $|F(z)| << e^{-2\pi y}$ (with $z = x+iy$) we see that for any $\alpha \in SL(2,\mathbb{Z})$

$$\begin{aligned} |G(\alpha z, s)| \ &\ll \ e^{-2\pi y} y^\sigma + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)\\ \gamma\notin\Gamma_\infty}} \text{Im}(\gamma z)^\sigma \\ &\ll \ e^{-2\pi y} y^\sigma \ + \ y^{1-\sigma}, \end{aligned}$$

using standard estimates for non–holomorphic Eisenstein series on the full modular group and the fact that the sum on the right (above) is invariant under $\alpha \in SL(2,\mathbb{Z})$. It immediately follows that for σ sufficiently large, $G(z,s)$ has rapid decay at every cusp, and is, therefore, an $\mathcal{L}^2$–function.

The Mellin transform of the product of an exponential and K–Bessel function is given by the formula [Gr–R]

$$\int_0^\infty e^{-y} K_v(y) y^s \, \frac{dy}{y} = \frac{\sqrt{\pi}}{2^s} \, \frac{\Gamma(s+v)\Gamma(s-v)}{\Gamma(s+\frac{1}{2})}.$$

Using this formula, we compute the inner product of G with the j^{th} Maass form η_j. We have

$$\begin{aligned} < G(*,\, s),\, \eta_j > &= \int_0^\infty \int_0^1 F(z)\, y^s\, \overline{\eta_j(z)}\, \frac{dxdy}{y^2} \\ &= \sum_{n=1}^\infty \frac{a(n)}{n} \overline{b_j(n)} \int_0^\infty e^{-2\pi n y} \sqrt{ny}\, K_{ir_j}(2\pi|n|y)\, y^s\, \frac{dy}{y^2} \\ &= 2^{1-2s} \pi^{1-s} \, \frac{\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j)}{\Gamma(s)} \, L_{f\otimes\eta_j}(s) \end{aligned}$$

where

$$L_{f\otimes\eta_j}(s) \ = \ \sum_{n=1}^\infty \frac{a(n)\overline{b_j(n)}}{n^s}.$$

The Rankin–Selberg zeta function $L_{f\otimes\eta_j}(s)$ was studied by Deshouillers and Iwaniec [D–I] who found the holomorphic continuation and functional equation. In particular they showed that $L_{f\otimes\eta_j}(s)$ has a polynomial growth in s in the region $\mathrm{Re}(s)\geq\frac{1}{2}$. They further proved that there exists a constant $c_f>0$, depending only on f, such that

$$L_{f\otimes\eta_j}(\frac{1}{2}+ir_j)\ \neq\ 0$$

for at least $c_f R(\log R)^{-2}$ values of $r_j\leq R$ (counted with multiplicity) as $R\to\infty$. This shows that $E^*(z,s)$ has infinitely many poles on the line $\mathrm{Re}(s)=\frac{1}{2}$.

To complete the Selberg spectral expansion, it remains to compute the contribution of the continuous spectrum. In a similar manner, we have

$$< G(*,s),\ E_{\mathfrak{a}}(*,\ \frac{1}{2}+ir) >= 2^{1-2s}\pi^{1-s}\,\frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{a(n)\overline{\phi_{\mathfrak{a}}(n,\frac{1}{2}+ir)}}{n^s},$$

where we have adopted the convention that $\phi_{\mathfrak{a}}=\phi_{\mathfrak{a}\infty}$.

The computations above are sufficient to enable one to show that for any fixed $\epsilon>0$, the generalized Eisenstein series $E^*(z,s)$ has a meromorphic continuation to $\mathrm{Re}(s)>\frac{1}{2}+\epsilon$ with at most finitely many simple poles on the interval $[\frac{1}{2},1]$. There will always be a simple pole at $s=1$, and there will be other real poles only if there are exceptional eigenvalues. To obtain the growth in s of $E^*(z,s)$ in the region $\mathrm{Re}(s)>\frac{1}{2}+\epsilon$ it is necessary to estimate the sum

$$\sum_{j=1}^{\infty} < G(*,s),\ \eta_j >= \sum_{j=1}^{\infty} 2^{1-2s}\pi^{1-s}\,\frac{\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j)}{\Gamma(s)}\,L_{f\otimes\eta_j}(s).$$

Since $\eta_j(z)$ is normalized so that $<\eta_j,\eta_j>=1$, it follows from [D–I], [H–L], that

$$|L_{f\otimes\eta_j}(s)|\ =\ O\left(|s|^c|r_j|^{\epsilon}e^{\frac{\pi}{2}|r_j|}\right)$$

for some constant $c>0$. Furthermore, the major contribution to the above sum comes from those r_j close to t where $s=\sigma+it$. There at most $O\left(|t|^{2+\epsilon}\right)$ such r_j. Also, the gamma factors (using Stirling's asymptotics) cancel the exponential growth in r_j of $|L_{f\otimes\eta_j}(s)|$. It follows that the above sum has a polynomial growth in s (in the region $\mathrm{Re}(s)>\frac{1}{2}+\epsilon$) for any fixed z. A similar argument can be made for the continuous spectrum integral in the Selberg spectral decomposition of $G(z,s)$. Assuming Selberg's conjecture (that there are no exceptional eigenvalues) these estimates imply that $(s-1)E^*(z,s)$ has a polynomial growth in s (in the region $\mathrm{Re}(s)>\frac{1}{2}+\epsilon$) for any fixed z.

Now, both E and E^* are eigenfunctions of the Laplacian

$$\Delta E=s(1-s)E,\qquad \Delta E^*=s(1-s)E^*.$$

Since

$$E^*(\gamma z,s)=E^*(z,s)-<\gamma,f>E(z,s)$$

for all $\gamma \in \Gamma_0(N)$ it follows that E^* must have a simple pole at $s = 1$ with residue $R(z)$ where $R(z)$ must be a harmonic function. Further, $R(z)$ must satisfy the automorphic relation

$$R(\gamma z) = R(z) - <\gamma, f> r_N$$

where

$$r_N = \frac{3}{N\pi} \prod_{p|N} \left(1 + \frac{1}{p}\right)^{-1}$$

denotes the residue at $s = 1$ of the Eisenstein series $E(z, s)$. It follows that

$$R(z) = \frac{3}{N\pi} \prod_{p|N} \left(1 + \frac{1}{p}\right)^{-1} F(z) \ + c,$$

for some constant c. In fact, it is shown in [G2] that $c = 0$. Thus the function

$$G(z, s) = E^*(z, s) - F(z)E(z, s)$$

is a non–holomorphic (in the variable z) modular form of weight zero for $\Gamma_0(N)$ which is entire for $\mathrm{Re}(s) > \frac{1}{2} + \epsilon$ with at most a polynomial growth in s in this region. Further, it has a Fourier expansion at $i\infty$ in z with vanishing constant term. The establishment of these properties is sufficient to obtain mean value estimates for the distribution of modular symbols.

We use the well known Mellin transform

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^s \Gamma(s)\, ds = e^{-\frac{1}{y}}.$$

It follows that for fixed $M > 0$, $X \to \infty$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} X^s M^{-s} E^*(iM, s)\Gamma(s)\, ds &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{<\gamma, f> \ X^s}{|c^2M^2 + d^2|^s} \Gamma(s)\, ds \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e^{-\frac{c^2M^2+d^2}{X}} <\gamma, f> \\ &= r_N \frac{X}{M} F(iM) \ + \ O\left(X^{\frac{1}{2}+\epsilon}\right). \end{aligned}$$

The asymptotic formula is established by shifting the line of integration in the above left–hand integral to the line $\mathrm{Re}(s) = \frac{1}{2} + \epsilon$. If there are no exceptional eigenvalues for the group $\Gamma_0(N)$ then $E^*(z, s)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$ except for a simple pole at $s = 1$ with residue given by $r_N F(iM)$. Since $E^*(z, s)$ has at most a polynomial growth in s in this region, the result follows by standard estimates in analytic number theory. If there are exceptional real eigenvalues, then there will be additional poles which will contribute additional terms to the final formula.

§3. Second moments of modular symbols:

In order to study sums of the form

$$\sum_{\gamma} |< \gamma, f >|^2$$

we introduce the generalized Eisenstein series

$$E^{**}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} |< \gamma, f >|^2 \, \mathrm{Im}(\gamma z)^s$$

which satisfies the automorphic relations

$$E^{**}(\gamma z, s) = E^{**}(z,s) - \overline{< \gamma, f >} E^*(z,s) - < \gamma, f > \overline{E^*(z,\bar{s})} + |< \gamma, f >|^2 \, E(z,s)$$

for all $\gamma \in \Gamma_0(N)$.

Define

$$r_N = \mathrm{Res}_{s=1} E(z,s) = \frac{3}{N\pi} \prod_{p|N} \left(1 + \frac{1}{p}\right)^{-1}$$

$$R(z) = \mathrm{Res}_{s=1} E^*(z,s) = r_N F(z)$$

$$R^*(z) = \mathrm{Res}_{s=1} E^{**}(z,s).$$

Then $R^*(z)$ must satisfy

$$R^*(\gamma z) = R^*(z) - \overline{< \gamma, f >} R(z) - < \gamma, f > \overline{R(z)} + |< \gamma, f >|^2 \, r_N$$

for all $\gamma \in \Gamma_0(N)$. We already know that

$$|F(\gamma z)|^2 = |F(z)|^2 - \overline{< \gamma, f >} F(z) - < \gamma, f > \overline{F(z)} + |< \gamma, f >|^2 .$$

It immediately follows that if we define

$$G^*(z,s) = |F(z)|^2 \, E(z,s) - F(z) \overline{E^*(z,\bar{s})} - \overline{F(z)} E^*(z,s) + E^{**}(z,s)$$

then

$$G^*(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} |F(\gamma z)|^2 \, \mathrm{Im}(\gamma z)^s$$

is an automorphic form of weight zero for $\Gamma_0(N)$.

Set

$$\eta_0(z) = \left(\iint_{\Gamma_0(N) \backslash \mathfrak{h}^*} \frac{dxdy}{y^2} \right)^{-\frac{1}{2}}$$

to be the constant function normalized to have Petersson norm equal to one. Let $\eta_j(z)$ (for $j = 1, 2, 3, \dots$) denote the Maass cusp forms as before. We may attempt to obtain the meromorphic continuation of $G^*(z,s)$ from the Selberg spectral decomposition as we did earlier for $G(z,s)$:

$$G^*(z,s) = \sum_{j=0}^{\infty} < G^*(*,s),\, \eta_j > \;+\; \sum_{\mathfrak{a}\in\mathcal{S}} \frac{1}{4\pi}\int_{-\infty}^{\infty} < G^*(*,s),\, E(*,\frac{1}{2}+ir > \; E(z,\frac{1}{2}+ir)\; dr.$$

We compute the inner products

$$\begin{aligned} < G^*(*,\, s),\, \eta_j > &= \int_0^\infty \int_0^1 |F(z)|^2 \; y^s\, \overline{\eta_j(z)}\, \frac{dxdy}{y^2} \\ &= \sum_{n=1}^{\infty} A(n)\overline{b_j(n)} \int_0^\infty e^{-2\pi ny}\, K_{ir_j}(2\pi|n|y)\, y^{s-\frac{1}{2}}\, \frac{dy}{y} \\ &= 2^{1-2s}\pi^{1-s}\, \frac{\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j)}{\Gamma(s)} \sum_{n\neq 0} \frac{A(n)\overline{b_j(n)}}{|n|^{s-\frac{1}{2}}} \end{aligned}$$

where

$$A(n) = \sum_{\substack{m_1,m_2\geq 1\\ m_1-m_2=n}} \frac{a(m_1)a(m_2)}{m_1 m_2}.$$

There does not seem to be any simple direct way to obtain the analytic continuation of the Dirichlet series

$$\sum_{n\neq 0} \frac{A(n)\overline{b_j(n)}}{n^s}$$

although one may easily obtain the continuation of the Dirichlet series

$$\sum_{n\neq 0} \frac{\tilde{A}(n)\overline{b_j(n)}}{n^s}$$

where

$$\tilde{A}(n) = \sum_{\substack{m_1,m_2\geq 1\\ m_1-m_2=n}} a(m_1)a(m_2).$$

We conjecture, however, that for $j>0$, the Dirichlet series

$$\sum_{n\neq 0} \frac{A(n)\overline{b_j(n)}}{n^s}$$

have holomorphic continuation to $\mathrm{Re}(s) > \frac{1}{2}$ and have a polynomial growth in s in this region. This suggests the asymptotic formula

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} e^{-\frac{c^2M^2+d^2}{X}} \;|<\gamma, f>|^2 \;\sim\; \frac{R^*(iM)}{M} X$$

as $X\to\infty$.

§4. Tables of modular symbols:

Jerzy Browkin has computed the modular symbols for generators of $\Gamma_0(p)$ for primes $p = 6t - 1$ which are conductors of elliptic curves E_p (defined over $\mathbb{Q}$) for $p < 250$. He has computed the generators using an algorithm of Kulkarni [K]. For a generator $\gamma \in \Gamma_0(N)$ and the holomorphic cusp form f of weight two associated to E_p, we write

$$< \gamma, f >= k_1\Omega_1 + k_2\Omega_2$$

where Ω_1, Ω_2 are the periods E_p. The coefficients k_1 and k_2 are usually very small (less than 5 in absolute value) in these tables. The tables list:

p

$[\Omega_1, \Omega_2]$

$[\gamma, \ \ k_1, k_2, \ \ < \gamma, f >, \ \ | < \gamma, f > | \,]$

where γ runs over a set of generators for $\Gamma_0(N)$ different from [1,1,0,1]. Note that we use the notation $[a, b, c, d] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$p = 11$
$[1.26920930427, 0.634604652139 + 1.45881661693 * i]$
$[[-4, 1, 11, -3], 0, -1, -0.634604652140 - 1.45881661693 * i, 1.59087051213]$
$[[-3, 1, 11, -4], 1, -1, 0.634604652140 - 1.45881661693 * i, 1.59087051213]$

$p = 17$
$[1.54707975355, 0.773539876775 + 1.37286955904 * i]$
$[[-4, -1, 17, 4], 0, 0, 0, 0]$
$[[-5, 2, 17, -7], 1, -1, 0.773539876776 - 1.37286955904 * i, 1.57579655004]$
$[[-7, 2, 17, -5], 0, -1, -0.773539876776 - 1.37286955904 * i, 1.57579655004]$
$[[4, -1, 17, -4], 0, 0, 0, 0]$

$p = 53$
$[4.68764104887, -2.34382052444 + 1.54059067013 * i]$
$[[-8, 3, 53, -20], 0, -1, 2.34382052443 - 1.54059067013 * i, 2.80480200080]$
$[[-10, 3, 53, -16], 0, -1, 2.34382052444 - 1.54059067013 * i, 2.80480200080]$
$[[-20, 3, 53, -8], -1, -1, -2.34382052444 - 1.54059067013 * i, 2.80480200080]$
$[[22, -5, 53, -12], -1, -1, -2.34382052444 - 1.54059067013 * i, 2.80480200080]$
$[[-7, -2, 53, 15], 1, 1, 2.34382052443 + 1.54059067013 * i, 2.80480200080]$
$[[-16, 3, 53, -10], -1, -1, -2.34382052444 - 1.54059067013 * i, 2.80480200080]$
$[[15, -2, 53, -7], 0, 1, -2.34382052443 + 1.54059067013 * i, 2.80480200080]$
$[[23, -10, 53, -23], 0, 0, 0, 0]$
$[[-23, -10, 53, 23], 0, 0, 0, 0]$
$[[-12, -5, 53, 22], 0, -1, 2.34382052444 - 1.54059067013 * i, 2.80480200080]$

$p = 83$
$[3.37446890009, 1.68723445004 + 1.95716430938 * i]$
$[[-18, -5, 83, 23], 0, 0, 0, 0]$

$[[-32,5,83,-13],0,-1,-1.68723445004-1.95716430937*i,2.58403796863]$
$[[-30,13,83,-36],1,-1,1.68723445004-1.95716430937*i,2.58403796863]$
$[[-13,5,83,-32],1,-1,1.68723445004-1.95716430937*i,2.58403796863]$
$[[-9,4,83,-37],1,0,3.37446890009,3.37446890009]$
$[[-25,3,83,-10],0,-1,-1.68723445004-1.95716430937*i,2.58403796863]$
$[[-37,4,83,-9],-1,0,-3.37446890009,3.37446890009]$
$[[15,-2,83,-11],0,0,0,0]$
$[[-19,8,83,-35],1,-1,1.68723445004-1.95716430937*i,2.58403796862]$
$[[-11,-2,83,15],0,0,0,0]$
$[[23,-5,83,-18],0,0,0,0]$
$[[-36,13,83,-30],0,-1,-1.68723445004-1.95716430937*i,2.58403796863]$
$[[-35,8,83,-19],0,-1,-1.68723445004-1.95716430937*i,2.58403796862]$
$[[-10,3,83,-25],1,-1,1.68723445004-1.95716430937*i,2.58403796863]$

$p=89$
$[5.55262656457,-2.77631328228+1.14967602209*i]$
$[[34,-13,89,-34],0,0,0,0]$
$[[-12,-5,89,37],1,1,2.77631328228+1.14967602209*i,3.00494099728]$
$[[-9,1,89,-10],0,0,0,0]$
$[[-14,-3,89,19],0,0,0,0]$
$[[-27,10,89,-33],-1,-2,-2.29935204419*i,2.29935204419]$
$[[-34,-13,89,34],0,0,0,0]$
$[[19,-3,89,-14],0,0,0,0]$
$[[-24,7,89,-26],-1,-2,-2.29935204419*i,2.29935204419]$
$[[-10,1,89,-9],0,0,0,0]$
$[[-16,7,89,-39],0,-1,2.77631328228-1.14967602209*i,3.00494099729]$
$[[37,-5,89,-12],0,1,-2.77631328228+1.14967602209*i,3.00494099728]$
$[[40,-9,89,-20],-1,-1,-2.77631328228-1.14967602209*i,3.00494099728]$
$[[-39,7,89,-16],-1,-1,-2.77631328228-1.14967602209*i,3.00494099729]$
$[[-33,10,89,-27],-1,-2,-2.29935204419*i,2.29935204419]$
$[[-20,-9,89,40],0,-1,2.77631328228-1.14967602209*i,3.00494099728]$
$[[-26,7,89,-24],-1,-2,-2.29935204419*i,2.29935204419]$

$p=101$
$[2.29512360593,2.72355956946*i]$
$[[44,-17,101,-39],0,0,0,0]$
$[[-12,-5,101,42],1,0,2.29512360593,2.29512360593]$
$[[31,-4,101,-13],-1,0,-2.29512360593,2.29512360593]$
$[[-16,3,101,-19],0,-1,-2.72355956946*i,2.72355956946]$
$[[28,-5,101,-18],-1,0,-2.29512360593,2.29512360593]$
$[[-19,3,101,-16],0,-1,-2.72355956946*i,2.72355956946]$
$[[10,-1,101,-10],0,0,0,0]$
$[[-10,-1,101,10],0,0,0,0]$
$[[-23,5,101,-22],0,-1,-2.72355956946*i,2.72355956946]$
$[[37,-11,101,-30],0,0,0,0]$
$[[42,-5,101,-12],-1,0,-2.29512360593,2.29512360593]$
$[[-18,-5,101,28],1,0,2.29512360593,2.29512360593]$
$[[-13,-4,101,31],1,0,2.29512360593,2.29512360593]$

[[−30, −11, 101, 37], 0, 0, 0, 0]
[[−46, 5, 101, −11], −1, 0, −2.29512360593, 2.29512360593]
[[−11, 5, 101, −46], 1, 0, 2.29512360593, 2.29512360593]
[[−22, 5, 101, −23], 0, −1, −2.72355956946 ∗ i, 2.72355956946]
[[−39, −17, 101, 44], 0, 0, 0, 0]

$p = 113$
[2.01836989318, 1.00918494659 + 1.42890601952 ∗ i]
[[43, −8, 113, −21], 0, 1, 1.00918494659 + 1.42890601951 ∗ i, 1.74935035628]
[[−12, −5, 113, 47], 0, 1, 1.00918494659 + 1.42890601952 ∗ i, 1.74935035628]
[[−35, 13, 113, −42], 0, −1, −1.00918494659 − 1.42890601951 ∗ i, 1.74935035628]
[[−13, −3, 113, 26], 0, 0, 0, 0]
[[−42, 13, 113, −35], 1, −1, 1.00918494659 − 1.42890601951 ∗ i, 1.74935035628]
[[−21, −8, 113, 43], −1, 1, −1.00918494659 + 1.42890601951 ∗ i, 1.74935035628]
[[−15, −2, 113, 15], 0, 0, 0, 0]
[[−33, 7, 113, −24], 0, −1, −1.00918494659 − 1.42890601952 ∗ i, 1.74935035629]
[[−20, 3, 113, −17], 0, −1, −1.00918494659 − 1.42890601952 ∗ i, 1.74935035629]
[[51, −14, 113, −31], −1, 0, −2.01836989318, 2.01836989318]
[[−17, 3, 113, −20], 1, −1, 1.00918494659 − 1.42890601952 ∗ i, 1.74935035629]
[[26, −3, 113, −13], 0, 0, 0, 0]
[[−11, −4, 113, 41], 0, 0, 0, 0]
[[−24, 7, 113, −33], 1, −1, 1.00918494659 − 1.42890601952 ∗ i, 1.74935035629]
[[47, −5, 113, −12], −1, 1, −1.00918494659 + 1.42890601952 ∗ i, 1.74935035628]
[[−30, 13, 113, −49], 2, −2, 2.01836989318 − 2.85781203904 ∗ i, 3.49870071258]
[[41, −4, 113, −11], 0, 0, 0, 0]
[[15, −2, 113, −15], 0, 0, 0, 0]
[[−31, −14, 113, 51], 1, 0, 2.01836989318, 2.01836989318]
[[−49, 13, 113, −30], 0, −2, −2.01836989318 − 2.85781203904 ∗ i, 3.49870071258]

$p = 131$
[4.17160927634, −2.08580463817 + 1.48258896129 ∗ i]
[[−39, −14, 131, 47], 0, 0, 0, 0]
[[40, −11, 131, −36], 0, 0, 0, 0]
[[−14, −3, 131, 28], 1, 1, 2.08580463817 + 1.48258896129 ∗ i, 2.55903321916]
[[−36, −11, 131, 40], 0, 0, 0, 0]
[[−57, 10, 131, −23], −1, −1, −2.08580463817 − 1.48258896130 ∗ i, 2.55903321916]
[[−59, 9, 131, −20], −1, −1, −2.08580463817 − 1.48258896129 ∗ i, 2.55903321916]
[[−20, 9, 131, −59], 0, −1, 2.08580463817 − 1.48258896129 ∗ i, 2.55903321916]
[[−17, 7, 131, −54], 0, −1, 2.08580463817 − 1.48258896129 ∗ i, 2.55903321916]
[[55, −21, 131, −50], 0, 0, 0, 0]
[[60, −11, 131, −24], −1, −1, −2.08580463817 − 1.48258896129 ∗ i, 2.55903321916]
[[28, −3, 131, −14], 0, 1, −2.08580463817 + 1.48258896129 ∗ i, 2.55903321916]
[[−30, −11, 131, 48], 0, 0, 0, 0]
[[47, −14, 131, −39], 0, 0, 0, 0]
[[−50, −21, 131, 55], 0, 0, 0, 0]
[[−11, 1, 131, −12], 0, 0, 0, 0]
[[−15, 4, 131, −35], 0, −1, 2.08580463817 − 1.48258896129 ∗ i, 2.55903321916]
[[48, −11, 131, −30], 0, 0, 0, 0]

$[[-23, 10, 131, -57], 0, -1, 2.08580463817 - 1.48258896130 * i, 2.55903321916]$
$[[-54, 7, 131, -17], -1, -1, -2.08580463817 - 1.48258896129 * i, 2.55903321916]$
$[[-12, 1, 131, -11], 0, 0, 0, 0]$
$[[-35, 4, 131, -15], -1, -1, -2.08580463817 - 1.48258896129 * i, 2.55903321916]$
$[[-24, -11, 131, 60], 0, -1, 2.08580463817 - 1.48258896129 * i, 2.55903321916]$

$p = 179$
$[2.26019825468, 1.13009912734 + 2.55454866451 * i]$
$[[-27, -8, 179, 53], 0, 0, 0, 0]$
$[[-24, -11, 179, 82], 0, -1, -1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[48, -11, 179, -41], 0, 0, 0, 0]$
$[[-19, 7, 179, -66], 1, -1, 1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-50, -19, 179, 68], 0, -1, -1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[68, -19, 179, -50], 1, -1, 1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-64, 5, 179, -14], -1, 0, -2.26019825468, 2.26019825468]$
$[[-17, -2, 179, 21], 1, 0, 2.26019825468, 2.26019825468]$
$[[78, -17, 179, -39], 0, -1, -1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-41, -11, 179, 48], 0, 0, 0, 0]$
$[[-14, 5, 179, -64], 1, 0, 2.26019825468, 2.26019825468]$
$[[-16, 5, 179, -56], 2, -1, 3.39029738202 - 2.55454866451 * i, 4.24497764633]$
$[[-63, 19, 179, -54], 0, 0, 0, 0]$
$[[-39, -17, 179, 78], 1, -1, 1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-32, 5, 179, -28], 1, 0, 2.26019825468, 2.26019825468]$
$[[-56, 5, 179, -16], -1, -1, -3.39029738202 - 2.55454866451 * i, 4.24497764633]$
$[[53, -8, 179, -27], 0, 0, 0, 0]$
$[[-42, 19, 179, -81], 0, 0, 0, 0]$
$[[-74, 31, 179, -75], 1, -1, 1.13009912734 - 2.55454866452 * i, 2.79335692617]$
$[[-75, 31, 179, -74], 0, -1, -1.13009912734 - 2.55454866452 * i, 2.79335692617]$
$[[-33, 7, 179, -38], 1, -1, 1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[83, -32, 179, -69], 0, 0, 0, 0]$
$[[82, -11, 179, -24], 1, -1, 1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[21, -2, 179, -17], -1, 0, -2.26019825468, 2.26019825468]$
$[[-38, 7, 179, -33], 0, -1, -1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-54, 19, 179, -63], 0, 0, 0, 0]$
$[[-81, 19, 179, -42], 0, 0, 0, 0]$
$[[-28, 5, 179, -32], -1, 0, -2.26019825468, 2.26019825468]$
$[[-66, 7, 179, -19], 0, -1, -1.13009912734 - 2.55454866451 * i, 2.79335692617]$
$[[-69, -32, 179, 83], 0, 0, 0, 0]$

$p = 197$
$[2.83478021112, 1.59772411663 * i]$
$[[86, -31, 197, -71], 0, -1, -1.59772411663 * i, 1.59772411663]$
$[[-46, 7, 197, -30], 0, 0, 0, 0]$
$[[-55, 12, 197, -43], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[-26, -7, 197, 53], -1, 1, -2.83478021112 + 1.59772411663 * i, 3.25402845689]$
$[[-83, 8, 197, -19], -1, -1, -2.83478021112 - 1.59772411664 * i, 3.25402845689]$
$[[-71, -31, 197, 86], 0, -1, -1.59772411663 * i, 1.59772411663]$
$[[-14, -1, 197, 14], 0, 0, 0, 0]$

$[[-19, 8, 197, -83], 1, -1, 2.83478021112 - 1.59772411664 * i, 3.25402845689]$
$[[-70, 27, 197, -76], 0, -3, -4.79317234991 * i, 4.79317234991]$
$[[-52, 19, 197, -72], 1, -1, 2.83478021112 - 1.59772411664 * i, 3.25402845689]$
$[[-35, -8, 197, 45], 0, 1, 1.59772411663 * i, 1.59772411663]$
$[[-42, 13, 197, -61], 1, -2, 2.83478021112 - 3.19544823327 * i, 4.27163531413]$
$[[-21, -8, 197, 75], 0, 1, 1.59772411663 * i, 1.59772411663]$
$[[-43, 12, 197, -55], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[-58, 5, 197, -17], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[14, -1, 197, -14], 0, 0, 0, 0]$
$[[-30, 7, 197, -46], 0, 0, 0, 0]$
$[[-31, 14, 197, -89], 1, 0, 2.83478021112, 2.83478021112]$
$[[-89, 14, 197, -31], -1, 0, -2.83478021112, 2.83478021112]$
$[[-90, 37, 197, -81], 0, -1, -1.59772411663 * i, 1.59772411663]$
$[[-61, 13, 197, -42], -1, -2, -2.83478021112 - 3.19544823327 * i, 4.27163531413]$
$[[75, -8, 197, -21], 0, 1, 1.59772411663 * i, 1.59772411663]$
$[[-16, 3, 197, -37], 0, 0, 0, 0]$
$[[-37, 3, 197, -16], 0, 0, 0, 0]$
$[[-76, 27, 197, -70], 0, -3, -4.79317234991 * i, 4.79317234991]$
$[[45, -8, 197, -35], 0, 1, 1.59772411663 * i, 1.59772411663]$
$[[-60, 7, 197, -23], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[-23, 7, 197, -60], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[-17, 5, 197, -58], 0, -2, -3.19544823327 * i, 3.19544823327]$
$[[-81, 37, 197, -90], 0, -1, -1.59772411663 * i, 1.59772411663]$
$[[53, -7, 197, -26], 1, 1, 2.83478021112 + 1.59772411663 * i, 3.25402845689]$
$[[-15, 7, 197, -92], 1, 0, 2.83478021112, 2.83478021112]$
$[[-72, 19, 197, -52], -1, -1, -2.83478021112 - 1.59772411664 * i, 3.25402845689]$
$[[-92, 7, 197, -15], -1, 0, -2.83478021112, 2.83478021112]$

$p = 233$

$[2.78426719713, -1.39213359856 + 0.819667807593 * i]$
$[[-86, 31, 233, -84], 2, 3, 1.39213359856 + 2.45900342278 * i, 2.82572712580]$
$[[-24, 7, 233, -68], -1, -3, 1.39213359856 - 2.45900342277 * i, 2.82572712580]$
$[[-19, -4, 233, 49], 2, 2, 2.78426719713 + 1.63933561518 * i, 3.23103158205]$
$[[-55, -17, 233, 72], -1, -2, -1.63933561518 * i, 1.63933561518]$
$[[-44, -17, 233, 90], -1, -3, 1.39213359856 - 2.45900342278 * i, 2.82572712580]$
$[[-71, -32, 233, 105], -1, 0, -2.78426719713, 2.78426719713]$
$[[-20, 3, 233, -35], 0, -1, 1.39213359856 - 0.819667807594 * i, 1.61551579103]$
$[[-107, 45, 233, -98], -1, -2, -1.63933561518 * i, 1.63933561518]$
$[[-32, 7, 233, -51], -1, -2, -1.63933561518 * i, 1.63933561518]$
$[[-84, 31, 233, -86], 1, 3, -1.39213359856 + 2.45900342278 * i, 2.82572712580]$
$[[90, -17, 233, -44], -2, -3, -1.39213359856 - 2.45900342278 * i, 2.82572712580]$
$[[-89, -34, 233, 89], 0, 0, 0, 0]$
$[[89, -34, 233, -89], 0, 0, 0, 0]$
$[[-95, -42, 233, 103], 0, 1, -1.39213359856 + 0.819667807597 * i, 1.61551579102]$
$[[49, -4, 233, -19], 0, 2, -2.78426719713 + 1.63933561518 * i, 3.23103158205]$
$[[-98, 45, 233, -107], -1, -2, -1.63933561518 * i, 1.63933561518]$
$[[-62, 29, 233, -109], -2, -4, -3.27867123037 * i, 3.27867123037]$
$[[-51, 7, 233, -32], -1, -2, -1.63933561518 * i, 1.63933561518]$

$[[-17, 7, 233, -96], 2, 3, 1.39213359856 + 2.45900342278 * i, 2.82572712580]$
$[[103, -42, 233, -95], 1, 1, 1.39213359856 + 0.819667807597 * i, 1.61551579102]$
$[[-43, -12, 233, 65], 0, -1, 1.39213359856 - 0.819667807596 * i, 1.61551579102]$
$[[-27, -8, 233, 69], 1, 1, 1.39213359856 + 0.819667807591 * i, 1.61551579102]$
$[[69, -8, 233, -27], 0, 1, -1.39213359856 + 0.819667807591 * i, 1.61551579102]$
$[[-35, 3, 233, -20], -1, -1, -1.39213359856 - 0.819667807594 * i, 1.61551579103]$
$[[-73, 26, 233, -83], 0, -1, 1.39213359856 - 0.819667807593 * i, 1.61551579102]$
$[[-83, 26, 233, -73], -1, -1, -1.39213359856 - 0.819667807593 * i, 1.61551579102]$
$[[-101, 13, 233, -30], 0, 0, 0, 0]$
$[[65, -12, 233, -43], -1, -1, -1.39213359856 - 0.819667807596 * i, 1.61551579102]$
$[[105, -32, 233, -71], 1, 0, 2.78426719713, 2.78426719713]$
$[[-108, 19, 233, -41], -3, -5, -1.39213359856 - 4.09833903796 * i, 4.32832748603]$
$[[-22, 5, 233, -53], -2, -5, 1.39213359856 - 4.09833903796 * i, 4.32832748603]$
$[[-53, 5, 233, -22], -3, -5, -1.39213359856 - 4.09833903796 * i, 4.32832748603]$
$[[-96, 7, 233, -17], 1, 3, -1.39213359856 + 2.45900342278 * i, 2.82572712580]$
$[[-41, 19, 233, -108], -2, -5, 1.39213359856 - 4.09833903796 * i, 4.32832748603]$
$[[-109, 29, 233, -62], -2, -4, -3.27867123037 * i, 3.27867123037]$
$[[-37, 10, 233, -63], 0, -2, 2.78426719712 - 1.63933561518 * i, 3.23103158205]$
$[[72, -17, 233, -55], -1, -2, -1.63933561518 * i, 1.63933561518]$
$[[-30, 13, 233, -101], 0, 0, 0, 0]$
$[[-63, 10, 233, -37], -2, -2, -2.78426719713 - 1.63933561518 * i, 3.23103158205]$
$[[-68, 7, 233, -24], -2, -3, -1.39213359856 - 2.45900342277 * i, 2.82572712580]$

§5. References:

[D–I] J.M. Deshouillers, H. Iwaniec, *Rankin–Selberg zeta functions*, in Contemporary Math. Vol 53, The Selberg trace formula and related topics, Amer. Math. Soc. 1986, 51–95.

[G1] D. Goldfeld, *Modular elliptic curves and diophantine problems*, Proc. of the First Canadian Number Theory Assoc., Banff, Canada 1988, de Gruyter, New York (1989).

[G2] D. Goldfeld, *Zeta functions formed with modular symbols*, to appear in AMS publications, A Conference in Honor of G. Shimura.

[G–L] D. Goldfeld, D. Lieman, *Effective bounds on the size of the Tate–Shafarevich group* , Math. Research Letters, Vol. **3**, (1996), 309–318.

[G–S] D. Goldfeld, L. Szpiro, *Bounds for the order of the Tate–Shafarevich group*, Compositio Math. **97** (1995), 71–87.

[Gr–R] I.S Gradshteyn, I.M. Ryzhik, Tables of integrals series and products, Academic Press 1965.

[H–L] J. Hoffstein, P. Lockhart, , *Coefficients of Maass forms and the Siegel zero*, Annals of Math. (1994), 161–181.

[I] H. Iwaniec, Introduction to the spectral theory of automorphic forms, Revista Matematica Iberoamericana 1995.

[K] R.S. Kulkarni, *An arithmetic–geometric method in the study of the subgroups of the modular group*, Amer. J. Math. **113** (1991), 1053-1133.

[M] Ju. I. Manin, *Parabolic points and zeta-functions of modular curves,* Math. USSR Izvestija **6** (1972), 19-64.

[O'S] C. O'Sullivan, *Properties of Eisenstein series formed with modular symbols*, to appear.

[Sh] G. Shimura, *On the factors of the jacobian variety of a modular function field,* J. Math. Soc. Japan **25** (1973), 523-544.

Dorian Goldfeld
Department of Mathematics
Columbia University
NY NY 10027