TRACE FORMULAE FOR SL$(2, \mathbb{R})$

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Contents

1. Trace Formula for finite groups

Let G be a finite group and let Γ be a subgroup of G. We consider the $\mathbb{C}$-vector space of functions

$$V := \Big\{\phi : G \to \mathbb{C} \;\Big|\; \phi(\gamma x) = \phi(x), \;\; \forall \gamma \in \Gamma, \, \forall x \in G\Big\}.$$

We can think of V as the $\mathbb{C}$-vector space of automorphic forms on $\Gamma\backslash G$.

Definition 1.1. (Inner product on V) *For $\phi_1, \phi_2 \in V$ we define*

$$\big\langle \phi_1, \phi_2 \big\rangle := \sum_{x \in \Gamma\backslash G} \phi_1(x)\overline{\phi_2(x)}.$$

Definition 1.2. (Orthonormal basis for V) *Let $\Gamma z \in \Gamma\backslash G$ and define $\mathbf{1}_{\Gamma z} \in V$ by*

$$\mathbf{1}_{\Gamma z}(x) := \begin{cases} 1 & \text{if } x \in \Gamma z, \\ 0 & \text{otherwise.} \end{cases}$$

The functions $\mathbf{1}_{\Gamma z}$ for distinct cosets Γz form an orthonormal basis for V with respect to the above inner product.

Definition 1.3. (The kernel function $K_f(x,y)$) *Let G be a finite group and $f : G \to \mathbb{C}$. Let Γ be a subgroup of G. For $x, y \in G$ we define the kernel function*

$$K_f(x,y) := \sum_{\gamma \in \Gamma} f\left(x^{-1}\gamma y\right).$$

Definition 1.4. (The linear map K_f) *Let $\phi \in V$. We define a linear map $K_f : V \to V$ where for $\phi \in V$ we have $K_f(\phi) \in V$ where*

$$\left(K_f \phi\right)(x) := \sum_{y \in \Gamma\backslash G} K_f(x,y) \cdot \phi(y).$$

Definition 1.5. (Trace of a linear map on a finite dimensional vector space) *Let V be a complex vector space with basis $v_1, v_2, \ldots, v_n$. A linear map $L : V \to V$ satisfies*

$$L(v_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} v_j, \qquad \big(\text{for all } 1 \le i \le n \text{ and } a_{i,j} \in \mathbb{C}\big).$$

Then associated to L we may define the matrix $A_L := \left(a_{i,j}\right)_{1 \le i,j \le n}$. The trace of L (denoted $\mathrm{Tr}(L)$) is defined to be

$$\mathrm{Tr}(L) = \sum_{i=1}^{n} a_{i,i},$$

i.e., it is the trace of the matrix A_L.

Remarks on eigenvalues of a matrix: It is well known that for a matrix A with complex coefficients $a_{i,j}$, $(1 \le i, j \le n)$ that

- $\mathrm{Tr}(A) =$ sum of the eigenvalues of A;
- $\mathrm{Det}(A) =$ product of the eigenvalues of A.

We now compute the action of the linear map $K_f : V \to V$ on the orthonormal basis given in definition 1.2. Let $z \in \Gamma\backslash G$. We have

$$\begin{aligned}\Big(K_f\, \mathbf{1}_{\Gamma z}\Big)(x) &= \sum_{y\in\Gamma\backslash G} K_f(x,y)\cdot \mathbf{1}_{\Gamma z}(y)\\ &= K_f(x,z)\\ &= \sum_{x_1\in\Gamma\backslash G} K_f(x_1,z)\cdot \mathbf{1}_{\Gamma x_1}(x).\end{aligned}$$

It follows that the matrix A_{K_f} associated to the linear map K_f is given by $\Big(\; K_f(x_1, z)\;\Big)_{x_1, z\in\Gamma\backslash G}$, and

$$\boxed{\mathrm{Tr}(K_f) = \sum_{z\in\Gamma\backslash G} K_f(z,z).}$$

Proposition 1.6. (Trace formula for a finite group G) *Let G be a finite group and let Γ be a subgroup of G. Let $\mathrm{Cl}[\Gamma]$ denote the set of conjugacy classes $\big\{\sigma^{-1}\gamma\sigma \;\big|\; \sigma\in\Gamma\big\}$ with $\gamma\in\Gamma$. For $\gamma\in\Gamma$ we also define*

$$\begin{aligned}\Gamma_\gamma &:= \big\{\sigma\in\Gamma \;\big|\; \sigma^{-1}\gamma\sigma = \gamma\big\},\\ G_\gamma &:= \big\{g\in G \;\big|\; g^{-1}\gamma g = \gamma\big\}.\end{aligned}$$

It follows that

$$\boxed{\mathrm{Tr}(K_f) = \sum_{\gamma\in\mathrm{Cl}[\Gamma]} \frac{\#\left(G_\gamma\right)}{\#\left(\Gamma_\gamma\right)} \sum_{z\in G_\gamma\backslash G} f\left(z^{-1}\gamma z\right).}$$

Proof. We have

$$\begin{aligned}
\mathrm{Tr}(K_f) &= \sum_{z\in\Gamma\backslash G} K_f(z,z)\\
&= \sum_{z\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f\left(z^{-1}\gamma z\right)\\
&= \sum_{z\in\Gamma\backslash G}\ \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} f\left(z^{-1}\cdot\sigma^{-1}\gamma\sigma\cdot z\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{z\in\Gamma\backslash G}\ \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} f\left(z^{-1}\cdot\sigma^{-1}\gamma\sigma\cdot z\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{z\in\Gamma_\gamma\backslash G} f\left(z^{-1}\gamma z\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{z\in G_\gamma\backslash G}\ \sum_{g\in\Gamma_\gamma\backslash G_\gamma} f\left((gz)^{-1}\cdot\gamma\cdot gz\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{z\in G_\gamma\backslash G}\ \sum_{g\in\Gamma_\gamma\backslash G_\gamma} f\left(z^{-1}\cdot g^{-1}\gamma g\cdot z\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]}\ \sum_{z\in G_\gamma\backslash G}\ \sum_{g\in\Gamma_\gamma\backslash G_\gamma} f\left(z^{-1}\gamma z\right)\\
&= \sum_{\gamma\in\mathrm{Cl}[\Gamma]} \frac{\#\left(G_\gamma\right)}{\#\left(\Gamma_\gamma\right)} \sum_{z\in G_\gamma\backslash G} f\left(z^{-1}\gamma z\right).
\end{aligned}$$

□

The trace formula has two sides:

$$\underbrace{\mathrm{Tr}(K_f)}_{\text{spectral side}} = \underbrace{\sum_{\gamma\in\mathrm{Cl}[\Gamma]} \frac{\#\left(G_\gamma\right)}{\#\left(\Gamma_\gamma\right)} \sum_{z\in G_\gamma\backslash G} f\left(z^{-1}\gamma z\right)}_{\text{geometric side}}.$$

where the geometric side consists of a sum over conjugacy classes.

2. Trace Formula for the infinite additive group $\mathbb{R}$ and subgroup $\mathbb{Z}$

Consider the additive group $G=\mathbb{R}$ and subgroup of rational integers $\Gamma=\mathbb{Z}$. Following the recipe in §1, we define the $\mathbb{C}$ vector space of smooth functions

$$V := \Big\{\phi:\mathbb{R}\to\mathbb{C}\ \Big|\ \phi(x+n)=\phi(x),\ \forall n\in\mathbb{Z},\ \forall x\in\mathbb{R}\Big\}.$$

Definition 2.1. (Inner product on V) *Let $\phi_1, \phi_2 \in V$. We define the inner product*

$$\langle \phi_2, \phi_2 \rangle := \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \phi_1(x)\,\overline{\phi_2(x)}\,dx = \int_0^1 \phi_1(x)\,\overline{\phi_2(x)}\,dx.$$

Definition 2.2. (Orthonormal basis for V) *Let $e_n(x) := e^{2\pi i n x}$. It is well known that every periodic function is a linear combination of $e_n(x)$ with $n \in \mathbb{Z}$. Furthermore*

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i(m-n)x}\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

This establishes that the $\{e_n\}_{n\in\mathbb{Z}}$ is an othonormal basis of V.

Definition 2.3. (The kernel function $K_f(x, y)$) *Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, i.e., a smooth function all of whose derivatives have rapid decay. For $x, y \in \mathbb{R}$ we define the kernel function*

$$K_f(x, y) := \sum_{n\in\mathbb{Z}} f\,(-x + n + y)\,.$$

Definition 2.4. (The linear map K_f) *Let $\phi \in V$. We define the linear map $K_f : V \to V$ by*

$$\big(K_f\,\phi\big)(x) := \int_0^1 K_f(x, y) \cdot \phi(y)\,dy.$$

Proposition 2.5. (The e_n are eigenfunctions of K_f) *Let $n \in \mathbb{Z}$. Then*

$$\big(K_f\,e_n\big)(x) = \widehat{f}(-n) \cdot e_n(x),$$

where $\widehat{f}$ is Fourier transform of f given by $\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i x\xi}\,dx$.

Proof.

$$\begin{aligned}
\big(K_f\,e_n\big)(x) &= \int_0^1 K_f(x, y) \cdot e_n(y)\,dy \\
&= \int\limits_0^1 \sum_{n\in\mathbb{Z}} f\,(-x + n + y) \cdot e_n(y)\,dy \\
&= \int\limits_{-\infty}^{\infty} f(-x + y) \cdot e_n(y)\,dy \;=\; \int\limits_{-\infty}^{\infty} f(y) \cdot e_n(x + y)\,dy \\
&= \widehat{f}(-n) \cdot e_n(x),
\end{aligned}$$

since $e_n(x + y) = e_n(x) \cdot e_n(y)$. ∎

Proposition 2.6. (The trace formula for the additive group $\mathbb{R}$) *The trace formula for the group $\mathbb{R}$ and subgroup $\Gamma = \mathbb{Z}$ is given by*

$$\underbrace{\sum_{n\in\mathbb{Z}} \widehat{f}(n)}_{\text{spectral side}} \quad = \quad \underbrace{\sum_{n\in\mathbb{Z}} f(n)}_{\text{geometric side}} .$$

Remark: The trace formula is the well known Poisson summation formula.

Proof. We have shown that e_n is an eigenfunction of the linear map K_f with the eigenvalue $\widehat{f}(n)$. The trace of K_f is given by the sum of the eigenvalues which is just $\sum\limits_{n\in\mathbb{Z}} \widehat{f}(n)$. To find the geometric side we must compute

$$\int_0^1 K(x,x)\,dx = \int\limits_0^1 \sum_{n\in\mathbb{Z}} f(n)\,dx = \sum_{n\in\mathbb{Z}} f(n).$$

Note that since the additive group $\mathbb{Z}$ is abelian there is only one conjugacy class, namely $\mathbb{Z}$ itself. □

3. The Selberg Trace Formula for $SL(2,\mathbb{R})$ (Spectral Side)

Let $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z})$. We also require the maximal compact subgroup

$$K = SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \;\middle|\; 0 \le \theta < 2\pi \right\}.$$

In this case we define the vector space V as a space of smooth functions ϕ given by

$$V := \left\{ \phi : G \to \mathbb{C} \;\middle|\; \phi(\gamma g k) = \phi(g),\ \forall \gamma \in \Gamma,\ \forall k \in K,\ \forall g \in G \right\}.$$

By the Iwasawa decomposition it is known that every $g \in G$ can be uniquely expressed in the form

$$g = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some $0 \le \theta < 2\pi,\ x \in \mathbb{R},\ y > 0$.

The action of $SL(2,\mathbb{R})$ on the upper half plane $\mathfrak{h} := \{x + iy \mid x \in \mathbb{R}, y > 0\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}, \qquad \Big(\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}),\ \forall z \in \mathfrak{h}\Big),$$

establishes a one-to-one correspondence between G/K and $\mathfrak{h}$ given by

$$\begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} i \;=\; \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} i \;=\; x+iy.$$

Notation involving z: *We shall use the notation* $z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in G/K$ *and also* $z = x+iy \in \mathfrak{h}$ *interchangeably. The usage will depend on the context of the discussion.*

Definition 3.1. (Petersson inner product on V) *For* $\phi_1, \phi_2 \in V$ *we define the inner product*

$$\big\langle \phi_1, \phi_2 \big\rangle \;:=\; \int\limits_{\Gamma\backslash G/K} \phi_1(g)\,\overline{\phi_2(g)}\; d^{\times}g$$

where $d^{\times}g$ *denotes the left invariant measure on the coset space* $\Gamma\backslash G/K$*. If* $z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in G/K$ *then the Petersson inner product can be written in a simple explicit manner as*

$$\big\langle \phi_1, \phi_2 \big\rangle \;:=\; \iint\limits_{z\in\Gamma\backslash\mathfrak{h}} \phi_1(z)\,\overline{\phi_2(z)}\;\frac{dxdy}{y^2}.$$

We may then define $\mathcal{L}^2\left(\Gamma\backslash G/K\right)$ as the Hilbert space completion of V. This space had been studied by Maass and elements of $\mathcal{L}^2\left(\Gamma\backslash G/K\right)$ which are eigenfunctions of the Laplacian with rapid decay at the cusp are termed Maass forms for $SL(2,\mathbb{Z})$.

Definition 3.2. (The space $\mathcal{L}^2_{\text{cusp}}$ of cusp forms) *We let* $\mathcal{L}^2_{\text{cusp}}$ *denote the Hilbert space of Maass forms for* $SL(2,\mathbb{Z})$*. It can be shown that each Maass form* ϕ *vanishes at the cusp, i.e.,* $\lim\limits_{y\to\infty} \phi(x+iy) = 0$*.*

It was not known before Selberg's work on the trace formula whether infinitely many such Maass forms existed or not. In addition to the Maass forms there are are also Eisenstein series which are eigenfunctions of the Laplacian but are not in $\mathcal{L}^2$.

Definition 3.3. *Let* $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right\}_{m\in\mathbb{Z}}$*. Let* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$*. We define the Eisenstein series*

$$E(z,s) := \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s.$$

Theorem 3.4. (Fourier expansion of Eisenstein series) *The Eisenstein series $E(z,s)$ has the Fourier expansion*

$$E(z,s) = y^s + M(s)y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

where

$$M(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}, \qquad \sigma_s(n) = \sum_{\substack{d|n\\ d>0}} d^s,$$

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}y\left(u+\frac{1}{u}\right)}\, u^s\, \frac{du}{u}.$$

Proof. See [Gol06]. □

Corollary 3.5. (Growth of the Eisenstein series at the cusp) *Let $s \in \mathbb{C}$ with with $\mathrm{Re}(s) > 1$. Then $|E(z,s)| \ll y^{\mathrm{Re}(s)}$ and $|E(z,s) - y^s| \ll y^{1-\mathrm{Re}(s)}$ as $y \to \infty$.*

Definition 3.6. (The kernel function K_f) *Let $f : K\backslash G/K \to \mathbb{C}$. For $z, z' \in SL(2,\mathbb{R})$ we define the kernel function*

$$K_f(z,z') := \sum_{\gamma\in\Gamma} f\left(z^{-1}\gamma z'\right),$$

provided the sum converges absolutely.

Proposition 3.7. (Properties of K_f) *For all $z, z' \in SL(2,\mathbb{R})$ the Kernel function K_f (given in the above definition) satisfies the following properties:*

- $K_f(zk, z'k') = K_f(z,z'), \qquad (\forall\, k, k' \in K),$
- $K_f(\gamma z, \gamma' z) = K_f(z,z'), \qquad (\forall\, \gamma, \gamma' \in \Gamma).$

Proof. Exercise for the reader. □

To facilitate the computation of the trace of the linear operator K_f Selberg chose a class of especially nice test functions f which we now describe.

Definition 3.8. (The function τ) *We define the function $\tau : G \to \mathbb{C}$ by*

$$\tau(g) := a^2 + b^2 + c^2 + d^2 - 2, \qquad \left(\text{for } g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in G\right).$$

Proposition 3.9. (Properties of τ) *Let* $z = \begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12} \end{pmatrix}$, *and let* $z' = \begin{pmatrix} y'^{\frac12} & x'y'^{-\frac12} \\ 0 & y'^{-\frac12} \end{pmatrix} \in G/K$. *Then we have*

(**i**) $\tau(k) = 0, \qquad (\forall\, k \in K)$,

(**ii**) $\tau(kz) = \tau(zk) = \tau(z), \qquad (\forall\, k \in K)$,

(**iii**) $\tau\big(z^{-1}z'\big) \ = \ \dfrac{(x-x')^2 + (y-y')^2}{yy'} \ = \ \dfrac{|z-z'|^2}{yy'}$,

(**iv**) $\dfrac{|\sigma z - \sigma z'|^2}{\mathrm{Im}(\sigma z)\cdot \mathrm{Im}(\sigma z')} = \dfrac{|z-z'|^2}{yy'}$ *for all* $\sigma \in SL(2,\mathbb{R})$.

Proof. Exercise for the reader. □

Definition 3.10. (Selberg's kernel function) *Let* $f : \mathbb{R}^+ \to \mathbb{C}$ *be a smooth function satisfying* $f(t) \ll_\epsilon |t+2|^{-1-\epsilon}$. *Then for* $g, g' \in SL(2,\mathbb{R})$, *we define*

$$K_f(g,g') = \sum_{\gamma\in\Gamma} f\Big(\tau\big(g^{-1}\gamma g'\big)\Big).$$

Theorem 3.11. (Growth of Selberg's kernel function at the cusps) *Let* $z = \begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12} \end{pmatrix}$, $z' = \begin{pmatrix} y'^{\frac12} & x'y'^{-\frac12} \\ 0 & y'^{-\frac12} \end{pmatrix} \in G/K$. *Then, for every* $\epsilon > 0$ *and* $y, y' > 1$, *we have*

- $\boxed{|K_f(z,z')| \ \le \ \sum_{\gamma\in\Gamma} \Big| f\Big(\tau\big(z^{-1}\gamma z'\big)\Big)\Big| \ \ll_\epsilon \ y^{-\epsilon}(y')^{1+\epsilon}}$

- $\boxed{\sum_{\gamma\,\in\,\Gamma-\Gamma_\infty} \Big| f\Big(\tau\big(z^{-1}\gamma z'\big)\Big)\Big| \ \ll_\epsilon \ (yy')^{-\epsilon}.}$

Proof. We compute

$$\begin{aligned} \Big| f\Big(\tau\big(z^{-1}z'\big)\Big)\Big| &= f\left(\frac{(x-x')^2 + (y-y')^2}{yy'}\right) \\ &\ll_\epsilon \left(\frac{(x-x')^2}{yy'} + \frac{y}{y'} + \frac{y'}{y}\right)^{-1-\epsilon} \\ &\ll_\epsilon \left(\frac{(x-x')^2}{yy'} + \frac{y}{y'}\right)^{-1-\epsilon} \end{aligned}$$

Next, for $\alpha \in \Gamma$, we adopt the notation that $z_\alpha = \alpha z' = x_\alpha + iy_\alpha$. It follows that

$$\begin{aligned}
K_f(z,z') &= \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\ \sum_{\delta\in\Gamma_\infty} f\left(\frac{|z-z_{\delta\alpha}|^2}{y\,y_{\delta\alpha}}\right)\\
&= \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} f\left(\frac{|z-z_\alpha-m|^2}{y\,y_\alpha}\right)\\
&= \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} f\left(\frac{(x-x_\alpha-m)^2+(y-y_\alpha)^2}{y\,y_\alpha}\right)\\
&\ll_\epsilon \sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} \left(\frac{(x-x_\alpha-m)^2}{y\,y_\alpha} + \frac{y^2}{yy_\alpha}\right)^{-1-\epsilon}\\
&\ll_\epsilon\ y^{1+\epsilon} \sum_{\alpha\in\Gamma_\infty\backslash\Gamma} y_\alpha^{1+\epsilon} \sum_{m\in\mathbb{Z}} \Big((x-x_\alpha-m)^2+y^2\Big)^{-1-\epsilon}
\end{aligned}$$

Now, we can bound the latter sum above as follows.

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \Big((x-x_\alpha-m)^2+y^2\Big)^{-1-\epsilon} &\ll\ y^{-2-2\epsilon} + \int_0^\infty (u^2+y^2)^{-1-\epsilon}\,du\\
&\ll y^{-1-2\epsilon}.
\end{aligned}$$

It immediately follows from corollary 3.5 and the above calculations that

$$K_f(z,z')\ \ll_\epsilon\ y^{-\epsilon}E(z',1+\epsilon)\ \ll_\epsilon\ y^{-\epsilon}(y')^{1+\epsilon}.$$

This proves the first assertion.

For the second assertion we calculate the partial sums over $\Gamma_\infty\backslash\Gamma$

$$\sum_{\gamma\,\in\,\Gamma-\Gamma_\infty} \left| f\Big(\tau\big(z^{-1}\gamma z'\big)\Big)\right| \ll_\epsilon\ y^{-\epsilon} \sum_{\Gamma_\infty\gamma\,\neq\,\Gamma_\infty} y_\gamma^{1+\epsilon} \ll (yy')^{-\epsilon}$$

since $\big|E(z,1+\epsilon)-y^{1+\epsilon}\big| \ll y^{-\epsilon}$ by corollary 3.5. □

Definition 3.12. (The integral operator K_f) *Let $z\in\mathfrak{h}$. Then for $\phi\in\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ we define the integral operator $K_f:\mathcal{L}^2(\Gamma\backslash\mathfrak{h})\to\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ by*

$$\big(K_f\,\phi\big)(z) = \int\limits_{\Gamma\backslash\mathfrak{h}} K_f(z,z')\,\phi(z')\,\frac{dx'dy'}{(y')^2}.$$

Remarks Since K_f is symmetric, i.e., $K_f\big(z,z'\big) = K_f\big(z',z\big)$, we see that K_f also satisfies the bound $|K_f(z,z')| \ll_\epsilon y^{1+\epsilon}(y')^{-\epsilon}$. It follows that for every $z \in \mathfrak{h}$ that

$$\begin{aligned}
\big|\big(K_f\phi\big)(z)\big| &\le \left(\int\limits_{\Gamma\backslash\mathfrak{h}} \big|K_f(z,z')\big|^2\,\frac{dx'dy'}{(y')^2}\cdot\int\limits_{\Gamma\backslash\mathfrak{h}} |\phi(z')|^2\,\frac{dx'dy'}{(y')^2}\right)^{\frac12}\\
&\ll_{z,\epsilon} \left(\int\limits_{\Gamma\backslash\mathfrak{h}} (y')^{-2\epsilon}\,\frac{dx'dy'}{(y')^2}\cdot\int\limits_{\Gamma\backslash\mathfrak{h}} |\phi(z')|^2\,\frac{dx'dy'}{(y')^2}\right)^{\frac12}\\
&= \mathcal{O}_{z,\epsilon}\big(1\big).
\end{aligned}$$

This shows that the integral defining the integral operator K_f converges absolutely.

Definition 3.13. (Hilbert-Schmidt Operator) *Let X be a locally compact space with a positive Borel measure. Assume that $\mathcal{L}^2(X)$ is a separable Hilbert space. Let $K : X \times X \to \mathbb{C}$ be $\mathcal{L}^2$ in each variable. We define the integral operator*

$$\big(K\,\phi\big)(x) := \int_X K(x,y)\,\phi(y)\,dy, \qquad \Big(\phi \in \mathcal{L}^2(X)\Big).$$

The integral operator K is said to be of Hilbert-Schmidt type if

$$\iint\limits_{X\times X} |K(x,y)|^2\,dxdy < \infty.$$

Theorem 3.14. (Hilbert-Schmidt) *A Hilbert-Schmidt operator as in definition 3.13 is a compact operator. If K is self adjoint then the space $\mathcal{L}^2(X)$ has an orthonormal basis of eigenfunctions $\phi_1, \phi_2, \ldots$ where $K\phi_i = \lambda_i\cdot\phi_i$, $(i = 1, 2, 3, \ldots)$ and $\lambda_i \to 0$ as $i \to \infty$. If the sum of the eigenvalues converges absolutely, then we say K is of trace class and the trace of the operator K is given by* $\boxed{\mathrm{Tr}(K) = \sum_{i=1}^{\infty}\lambda_i = \int_X K(x,x)\,dx.}$

Proof. See [Bum97] □

Warning: The integral operator K_f given in definition 3.12 is not Hilbert-Schmidt since it can be shown that

$$\int\limits_{\Gamma\backslash\mathfrak{h}}\int\limits_{\Gamma\backslash\mathfrak{h}} |K_f(z,z')|^2\,\frac{dxdy}{(y)^2}\,\frac{dx'dy'}{(y')^2} = \infty.$$

We shall now construct a modification of the integral operator K_f. The modification will be done in two steps.

Definition 3.15. (First modification of K_f) *We define*

$$\boxed{\widetilde{K}_f(z,z') := K_f(z,z') - \int_0^1 K_f(z, z'+t)\, dt.}$$

Proposition 3.16. *Let $z \in \mathfrak{h}$. Fix a fundamental domain for $\Gamma\backslash\mathfrak{h}$ given by*

$$\mathcal{D} = \Big\{ z \in \mathfrak{h} \;\Big|\; -1/2 \le \mathrm{Re}(z) \le 1/2, \;\; |z| \ge 1. \Big\}$$

If $\phi \in \mathcal{L}^2_{\text{cusp}}$ then we have

$$\Big(\widetilde{K}_f \phi\Big)(z) = \int\limits_{\mathcal{D}} \widetilde{K}_f(z,z')\, \phi(z')\, \frac{dx'dy'}{(y')^2} = \big(K_f\, \phi\big)(z).$$

Proof.

$$\int\limits_{\mathcal{D}} \bigg(\int_0^1 K_f(z, z'+t)\; dt \bigg)\, \phi(z')\, \frac{dx'dy'}{(y')^2} = \int\limits_{\mathcal{D}} K_f(z,z') \bigg(\int_0^1 \phi(z'-t)\; dt \bigg) \frac{dx'dy'}{(y')^2} \;=\; 0.$$

□

Definition 3.17. (Second modification of K_f) *We define*

$$\boxed{K_f^{\#}(z,z') := K_f(z,z') - \sum_{n\in\mathbb{Z}} f\left(\frac{|z-z'+n|^2}{yy'} \right).}$$

Theorem 3.18. (Growth of the Modified Kernel at the Cusps) *Let $z = x+iy,\, z' = x'+iy' \in \mathfrak{h}$. Then for every $\epsilon > 0$ and all $y, y' > 1$*

$$\boxed{\Big|\widetilde{K}_f(z,z')\Big| \;\ll\; (yy')^{-\epsilon} + \int_0^\infty |f'(r)|\; dr.}$$

Proof. We calculate

$$K_f^{\#}(z,z') - \widetilde{K}_f(z,z') = \int\limits_0^1 K_f(z,z'+t)\, dt - \sum_{n\in\mathbb{Z}} f\left(\frac{|z-z'+n|^2}{yy'} \right)$$

$$= \int\limits_0^1 K_f^{\#}(z,z'+t)\, dt + \int\limits_0^1 \left[\sum_{n\in\mathbb{Z}} f\left(\frac{|z-z'+n+t|^2}{yy'} \right) - \sum_{n\in\mathbb{Z}} f\left(\frac{|z-z'+n|^2}{yy'} \right) \right] dt$$

$$= \int\limits_0^1 K_f^{\#}(z,z'+t)\, dt + \int\limits_{-\infty}^{\infty} f\left(\frac{|z-z'+t|^2}{yy'} \right)\; d\big(t - [t]\big).$$

The first term above is bounded by $(yy')^{-\epsilon}$ by theorem 3.11. For the second term we apply integration by parts.

$$\int_{-\infty}^{\infty} f\left(\frac{(x-x'+t)^2+(y-y')^2}{yy'}\right)\, d\big(t-[t]\big)$$

$$= -\int_{-\infty}^{\infty} \big(t-[t]\big)\cdot df\left(\frac{(x-x'+t)^2+(y-y')^2}{yy'}\right)$$

$$\ll \int_{-\infty}^{\infty} \left|df\left(\frac{t^2+(y-y')^2}{yy'}\right)\right| = \int_{-\infty}^{\infty} \underbrace{\left|f'\left(\frac{t^2+(y-y')^2}{yy'}\right)\frac{2t}{yy'}\, dt\right|}_{\text{put } r=\frac{t^2+(y-y')^2}{yy'}}$$

$$\ll \int_{0}^{\infty} |f'(r)|\, dr.$$

We have proved that $\left|K_f^{\#}(z,z') - \widetilde{K}_f(z,z')\right| \ll (yy')^{-\epsilon} + \int_0^\infty |f'(r)|\, dr.$ The theorem follows from theorem 3.11 which says $|K_f^{\#}(z,z')| \ll (yy')^{-\epsilon}$. □

Theorem 3.19. (The Kernel function $\widetilde{K}_f$ is Hilbert-Schmidt and of Trace Class) *The kernel function $\widetilde{K}_f$ defines an integral operator $\widetilde{K}_f : \mathcal{L}^2_{\text{cusp}} \to \mathcal{L}^2_{\text{cusp}}$ which is Hilbert-Schmidt i.e., it satisfies*

$$\int_{\mathcal{D}}\int_{\mathcal{D}} \left|\widetilde{K}_f(z,z')\right|^2 \frac{dxdy}{(y)^2}\,\frac{dx'dy'}{(y')^2} < \infty.$$

Furthermore, when restricted to the space $\mathcal{L}^2_{\text{cusp}}$ we have $K_f = \widetilde{K}_f$. For f real valued the integral operator $\widetilde{K}_f : \mathcal{L}^2_{\text{cusp}} \to \mathcal{L}^2_{\text{cusp}}$ is self-adjoint and of trace class with trace given by

$$\boxed{\mathrm{Tr}\big(\widetilde{K}_f\big) = \int_{\mathcal{D}} \widetilde{K}_f(z,z)\, \frac{dxdy}{(y)^2}.}$$

Proof. It follows from theorem 3.18 that

$$\int_{\mathcal{D}}\int_{\mathcal{D}} \left|\widetilde{K}_f(z,z')\right|^2 \frac{dxdy}{(y)^2}\,\frac{dx'dy'}{(y')^2} \ll \int_{\mathcal{D}}\int_{\mathcal{D}} \Big((yy')^{-\epsilon}+1\Big)\frac{dxdy}{(y)^2}\,\frac{dx'dy'}{(y')^2} \ll 1.$$

□

Proposition 3.20. (The integral operator K_f commutes with the Laplacian) *For $z = x + iy \in \mathfrak{h}$ let*

$$\Delta_z := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

denote the Laplacian on $\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. Then $\Delta_z K_f = K_f \Delta_z$.

Proof. First note (by a brute force verification) that

$$\Delta_z K_f(z, z') = \Delta_{z'} K_f(z, z').$$

We compute, using integration by parts (Green's theorem), that

$$\begin{aligned}
\big(\Delta_z K_f\, \phi\big)(z) &= \int_{\Gamma\backslash\mathfrak{h}} \Big(\Delta_{z'} K_f(z, z')\Big) \cdot \phi(z')\, \frac{dx'dy'}{(y')^2} \\
&= \int_{\Gamma\backslash\mathfrak{h}} \left(- \left(\frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial (y')^2} \right) K_f(z, z') \right) \cdot \phi(z')\, dx'dy' \\
&= \int_{\Gamma\backslash\mathfrak{h}} K_f(z, z') \cdot \left(- \left(\frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial (y')^2} \right) \phi(z') \right)\, dx'dy' \\
&= \int_{\Gamma\backslash\mathfrak{h}} K_f(z, z') \cdot \Big(\Delta_{z'}\, \phi(z')\Big)\, \frac{dx'dy'}{(y')^2} \\
&= \big(K_f \Delta_z\, \phi\big)(z).
\end{aligned}$$

□

Let $\lambda \in \mathbb{C}$ and assume the eigenspace $\ker(\Delta - \lambda)$ is one dimensional. Then since Δ and K_f commute, it would follow that if $\Delta\phi = \lambda \cdot \phi$ for some $\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ and $\lambda \in \mathbb{C}$, then for each f there would exist a function $h_f(\lambda) \in \mathbb{C}$ such that

$$K_f \phi = h_f(\lambda) \cdot \phi.$$

We will now show the existence and uniqueness of $h_f(\lambda)$ using differential equations.

Proposition 3.21. (Eigenfunctions of Δ are eigenfunctions of K_f) *Assume that $\Delta_z\phi(z) = \lambda \cdot \phi(z)$ for some $\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. Then for every $f : \mathbb{R}^+ \to \mathbb{C}$ (satisfying $f(t) \ll_\epsilon (t+2)^{-1-\epsilon}$) there exists a unique function $h_f : \mathbb{C} \to \mathbb{C}$ such that*

$$\big(K_f \phi\big)(z) = h_f(\lambda) \cdot \phi(z).$$

Proof. We follow [Hej76]. We want to prove $(K_f\phi)(z) = h_f(\lambda) \cdot \phi(z)$ which is equivalent to:

(3.22)
$$\int\limits_{\Gamma\backslash\mathfrak{h}} \sum_{\gamma\in\Gamma} f\left(\frac{|z-\gamma z'|^2}{y\cdot \mathrm{Im}(\gamma z')}\right)\phi(z')\,\frac{dx'dy'}{(y')^2} = \int\limits_{\mathfrak{h}} f\left(\frac{|z-z'|^2}{y\cdot y'}\right)\phi(z')\,\frac{dx'dy'}{(y')^2}$$
$$= h_f(\lambda)\cdot\phi(z).$$

The Cayley transformation

$$\mathfrak{c}:\mathfrak{h}\to\mathfrak{U}:=\{u\in\mathbb{C}\mid |u|\le 1\}$$

which maps the upper half plane to the unit disk (taking i to 0) is given by

$$\mathfrak{c}(z'):=\begin{pmatrix} i & 1\\ i & -1\end{pmatrix} z' = \frac{z'-i}{z'+i}, \qquad (\forall\, z'\in\mathfrak{h}).$$

It is easy to see that under this transformation

$$\frac{|dz'|}{\mathrm{Im}(z')} = \frac{2|d\mathfrak{c}|}{1-|\mathfrak{c}|^2}.$$

For fixed $z=x+iy\in\mathfrak{h}$ we define the modified Cayley transformation

$$w(z'):=\begin{pmatrix} i & 1\\ i & -1\end{pmatrix}\begin{pmatrix} y^{-\frac12} & -xy^{-\frac12}\\ 0 & y^{\frac12}\end{pmatrix} z'$$
$$= 1-\frac{2y}{i(x-x')+y+y'}$$

which maps the upper half plane $\mathfrak{h}$ to the unit disk $\mathfrak{U}$ (taking z to 0). Furthermore

$$\frac{|z'-z|^2}{yy'} = \frac{4\,|w|^2}{1-|w|^2}.$$

Let us now make the change of coordinates $z'\to w\,(z')$ in the integral 3.22 and define $\phi(z'):=\Psi(w)$. It follows that

$$\int\limits_{\mathfrak{U}} f\left(\frac{4\,|w|^2}{1-|w|^2}\right)\Psi\big(w\big)\; d_A\big(w\big) = h_f(\lambda)\cdot\Psi(0)$$

where $d_A\big(w\big)$ denotes the area differential on $\mathfrak{U}$. We must prove that $h_f(\lambda)$ depends solely on f and λ.

If we convert to polar coordinates on $\mathfrak{U}$ the above can be rewritten as

$$\int\limits_{r=0}^{1} f\left(\frac{4r^2}{1-r^2}\right)\left(\int_0^{2\pi}\Psi\left(re^{i\theta}\right)\,d\theta\right)\frac{rdr}{(1-r^2)^2} = h_f(\lambda)\cdot\Psi(0).$$

This can be rewritten as

$$\int\limits_{r=0}^{1} f\left(\frac{4r^2}{1-r^2}\right)\Psi^*(r)\frac{rdr}{(1-r^2)^2} = h_f(\lambda)\cdot\Psi^*(0)$$

where

$$\Psi^*(w) := \int_0^{2\pi} \Psi\left(w\cdot e^{i\theta}\right)\,d\theta.$$

Note that

$$\int_0^{2\pi} \Psi\left(re^{i\theta}\right)\,d\theta = \int_0^{2\pi} \Psi\left(w\cdot e^{i\theta}\right)\,d\theta$$

for all $w \in \mathfrak{U}$ satisfying $|w| = r$ from which it follows that $\Psi^*(w)$ is a radially symmetric function which depends only on $r = |w|$ and, hence, satisfies $\frac{\partial}{\partial\theta}\Psi^*(w) = 0$.

Furthermore, Ψ^* is also a radially symmetric eigenfunction of the Laplacian and, therefore, satisfies the differential equation

$$\frac{d^2\Psi^*}{dr^2} + \frac{1}{r}\frac{d\Psi^*}{dr} + \frac{4\lambda}{(1-r^2)^2}\Psi^* = 0. \tag{3.23}$$

Let $\lambda \in \mathbb{C}$ be fixed. A possible solution (up to a constant factor) to (3.23) must be of the form

$$\Psi^*(r) = r^c\left(1 + a_1 r + a_2 r^2 + \cdots\right).$$

We calculate

$$\frac{1}{r}\Psi^{*\prime}(r) = cr^{c-2} + a_1(c+1)r^{c-1} + \cdots$$

$$\Psi^{*\prime\prime}(r) = c(c-1)r^{c-2} + a_1(c+1)cr^{c-1} + \cdots$$

Now substitue into the differential equation and we get

$$0 = \Psi^{*\prime\prime}(r) + \frac{1}{r}\Psi^{*\prime}(r) + \frac{4\lambda}{(1-r^2)^2}\Psi^*(r) = c^2r^{c-2} + a_1(c+1)^2r^{c-1} + \cdots$$

It follows that $c^2 = 0$ and higher order terms are defined recursively from the differential equation. There will be a second solution but it will be of the form $\log r(1 + b_1 r + \cdots)$ which is singular at the origin. Since a regular solution Ψ^* exists and is uniquely determined it follows that $h_f(\lambda)$ exists and is also uniquely determined. $\square$

Remark: The above proof shows that a solution to the equations

$$\Delta\phi = \lambda\phi, \qquad (K_f\phi)(z) = h_f(\lambda)\phi(z)$$

can be made radially symmetric around z in the non euclidean sense. Then $\Delta\phi = \lambda\phi$ becomes an ordinary differential equation with a 1-dimensional space of solutions regular near z. This yields the constant $h_f(\lambda)$ as in the case of a one-dimensional eigenspace $\ker(\Delta - \lambda)$.

Definition 3.24. (The Abel Transform) *Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a smooth function such that f and f' are integrable on $\mathbb{R}^+$. For $w \geq 0$ we define the Abel transform*

$$Q(w) := \int\limits_{-\infty}^{\infty} f\left(w + \xi^2\right) d\xi = \int\limits_{w}^{\infty} \frac{f(t)}{\sqrt{t - w}}\, dt,$$

provided the integral converges absolutely.

Proposition 3.25. (Inverse Abel Transform) *Let $Q(w)$ be the Abel transform of f. Then*

$$f(t) = \frac{-1}{\pi} \int\limits_{-\infty}^{\infty} Q'(t + w^2)\, dw.$$

Proof. We have

$$\begin{aligned}
\frac{-1}{\pi} \int\limits_{-\infty}^{\infty} Q'(t + w^2)\, dw &= \frac{-1}{\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f'(t + w^2 + \xi^2)\, dw\, d\xi \\
&= \frac{-1}{\pi} \int\limits_{0}^{2\pi} \int\limits_{0}^{\infty} f'(t + r^2)\, r dr d\theta \\
&= -\int_0^\infty f'(t + u)\, du \;=\; f(t).
\end{aligned}$$

□

Definition 3.26. (Selberg Transform h_f) *Assume $\Delta\phi = \left(\frac{1}{4} + r^2\right) \cdot \phi$ for some $\phi \in \mathcal{L}^2\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}\right)$. Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a smooth function satisfying $f(t) \ll_\epsilon (t + 2)^{-1-\epsilon}$ for $t \geq 0$. Then the Selberg transform is defined as the unique function $h = h_f : \mathbb{C} \to \mathbb{C}$ for which we have $K_f\phi = h_f(r) \cdot \phi$ as in proposition 3.21.*

Proposition 3.27. (Evaluation of the Selberg transform) *Let Q be the Abel transform of f as in definition 3.24. Then we have*

$$h(r) = \int_{-\infty}^{\infty} Q\left(e^u - 2 + e^{-u}\right) e^{iru}\, du.$$

Proof. Choose $\phi(z) = y^{\frac{1}{2}+ir}$ in proposition 3.21. Now

$$\Delta y^{\frac{1}{2}+ir} = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)y^{\frac{1}{2}+ir} = \left(\frac{1}{4}+r^2\right)\cdot y^{\frac{1}{2}+ir}.$$

Although $y^{\frac{1}{2}+ir}$ is not in $\mathcal{L}^2\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}\right)$, the proof of proposition 3.21 still goes through. It follows that

$$h(r)\cdot y^{\frac{1}{2}+ir} = \int\limits_0^\infty\int\limits_{-\infty}^\infty f\left(\frac{(x-x')^2+(y-y')^2}{yy'}\right)(y')^{\frac{1}{2}+ir}\,\frac{dx'dy'}{y'^2}$$

Make the transformation

$$\xi = \frac{x-x'}{\sqrt{yy'}},\qquad d\xi = \frac{dx'}{\sqrt{yy'}}.$$

Then

$$\begin{aligned} h(r)\cdot y^{\frac{1}{2}+ir} &= \int\limits_0^\infty\int\limits_{-\infty}^\infty f\left(\frac{(y-y')^2}{yy'}+\xi^2\right)(y')^{\frac{1}{2}+ir}\sqrt{yy'}\,d\xi\,\frac{dy'}{y'^2}\\ &= \int\limits_0^\infty Q\left(\frac{(y-y')^2}{yy'}\right)(y')^{\frac{1}{2}+ir}\sqrt{yy'}\,\frac{dy'}{y'^2}\\ &= y^{\frac{1}{2}}\int\limits_0^\infty Q\left(\frac{y}{y'}-2+\frac{y'}{y}\right)(y')^{ir}\,\frac{dy'}{y'}\\ &= y^{\frac{1}{2}+ir}\int_0^\infty Q\left(\frac{1}{y'}-2+y'\right)(y')^{ir}\,\frac{dy'}{y'} \end{aligned}$$

The result follows upon making the transformation $y' = e^u$. $\square$

Definition 3.28. (The Selberg transform Functions) *Start with a smooth function $f:\mathbb{R}^+\to\mathbb{C}$ satisfying $f(t)\ll_\epsilon (t+2)^{-1-\epsilon}$. Selberg defines the following functions:*

$$\begin{aligned} Q(w) &= \int_{-\infty}^\infty f(w+\xi^2)\,d\xi, &&\text{(Abel transform of } f),\\ g(u) &= Q\left(e^u-2+e^{-u}\right), &&(u\in\mathbb{R}),\\ h(r) &= \int\limits_{-\infty}^\infty g(u)e^{iru}\,du, &&\text{(Fourier transform of } g)\\ g(u) &= \frac{-1}{2\pi}\int\limits_{-\infty}^\infty h(r)e^{iru}\,dr, &&\text{(inverse Fourier transform of } h). \end{aligned}$$

Explicit Example of the Selberg Transform

$$f(t) = (t+2)^{-\frac{3}{2}}, \qquad Q(w) = \left(1 + \frac{w}{2}\right)^{-1},$$
$$g(u) = \operatorname{sech}(u), \qquad h(r) = \pi \cdot \operatorname{sech}\left(\frac{\pi r}{2}\right).$$

Since $\operatorname{sech}(u) = \frac{2}{e^u + e^{-u}}$ we see that g, h have exponential decay.

Proposition 3.29. Growth of the Selberg Transform Functions *Let $\epsilon > 0$. Then*

$$f(t) \ll_\epsilon (t+2)^{-1-\epsilon}, \quad Q(w) \ll_\epsilon w^{-\frac{1}{2}-\epsilon},$$
$$g(u) \ll_\epsilon e^{-\left(\frac{1}{2}-\epsilon\right)|u|}, \qquad h(r) \ll_\epsilon e^{-\left(\frac{1}{2}-\epsilon\right)\left|\frac{\pi r}{2}\right|}.$$

Proof. Exercise for the reader. □

For $z \in \mathfrak{h}$ let $\phi_0(z) := \sqrt{\frac{3}{\pi}}$ be the constant function of Petersson norm one, and let $\phi_1, \phi_2, \ldots$ denote an orthonormal basis of Maass forms consisting of eigenfunctions of the Laplacian where $\Delta\phi_i = \left(\frac{1}{4} + r_i^2\right)\phi_i$ for $i = 1, 2 \ldots$

Theorem 3.30. (Selberg Spectral Decomposition) *Let ϕ_0 be the constant function of Petersson norm one, and let $\phi_1, \phi_2, \phi_3, \ldots$ denote an orthonormal basis of Maass forms consisting of eigenfunctions of the Laplacian. Let $F \in \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h})$. Then for $z \in \mathfrak{h}$*

$$F(z) = \sum_{i=0}^{\infty} \langle F, \phi_i \rangle \phi_i(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle F, E(*, 1/2 + ir) \rangle E(z, 1/2 + ir)\, dr.$$

Proof. See [Gol06]. □

Theorem 3.31. (Spectral Decomposition of K_f) *The Selberg kernel function $K_f : \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}) \to \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h})$ defined in 3.12 has the following spectral decomposition.*

$$K_f(z, z') =$$
$$= \sum_{i=0}^{\infty} h(r_i)\phi_i(z)\overline{\phi_i(z')} + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) E(z, 1/2 + ir)\overline{E(z', 1/2 + ir)}\, dr$$

for $z, z' \in \mathfrak{h}$.

It follows from theorem 3.31, that

$$\int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \left(K_f(z,z) - \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r)\cdot |E(z,1/2+ir)|^2 \, dr \right) \frac{dxdy}{y^2} \tag{3.32}$$
$$= \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \sum_{i=0}^{\infty} h(r_i)\phi_i(z)\overline{\phi_i(z)}\frac{dxdy}{y^2} \;=\; \sum_{j=0}^{\infty} h(r_j).$$

Unfortunately, each individual integral

$$\int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} K_f(z,z)\, \frac{dxdy}{y^2}$$

and

$$\frac{1}{4\pi} \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \int\limits_{-\infty}^{\infty} h(r)\cdot |E(z,1/2+ir)|^2 \, dr \, \frac{dxdy}{y^2}$$

does not converge! Nevertheless, it turns out that each of these integrals blows up in exactly the same way, so that when they are subtracted one is left with a finite value.

We, therefore, modify (3.32) as follows. First we modify the kernel function K_f by subtracting the contribution from the parabolic conjugacy classes of $SL(2,\mathbb{Z})$. A matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ is parabolic if $|\mathrm{Tr}(\gamma)| = |a+d| = 2$. Associated to γ we have the conjugacy class

$$[\gamma] := \{\sigma\gamma\sigma^{-1} \mid \sigma \in SL(2,\mathbb{Z})\}$$

and the centralizer

$$\Gamma_\gamma := \{\delta \in SL(2,\mathbb{Z}) \mid \delta\gamma = \gamma\delta\}.$$

We define

$$\begin{aligned} K_f^*(z,z') \;&:=\; K_f(z,z') - \sum_{\gamma\in \mathrm{Conj}_{\mathrm{par}}(\Gamma)} \; \sum_{\delta\in[\gamma]} f\left(\frac{|z-\delta z'|^2}{(\mathrm{Im} z)(\mathrm{Im}\delta z')} \right) \\ &=\; K_f(z,z') - f\left(\frac{|z-z'|^2}{(\mathrm{Im} z)(\mathrm{Im} z')} \right) \\ &\qquad - \sum_{\substack{\gamma\in \mathrm{Conj}_{\mathrm{par}}(\Gamma) \\ \gamma \neq \pm I}} \; \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} f\left(\frac{|z-\sigma\gamma\sigma^{-1}z'|^2}{(\mathrm{Im} z)(\mathrm{Im}\sigma\gamma\sigma^{-1}z')} \right), \end{aligned}$$

where $\mathrm{Conj}_{\mathrm{par}}(\Gamma)$ is a set of representatives of parabolic matrices in $\Gamma = SL(2,Z)$ under conjugation.

Lemma 3.33. *Let $\gamma \neq \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be a parabolic matrix. Then*

$$\Gamma_\gamma = \{\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\} = \pm\Gamma_\infty.$$

Furthermore, $\{\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\}$ constitutes a set of representatives of parabolic matrices in $SL(2, \mathbb{Z})$.

Proof. Exercise for the reader. □

Let

$$\mathcal{D}^Y := \left\{x + iy \in \mathfrak{h} \;\middle|\; -\frac{1}{2} \leq x \leq \frac{1}{2},\ x^2 + y^2 \geq 1,\ y \leq Y\right\}$$

denote a fundamental domain for $SL(2, \mathbb{Z})\backslash\mathfrak{h}$ truncated at $Y > 1$. We rewrite the left side of (3.32) as

$$\int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \left(K_f(z,z) - \frac{1}{4\pi}\int\limits_{-\infty}^{\infty} h(r)\cdot|E(z,1/2+ir)|^2\,dr\right)\frac{dxdy}{y^2} \tag{3.34}$$
$$= \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} K_f^*(z,z)\,\frac{dxdy}{y^2} \;+\; \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} f(0)\,\frac{dxdy}{y^2} \;+\; \lim_{Y\to\infty}\mathcal{I}(Y),$$

where

$$\int_{\mathcal{D}^Y} \sum_{\substack{\gamma\in\mathrm{Conj}_{\mathrm{par}}(\Gamma)\\ \gamma\neq\pm I}} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} f\left(\frac{|z-\sigma\gamma\sigma^{-1}z|^2}{(\mathrm{Im}z)(\mathrm{Im}\sigma\gamma\sigma^{-1}z)}\right)\frac{dx\,dy}{y^2}$$
$$= \sum_{\substack{\gamma\in\mathrm{Conj}_{\mathrm{par}}(\Gamma)\\ \gamma\neq\pm I}} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} \int_{\mathcal{D}^Y} f\left(\frac{|\sigma^{-1}z-\gamma\sigma^{-1}z|^2}{(\mathrm{Im}\sigma^{-1}z)(\mathrm{Im}\gamma\sigma^{-1}z)}\right)\,\frac{dx\,dy}{y^2}$$
$$= \sum_{\substack{m\in\mathbb{Z}\\ m\neq0}} \sum_{\sigma\in\pm\Gamma_\infty\backslash\Gamma} \int_{\mathcal{D}^Y} f\left(\frac{|\sigma^{-1}z-(\sigma^{-1}z+m)|^2}{(\mathrm{Im}\sigma^{-1}z)^2}\right)\frac{dx\,dy}{y^2}$$
$$= \int\limits_{x=0}^{1}\int\limits_{y=0}^{Y} \sum_{\substack{m\in\mathbb{Z}\\ m\neq0}} f\left(\frac{m^2}{y^2}\right)\;\frac{dx\,dy}{y^2}..$$

Hence

$$\mathcal{I}(Y) := \int\limits_0^Y\int\limits_0^1 \sum_{\substack{m\in\mathbb{Z}\\ m\neq0}} f\left(\frac{m^2}{y^2}\right)\frac{dxdy}{y^2} - \int\limits_{\mathcal{D}^Y}\int\limits_{-\infty}^{\infty} h(r)\cdot|E(z,1/2+ir)|^2\,dr\frac{dxdy}{4\pi y^2}.$$

On combining (3.32) and (3.34) we obtain the following preliminary version of the Selberg Trace Formula.

Proposition 3.35. (Preliminary Trace Formula) *Let ϕ_0 be the constant function of Petersson norm one, and let $\phi_1, \phi_2, \phi_3, \ldots$ denote an orthonormal basis of Maass forms satisfying $\Delta\phi_j = \left(\frac{1}{4} + r_j^2\right)\phi_j$. Let*

$$\mathcal{D}^Y := \left\{ x + iy \in \mathfrak{h} \;\middle|\; -\frac{1}{2} \le x \le \frac{1}{2},\; x^2 + y^2 \ge 1,\; y \le Y \right\}$$

denote a fundamental domain for $SL(2,\mathbb{Z})\backslash\mathfrak{h}$ truncated at $Y > 1$.

Then

$$\boxed{\underbrace{\sum_{j=0}^{\infty} h(r_j)}_{\text{Spectral Side}} \;=\; \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} K_f^*(z,z)\,\frac{dxdy}{y^2} \;+\; \frac{\pi f(0)}{3} \;+\; \lim_{Y\to\infty} \mathcal{I}(Y),}$$

where

$$\boxed{\mathcal{I}(Y) := \int\limits_0^Y \int\limits_0^1 \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f\left(\frac{m^2}{y^2}\right)\frac{dxdy}{y^2} \;-\; \int\limits_{\mathcal{D}^Y}\int\limits_{-\infty}^{\infty} h(r)\cdot |E(z, 1/2 + ir)|^2 \, dr\,\frac{dxdy}{4\pi y^2}.}$$

4. The Selberg Trace Formula for $SL(2,\mathbb{R})$ (Geometric Side)

A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ is of 3 types:

- Parabolic with $|a+d| = 2$,
- Hyperbolic with $|a+d| > 2$,
- Elliptic with $|a+d| < 2$.

We can decompose $SL(2,\mathbb{Z})$ as a union over distinct conjugacy classes

$$SL(2,\mathbb{Z}) = \bigcup_{\text{conj. classes } \alpha} [\alpha]$$

where $[\alpha]$ denotes the conjugacy class given by

$$[\alpha] := \left\{ \sigma\alpha\sigma^{-1} \;\middle|\; \sigma \in SL(2,\mathbb{Z}) \right\}.$$

Let $\phi_0 = \sqrt{3/\pi}$ and let $\phi_1, \phi_2, \ldots$ be an orthonormal basis of Maass cusp forms for $\mathcal{L}^2_{\text{cusp}}(SL(2,\mathbb{Z})\backslash\mathfrak{h})$ each satisfying $\Delta\phi_j = \left(1/4 + r_j^2\right)\phi_j$. Fix Selberg transform functions as in definition 3.28. We assume that $h(r) = h(-r)$. The Selberg Trace Formula for $SL(2,\mathbb{Z})$ is the identity

$$\boxed{\underbrace{\sum_{j=0}^{\infty} h(r_j)}_{\text{Spectral Side}} \;=\; \underbrace{C(\mathrm{Id}) \;+\; \sum_{\substack{P \text{ hyperbolic} \\ \text{conj. classes}}} C(P) \;+\; \sum_{\substack{R \text{ elliptic} \\ \text{conj. classes}}} C(R) \;+\; C(\infty)}_{\text{Geometric Side}},}$$

where $C(\mathrm{Id})$ is the contribution of the Identity element, $C(P)$ is the contribution of a hyperbolic conjugacy class $[P]$, while $C(R)$ is the contribution of an elliptic conjugacy class $[R]$, and $C(\infty)$ is the contribution of the sum of non-identity parabolic conjugacy classes minus the contribution of the continuous spectrum. In the following sections of these notes we will explicitly evaluate each of these contributions based on the preliminary trace formula derived in proposition 3.35.

4.1. The Identity Contribution $C(\mathrm{Id})$.

The identity contribution $C(\mathrm{Id})$ is just the term $\frac{\pi f(0)}{3}$ in proposition 3.35. We want to express $C(\mathrm{Id})$ in terms of the test function h. The final result is.

Proposition 4.1. $\boxed{C(\mathrm{Id}) = \dfrac{1}{12} \displaystyle\int_{-\infty}^{\infty} r \tanh(\pi r)\, h(r)\, dr.}$

Proof.

$$C(\mathrm{Id}) = \frac{\pi}{3} f(0) = -\frac{1}{3}\int_0^\infty \frac{Q'(w)}{\sqrt{w}}\, dw = -\frac{1}{3}\int_0^\infty \frac{g'(u)}{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}\, du$$

$$= \frac{1}{6\pi}\int_{-\infty}^\infty \int_0^\infty r h(r)\, \frac{\sin(ru)}{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}\, du\, dr.$$

Now $\displaystyle\int_0^\infty \frac{\sin(ru)}{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}\, du = \frac{\pi \tanh(\pi r)}{2}$. The result follows. $\square$

4.2. **The Contribution $C(\infty)$.** Following proposition 3.35, we define

$$C(\infty) = \lim_{Y\to\infty}\left[\int_0^Y \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f\left(\frac{m^2}{y^2}\right)\frac{dy}{y^2} \;-\; \int_{\mathcal{D}^Y}\int_{-\infty}^\infty h(r)|E(z,1/2+ir)|^2\, dr\frac{dxdy}{4\pi y^2}\right].$$

Proposition 4.2. *For $s \in \mathbb{C}$, let*

$$M(s) := \frac{\sqrt{\pi}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Let $h(r)$ be an even function, holomorphic in some strip $|\mathrm{Im}(r)| \le \frac{1}{2}+\epsilon$ for $\epsilon > 0$ which satisfies $h(r) = \mathcal{O}\left(e^{-5|r|}\right)$ for $r \in \mathbb{R}$. Then

$$\boxed{\begin{aligned} C(\infty) = -\frac{1}{2\pi}\int_{-\infty}^\infty h(r)\left[\frac{\Gamma'}{\Gamma}(1+ir) + \mathrm{Re}\left(\frac{M'}{M}(1/2+ir)\right)\right] dr \\ + \frac{1-M(1/2)}{4} - g(0)\log 2. \end{aligned}}$$

Proof. The result follows from the next two lemmas.

Lemma 4.3. *For $Y \to \infty$ we have*

$$\int_0^Y \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f\left(\frac{m^2}{y^2}\right)\frac{dy}{y^2} \;=\; g(0)\log(Y/2) + \frac{h(0)}{4}$$

$$- \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\,\frac{\Gamma'}{\Gamma}(1+ir)\, dr \;+\; \mathcal{O}\left(\frac{\log Y}{Y}\right).$$

Proof. After transforming, $y \mapsto \frac{|m|}{y}$, and using

$$\sum_{m=1}^{x} \frac{1}{m} = \log x + c_0 + \mathcal{O}\left(\frac{1}{x}\right),$$

along with

$$\sum_{0<m\leq yY} \frac{1}{m} = 0$$

for $y < 1/Y$ we obtain

$$\int_0^Y \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f\left(\frac{m^2}{y^2}\right)\frac{dy}{y^2} = 2\int_{1/Y}^{\infty}\left(\sum_{1\leq m\leq yY}\frac{1}{m}\right) f\left(y^2\right)\, dy$$

$$= 2\int_{1/Y}^{\infty} [\log(yY) + c_0]\, f\left(y^2\right)\, dy + \mathcal{O}\left(\frac{1}{Y}\int_{1/Y}^{\infty}\frac{|f(y^2)|}{y}\, dy\right)$$

Since $|f| \ll 1$, and $f(y) \ll y^{-1-\epsilon}$ as $y \to \infty$ we have

$$\frac{1}{Y}\int_{1/Y}^{\infty}\frac{|f(y^2)|}{y}\, dy = \frac{1}{Y}\int_{1/Y}^{Y}\frac{1}{y}\, dy \; + \; \frac{1}{Y}\int_{Y}^{\infty}\frac{y^{-2-2\epsilon}}{y}\, dy \; \ll \; \frac{\log Y}{Y}$$

Further, using the bound $|f| \ll 1$, we see that

$$\int_0^{1/Y} \Big|\log(yY) + c_0\Big| \cdot |f(y^2)|\, dy \; \ll \; \frac{1}{Y}.$$

It follows that

$$\int_0^Y \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f\left(\frac{m^2}{y^2}\right)\frac{dy}{y^2} = 2\int_{1/Y}^{\infty}\left(\sum_{1\leq m\leq yY}\frac{1}{m}\right) f\left(y^2\right)\, dy$$

$$= 2(\log Y)\int_0^{\infty} f\left(y^2\right)\, dy + 2\int_0^{\infty} f\left(y^2\right)(\log y + c_0)\, dy \; + \; \mathcal{O}\left(\frac{\log Y}{Y}\right)$$

$$= g(0)\big(\log Y + c_0\big) + \frac{1}{2}\int_0^{\infty} f(y)\log y\, \frac{dy}{\sqrt{y}} \; + \; \mathcal{O}\left(\frac{\log Y}{Y}\right).$$

Next, by the Selberg transform,

$$\begin{aligned}
\int_0^\infty (\log y) f(y) \frac{dy}{\sqrt{y}} &= -\frac{1}{\pi} \int_0^\infty \int_y^\infty \frac{\log y}{\sqrt{y}\sqrt{w-y}}\, Q'(w)\, dw\, dy \\
&= -\frac{1}{\pi} \int_{w=0}^\infty \int_{y=0}^w \frac{\log y}{\sqrt{y}\sqrt{w-y}}\, Q'(w)\, dy\, dw \\
&= -\frac{1}{\pi} \int_0^\infty \underbrace{\left[\int_0^1 \frac{\log(wy)}{\sqrt{y}\sqrt{1-y}}\, dy \right]}_{\text{This integral } = \pi\left(\log w - 2\log 2\right).} Q'(w)\ dw \\
&= -\int_0^\infty \left(\log w - 2\log 2\right) Q'(w) dw \\
&= -\int_0^\infty \underbrace{(\log w)\ Q'(w)\, dw}_{\text{Let } w = e^u + e^{-u} - 2} \ - \ 2(\log 2) g(0) \\
&= -\int_0^\infty \log\left(e^u + e^{-u} - 2\right) g'(u)\ du \ - \ 2(\log 2) g(0)
\end{aligned}$$

Observe that $e^u + e^{-u} - 2 = e^u \left(1 - e^{-u}\right)^2$ so that

$$\log\left(e^u + e^{-u} - 2\right) = u + 2\log\left(1 - e^{-u}\right).$$

It follows that

$$\begin{aligned}
\int_0^\infty (\log y) f(y) \frac{dy}{\sqrt{y}} &= -2\int_0^\infty \log\left(1 - e^{-u}\right)\ g'(u)\ du \ - \ \int_0^\infty u\, g'(u)\, du - 2(\log 2) g(0) \\
&= -2\int_0^\infty \log\left(1 - e^{-u}\right)\ g'(u)\ du \ + \ \frac{h(0)}{2} - 2(\log 2) g(0).
\end{aligned}$$

Now

$$g'(u) = \frac{1}{2\pi} \int_{-\infty}^\infty ir\, h(r) e^{iru}\, dr.$$

It follows that

$$-2\int_0^\infty \log\left(1-e^{-u}\right)\,dg(u) = \frac{i}{\pi}\int_{-\infty}^{\infty} h(r)r\left[\int_0^\infty e^{iru}\log\left(1-e^{-u}\right)\,du\right]dr.$$

Further,

$$-ir\int_0^\infty e^{iru}\log\left(1-e^{-u}\right)\,du = -\frac{\Gamma'(1-ir)}{\Gamma(1-ir)} - c_0.$$

If we insert this into the above we obtain

$$-2\int_0^\infty \log\left(1-e^{-u}\right)\,dg(u) = -2c_0 g(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\,dr$$

since h is even.

Combining all the previous computations completes the proof. □

Definition 4.4. (Truncated Eisenstein series) *Let $Y > \frac{\sqrt{3}}{2}$. We define the truncated Eisenstein series $E^Y(z,s)$ for $z \in \mathcal{D}$ by*

$$E^Y(z,s) := \begin{cases} E(z,s) & \text{for } y < Y, \\ E(z,s) - y^s - M(s)y^{1-s} & \text{for } y \geq Y. \end{cases}$$

Lemma 4.5. (Maass-Selberg relation) *Let $s = \sigma + ir$ with $\sigma > \frac{1}{2}$ and $r \neq 0$. Then*

$$\int_{\mathcal{D}} \left|E^Y(z,s)\right|^2 \frac{dxdy}{y^2} = \frac{Y^{2\sigma-1} - |M(s)|^2 Y^{1-2\sigma}}{2\sigma - 1} + \frac{M(\overline{s})Y^{2ir} - M(s)Y^{-2ir}}{2ir}$$

where $\int_0^1 E(x+iy,s)\,dx = y^s + M(s)y^{1-s}$, $M(s)M(1-s) = 1$, *and*

$$M(s) = \frac{\sqrt{\pi}\,\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Proof. See [Iwa95]. □

We would like to use the Maass-Selberg relation to evaluate the term

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\left[\int_{\mathcal{D}^Y} |E(z,1/2+ir)|^2 \frac{dxdy}{y^2}\right]dr$$

which occurs in $C(\infty)$. There is an issue, however, because the Maass-Selberg relation is for $\int_{\mathcal{D}} \left|E^Y(z,s)\right|^2 \frac{dxdy}{y^2}$ while our goal is to compute $\int_{\mathcal{D}^Y} |E(z,1/2+ir)|^2 \frac{dxdy}{y^2}$. We get around this following a method which was introduced by Selberg (see the Gottingen Lecture notes [Sel89]) Define

$$E^*(z,s) := E(z,s) - y^s - M(s)y^{1-s}.$$

Then for $Y \to \infty$ we need to estimate the difference

$$\int\limits_{\mathcal{D}} \left|E^Y(z,s)\right|^2 - \int\limits_{\mathcal{D}^Y} |E(z,s)|^2 \, \frac{dxdy}{y^2} = \int\limits_{x=0}^{1} \int\limits_{y=Y}^{\infty} |E^*(z,s)|^2 \, \frac{dxdy}{y^2}$$

when $s = u + ir$ and $\frac{1}{2} \le u \ll 1$. Now, with this choice of s,

$$\begin{aligned} E^*(z,s) &= \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n\neq 0} \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x} \\ &\ll \; e^{2r}e^{-\pi y}. \end{aligned}$$

The constants $2,\pi$ in the bound $e^{2r}e^{-\pi y}$ are not optimal.

This bound follows from the exponential decay of the K-Bessel function (here we use the fact that $t + t^{-1} \ge 2$) given by

$$\begin{aligned} \sqrt{y} \cdot K_{u-\frac{1}{2}+ir}(2\pi|n|y) &= \frac{\sqrt{y}}{2} \int\limits_0^\infty e^{-\pi|n|y\left(t+\frac{1}{t}\right)} t^{u-\frac{1}{2}+ir} \, \frac{du}{u} \\ &\ll \; e^{-\frac{3}{2}|n|\pi} \end{aligned}$$

and the exponential decay of the Gamma function given by Stirling's asymptotic formula $|\Gamma(u+ir)|^2 \sim 2\pi \cdot |r|^{2u-1} \, e^{-\pi|r|}$ together with the prime number theorem which says $|\zeta(1+2ir)|^{-1} \ll \log|r|$, for $|r| \gg 1$.

We immediately deduce that

$$\int\limits_{\mathcal{D}^Y} |E(z,u+ir)|^2 \, \frac{dxdy}{y^2} = \int\limits_{\mathcal{D}} \left|E^Y(z,u+ir)\right|^2 \; + \; \mathcal{O}\left(e^{4r}e^{-2\pi Y}\right)$$

It follows from the Maass-Selberg relation and the above computations that

$$\int\limits_{\mathcal{D}^Y} \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot |E(z, 1/2 + ir)|^2 \, dr \, \frac{dxdy}{y^2}$$

$$= \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot \lim_{\sigma \to \frac{1}{2}} \left(\frac{Y^{2\sigma-1} - Y^{1-2\sigma}}{2\sigma - 1} + \frac{\left(1 - |M(\sigma + ir)|^2\right) \cdot Y^{1-2\sigma}}{2\sigma - 1} \right) dr$$

$$+ \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot \frac{M(\frac{1}{2} - ir) Y^{2ir} - M(\frac{1}{2} + ir) Y^{-2ir}}{2ir} \, dr + \mathcal{O}\left(e^{-2\pi Y} \int\limits_{-\infty}^{\infty} h(r) e^{4r} \, dr \right)$$

$$= I_1(Y) + I_2(Y) + \mathcal{O}\left(e^{-2\pi Y}\right)$$

since we have assumed that $h(r) \ll e^{-5|r|}$.

Evaluation of $I_1(Y)$:

First of all, by the Selberg transform (see definition 3.28), we have

$$\frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot \lim_{\sigma \to \frac{1}{2}} \left(\frac{Y^{2\sigma-1} - Y^{1-2\sigma}}{2\sigma - 1} \right) dr = g(0) \log Y.$$

It follows that

$$I_1(Y) = g(0) \log Y + \lim_{\sigma \to \frac{1}{2}} \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot \frac{\left(1 - |M(\sigma + ir)|^2\right) \cdot Y^{1-2\sigma}}{2\sigma - 1} \, dr.$$

To evaluate the limit in the integral above above we consider the Taylor expansion around $\sigma = \frac{1}{2}$ given by

$$1 - M(\sigma + ir) M(\sigma - ir) = \underbrace{1 - M\left(1/2 + ir\right) M\left(1/2 - ir\right)}_{\text{this term} = 0}$$

$$+ \left[M'\left(\frac{1}{2} + ir\right) M\left(\frac{1}{2} - ir\right) + M\left(\frac{1}{2} + ir\right) M'\left(\frac{1}{2} - ir\right) \right] \cdot \left(\sigma - \frac{1}{2}\right)$$

$$+ \Big\{ \text{higher powers of } \sigma - 1/2 \Big\}.$$

We obtain

$$\boxed{I_1(Y) = g(0) \log Y + \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} h(r) \cdot \left[\frac{M'\left(\frac{1}{2} + ir\right)}{M\left(\frac{1}{2} + ir\right)} + \frac{M'\left(\frac{1}{2} - ir\right)}{M\left(\frac{1}{2} - ir\right)} \right] dr.}$$

Evaluation of $I_2(Y)$:

For the evaluation of $I_2(Y)$, it is necessary to interpret the integrals involved in the sense of Cauchy Principal Value, i.e.,

$$\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \;:=\; \lim_{\epsilon\to 0}\int_{|x|\ge\epsilon}\frac{f(x)}{x}\,dx. \tag{4.6}$$

It is easy to see that if f is C^1 on $\mathbb{R}$ and decays moderately at $\pm\infty$, then the limit (4.6) exists.

Next, we compute

$$\begin{aligned} I_2(Y) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\cdot\frac{M(\frac12 - ir)Y^{2ir} - M(\frac12+ir)Y^{-2ir}}{2ir}\,dr \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\cdot\frac{\mathrm{Re}\Big(M\big(\frac12 - ir\big)\sin\big(2r\log Y\big)\Big)}{r}\,dr \\ &= \frac{h(0)M\left(\frac12\right)}{4} \;+\; o(1) \end{aligned}$$

as $Y\to\infty$. This follows since $F(0) = \lim\limits_{T\to\infty}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}F(r)\frac{\sin\left(r\cdot T\right)}{r}\,dr$ which is a well known result in the theory of Fourier integrals.

We have

$$\boxed{I_2(Y) = \frac{h(0)M\left(\frac12\right)}{4} \;+\; o(1).}$$

□

4.3. **Contribution of the Hyperbolic Conjugacy Classes** $C(P)$.

A matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$ is termed hyperbolic if one of the following three equivalent conditions is satisfied:

(**1**) $|a+d|>2$,

(**2**) $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ has 2 distinct real fixed points w_1, w_2 with $\dfrac{aw_i+b}{cw_i+b} = w_i$,

(**3**) There exists unique $\eta\in SL(2,\mathbb{R})$ and $\rho\in\mathbb{R}$ with $|\rho|>1$ such that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = \eta\left(\begin{smallmatrix} \rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)\eta^{-1}$.

Definition 4.7. *If* $P = \eta\left(\begin{smallmatrix} \rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)\eta^{-1}\in SL(2,\mathbb{Z})$ *is hyperbolic with* $|\rho|>1$, *we define* $N(P) := \rho^2$ *to be the norm of* P.

Definition 4.8. (Centralizer Γ_P) *Let $P \in SL(2,\mathbb{Z})$ be hyperbolic. We define the centralizer of P (denoted Γ_P) by*

$$\Gamma_P := \{\delta \in SL(2,\mathbb{Z}) \mid \delta P = P\delta\}.$$

Lemma 4.9. *Let $P \in SL(2,\mathbb{Z})$ be hyperbolic. Then the centralizer Γ_P is the direct product of an infinite cyclic group generated by some $P_0 \in SL(2,\mathbb{Z})$ and the group $\{\pm 1\}$ of order 2. Here P_0 is itself hyperbolic with the same fixed points as P. Hence $\Gamma_P = \{\pm P_0^\ell \mid \ell \in \mathbb{Z}\}$.*

Furthermore $P = \pm P_0^m$ for some $m \geq 1$, and this uniquely determines P_0. The element P_0 is called a primitive hyperbolic element.

Proof. Assume $\delta \in \Gamma_P \iff \delta P = P\delta$.

We now show that this condition forces δ to have the same fixed points w_1, w_2 as P. In fact, for $i = 1, 2$, we have

$$\delta(Pw_i) = \delta\, w_i = P(\delta w_i).$$

This implies that one of the following two cases must hold:

$$\boxed{\textbf{case 1: } \delta w_1 = w_1,\ \delta w_2 = w_2} \quad \text{or} \quad \boxed{\textbf{case 2: } \delta w_1 = w_2,\ \delta w_2 = w_1}.$$

We want to show case 2 cannot happen. In either case $\delta^2 w_i = w_i$.

Now δ must have at least one fixed point which cannot be w_1 or w_2. Further, fixed points of δ must be fixed points of δ^2. It follows that δ^2 must have at least 3 distinct fixed points which is not possible unless δ^2 is $\pm I$ where I is the identity matrix.

So, we may assume $\delta^2 = \pm I$ and $\delta \neq \pm I$ since we are in case 2. It follows that $\delta = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, hence $\mathrm{tr}(\delta) = 0$.

It remains to show that $\mathrm{tr}(\delta) \neq 0$. Let $\eta \in SL(2,\mathbb{R})$ such that $P = \eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \eta^{-1}$. Then $\delta \cdot \left(\eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \eta^{-1}\right) = \left(\eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \eta^{-1}\right) \cdot \delta$ which implies that

$$\left(\eta^{-1}\delta\eta\right) \cdot \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \cdot \left(\eta^{-1}\delta\eta\right).$$

The only matrices in $SL(2,\mathbb{R})$ that commute with $\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right)$ are diagonal matrices of the form $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ with $u \in \mathbb{R}$ and $u \neq 0$. This implies that

$$\eta^{-1}\delta\eta = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$$

for some $u \in \mathbb{R}^\times$. This is a contradiction since the trace of $\eta^{-1}\delta\eta$ is equal to the trace of δ which is zero. We have thus proved that any matrix δ which commutes with a hyperbolic matrix P must have the same fixed points w_1, w_2 as P. It follows that $\Gamma_P = \{\pm P_0^\ell \mid \ell \in \mathbb{Z}\}$

□

Proposition 4.10. (Computation of $C(P)$) *Let $P \in SL(2,\mathbb{Z})$ be a hyperbolic matrix where $P = \pm P_0^m$ $(m \geq 1)$and $P_0 = \eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{smallmatrix}\right) \eta^{-1}$ as in lemma 4.9. Then*

$$\boxed{C(P) = \frac{\log N(P_0)}{\Big(N(P)^{\frac{1}{2}} - N(P)^{-\frac{1}{2}}\Big)} \cdot g\big(\log N(P)\big).}$$

Proof. The proof of proposition 4.10 is based on the following lemma.

Lemma 4.11. *Let $P_0 = \eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{smallmatrix}\right) \eta^{-1}$ be a primitive hyperbolic matrix in $SL(2,\mathbb{Z})$. Then a fundamental domain for $(\eta^{-1}\Gamma_{P_0}\eta)\backslash\mathfrak{h}$ is given by*

$$\mathcal{D}_\rho := \Big\{ z = x + iy \in \mathfrak{h} \;\Big|\; x \in \mathbb{R},\; 1 \leq y \leq \rho^2 \Big\}.$$

Proof. First of all $\eta^{-1}\Gamma_{P_0}\eta = \left\{ \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{smallmatrix}\right)^\ell \;\middle|\; \ell \in \mathbb{Z}\right\}$. Let $z \in \mathfrak{h}$. Then $\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{smallmatrix}\right) z = \rho^2 z$. Clearly $\bigcup\limits_{\ell\in\mathbb{Z}} \rho^{2\ell} \cdot \mathcal{D}_\rho = \mathfrak{h}$. ◻

We now go on to prove proposition 4.10.

$$\begin{aligned}
C(P) &= \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \sum_{\sigma\in\Gamma_{P_0}\backslash SL(2,\mathbb{Z})} f\left(\frac{\left|z - \sigma P_0^m \sigma^{-1} z\right|^2}{\mathrm{Im}(z)\cdot \mathrm{Im}(\sigma P_0^m \sigma^{-1} z)}\right) \frac{dxdy}{y^2} \\
&= \sum_{\sigma\in\Gamma_{P_0}\backslash SL(2,\mathbb{Z})} \int\limits_{\sigma\cdot\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}\right)} \underbrace{f\left(\frac{\left|\sigma z - \sigma P_0^m z\right|^2}{\mathrm{Im}(\sigma z)\cdot \mathrm{Im}(\sigma P_0^m z)}\right)}_{z\mapsto\sigma z} \frac{dxdy}{y^2} \\
&= \int\limits_{\Gamma_{P_0}\backslash\mathfrak{h}} \underbrace{f\left(\frac{\left|z - P_0^M z\right|^2}{\mathrm{Im}(z)\cdot \mathrm{Im}\left(P_0^m z\right)}\right)}_{\text{This follows from proposition 3.9 (iv)}} \frac{dxdy}{y^2}
\end{aligned}$$

Next, make the transformation $z \mapsto \eta^{-1}z$. It follows from lemma 4.11 that

$$\begin{aligned}
C(P) &= \int\limits_{\eta^{-1}\cdot\left(\Gamma_{P_0}\backslash\mathfrak{h}\right)} f\left(\frac{\left|\eta^{-1}z - P_0^m\eta^{-1}z\right|^2}{\mathrm{Im}(\eta^{-1}z)\cdot\mathrm{Im}\left(P_0^m\eta^{-1}z\right)}\right)\frac{dxdy}{y^2} \\
&= \int\limits_{\left(\eta^{-1}\Gamma_{P_0}\eta\right)\backslash\mathfrak{h}} f\left(\frac{\left|z - \eta P_0^m\eta^{-1}z\right|^2}{\mathrm{Im}(z)\cdot\mathrm{Im}\left(\eta P_0^m\eta^{-1}z\right)}\right)\frac{dxdy}{y^2} \\
&= \int\limits_1^{\rho^2}\int\limits_{-\infty}^{\infty} f\left(\frac{\left|z-\rho^{2m}z\right|^2}{\rho^{2m}\,y^2}\right)\frac{dxdy}{y^2} \\
&= \int\limits_1^{\rho^2}\int\limits_{-\infty}^{\infty} f\left(\frac{\left(\rho^{2m}-1\right)^2}{\rho^{2m}}\cdot\frac{x^2+y^2}{y^2}\right)\frac{dxdy}{y^2}.
\end{aligned}$$

To proceed further let $x = y\xi$ in the last integral above. We obtain

$$\begin{aligned}
C(P) &= 2\int\limits_1^{\rho^2}\int\limits_0^{\infty} f\left(\frac{\left(\rho^{2m}-1\right)^2}{\rho^{2m}}\cdot\left(1+\xi^2\right)\right)\frac{d\xi dy}{y} \\
&= 2\log N(P_0)\int\limits_0^{\infty} f\left(\frac{\left(N(P)-1\right)^2}{N(P)}\cdot\left(1+\xi^2\right)\right)d\xi
\end{aligned}$$

Next, let

$$u = \frac{\left(N(P)-1\right)^2}{N(P)}\cdot\left(1+\xi^2\right), \qquad \xi = \left(\frac{N(P)}{(N(P)-1)^2}\,u - 1\right)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned}
C(P) &= \log N(P_0)\frac{N(P)^{\frac{1}{2}}}{N(P)-1}\,Q\Big(N(P)+N(P)^{-1}-2\Big) \\
&= \frac{\log N(P_0)}{N(P)^{\frac{1}{2}} - N(P)^{-\frac{1}{2}}}\,g\big(\log N(P)\big).
\end{aligned}$$

□

4.4. **Contribution of the Elliptic Conjugacy Class $C(R)$.**

A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ is termed elliptic if one of the following conditions is satisfied:

(**1**) $|a+d| < 2$,

(**2**) $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has 2 complex conjugate fixed points,

(**3**) There exists unique $\eta \in SL(2,\mathbb{R})$ and $k(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \eta\, k(\theta)\, \eta^{-1}$.

Definition 4.12. (Centralizer Γ_R) *Let $R \in SL(2,\mathbb{Z})$ be elliptic. We define the centralizer of R (denoted Γ_R) by*

$$\Gamma_R := \{\delta \in SL(2,\mathbb{Z}) \mid \delta R = R\delta\}.$$

Lemma 4.13. *Let $R \in SL(2,\mathbb{Z})$ be elliptic. Then the centralizer Γ_R is a finite cyclic group of order $e > 1$ generated by some $R_0 \in SL(2,\mathbb{Z})$ which is itself elliptic with the same fixed points as R. Hence*

$$\Gamma_R = \Gamma_{R_0} = \{\pm R_0^\ell \mid \ell \in \mathbb{Z}, 1 \le \ell \le e\} = \{R_0^\ell \mid \ell \in \mathbb{Z}, 1 \le \ell \le 2e\}.$$

Furthermore $R = R_0^m$ for some $1 \le m < e$, and this uniquely determines R_0. The element R_0 is called a primitive elliptic element and takes the form $R_0 = \eta\, k(\pi/e)\, \eta^{-1}$.

Proof. Exercise for the reader. $\square$

Proposition 4.14. (Computation of $C(R)$) *Let $R \in SL(2,\mathbb{Z})$ be an elliptic matrix of order e where $R = \pm R_0^m$ (with $1 \le m < e$) and*

$$R_0 = \eta \begin{pmatrix} \cos(\pi/e) & \sin(\pi/e) \\ -\sin(\pi/e) & \cos(\pi/e) \end{pmatrix} \eta^{-1}$$

as in lemma 4.9. Then

$$\boxed{C(R) = \frac{1}{2e\sin\left(\frac{m\pi}{e}\right)} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{2\pi m r}{e}}}{1+e^{-2\pi r}} \cdot h(r)\, dr.}$$

Proof. We follow Kubota's proof in [Kub73]. Our goal is to compute

$$C(R) = \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}} \sum_{\sigma \in \Gamma_{R_0}\backslash SL(2,\mathbb{Z})} f\left(\frac{|z - \sigma R_0^m \sigma^{-1} z|^2}{\operatorname{Im}(z)\cdot \operatorname{Im}(\sigma R_0^m \sigma^{-1} z)}\right) \frac{dxdy}{y^2}.$$

Now $\left(\begin{smallmatrix} \cos(\pi/e) & \sin(\pi/e) \\ -\sin(\pi/e) & \cos(\pi/e) \end{smallmatrix}\right)$ acts as a rotation of angle $\frac{2\pi}{e}$ around i. A fundamental domain for $\eta^{-1}\Gamma_{R_0}\eta$ is a hyperbolic sector of angle $\frac{2\pi}{e}$.

It follows (as in the proof of proposition 4.10) that

$$C(R) = \int\limits_{\left(\eta^{-1}\Gamma_{R_0}\eta\right)\backslash\mathfrak{h}} f\left(\frac{\left|z-\eta R_0^m\eta^{-1}z\right|^2}{\operatorname{Im}(z)\cdot\operatorname{Im}\left(\eta P_0^m\eta^{-1}z\right)}\right)\frac{dxdy}{y^2} = \frac{1}{e}\int\limits_{\mathfrak{h}} f\left(\frac{\left|z-\eta R_0^m\eta^{-1}z\right|^2}{\operatorname{Im}(z)\cdot\operatorname{Im}\left(\eta R_0^m\eta^{-1}z\right)}\right)\frac{dxdy}{y^2} \tag{4.15}$$

because it takes e images of $\left(\eta^{-1}\Gamma_{R_0}\eta\right)\backslash\mathfrak{h}$ to cover $\mathfrak{h}$ exactly.

Now, let

$$\eta R_0^m\eta^{-1} = \begin{pmatrix}\cos(m\pi/e) & \sin(m\pi/e)\\ -\sin(m\pi/e) & \cos(m\pi/e)\end{pmatrix} := \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}.$$

We see that

$$\begin{aligned} C(R) &= \frac{1}{e}\int_0^\infty\int_{-\infty}^\infty f\left(\frac{\left|z-\frac{\alpha z-\beta}{\beta z+\alpha}\right|^2}{\operatorname{Im}(z)\cdot\operatorname{Im}\left(\frac{\alpha z-\beta}{\beta z+\alpha}\right)}\right)\frac{dxdy}{y^2}\\ &= \frac{1}{e}\int_0^\infty\int_{-\infty}^\infty f\left(\frac{\left|\beta z^2+\beta\right|^2}{y^2}\right)\frac{dxdy}{y^2}\\ &= \frac{2}{e}\int_0^\infty\int_0^\infty f\left(\beta^2\left(\frac{(x^2+1)^2}{y^2}+2(x^2-1)+y^2\right)\right)\frac{dxdy}{y^2}\end{aligned}$$

Next, make the change of variables (recall that $\beta = -\sin\left(\frac{m\pi}{e}\right) < 0$):

$$\begin{aligned} t &= \beta^2\left(\frac{(x^2+1)^2}{y^2}+2(x^2-1)+y^2\right),\\ dt &= 4\beta^2\left(\frac{x^2+1+y^2}{y^2}\right)x\,dx = \frac{4\beta^2}{y}\sqrt{4+\frac{t}{\beta^2}}\,x\,dx,\\ \frac{dx}{dt} &= \frac{y}{4|\beta|x\sqrt{t+4\beta^2}}.\end{aligned}$$

It is clear that $t = t(x)$ is strictly increasing for $0 \le x \le \infty$, so the minimum value of t is $\beta^2\left(y^{-1}-y\right)^2$. Now $t = \beta^2\left(y^{-1}-y\right)^2$ when $y - y^{-1} = \pm\frac{\sqrt{t}}{\beta}$. There are 2 cases.

(1) $y - y^{-1} = \frac{\sqrt{t}}{|\beta|} \implies y = \frac{\sqrt{t}+\sqrt{t+4|\beta|^2}}{2|\beta|} := y_2$,

(2) $y - y^{-1} = -\frac{\sqrt{t}}{|\beta|} \implies y = \frac{-\sqrt{t}+\sqrt{t+4\beta^2}}{2|\beta|} := y_1$.

Hence $0 \le t < \infty$ and $y_1 < y < y_2$. Furthermore

$$x = \left(y\sqrt{4+\frac{t}{\beta^2}} - y^2 - 1\right)^{\frac{1}{2}} = \Big((y-y_1)(y_2-y)\Big)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned}
C(R) &= \frac{2}{e}\int\limits_0^\infty \int\limits_{\beta^2(y^{-1}-y)^2}^\infty f(t)\,\left(\frac{dx}{dt}\right) dt\,\frac{dy}{y^2}\\
&= \frac{2}{e}\int\limits_0^\infty \int\limits_{\beta^2(y^{-1}-y)^2}^\infty f(t)\,\left(\frac{y}{4|\beta|x\sqrt{t+4\beta^2}}\right) dt\,\frac{dy}{y}\\
&= \frac{1}{2e|\beta|}\int\limits_0^\infty \frac{f(t)}{\sqrt{t+4\beta^2}}\underbrace{\left[\int\limits_{y_1}^{y_2}\frac{dy}{y(y-y_1)^{\frac{1}{2}}(y_2-y)^{\frac{1}{2}}}\right]}_{\text{This integral } = \pi \text{ because } y_1y_2=1} dt\\
&= \frac{\pi}{2e|\beta|}\int\limits_0^\infty \frac{f(t)}{\sqrt{t+4\beta^2}}\,dt\\
&= \frac{\pi}{2e|\beta|}\int\limits_0^\infty \frac{1}{\sqrt{t+4\beta^2}}\left[\frac{-1}{\pi}\int\limits_t^\infty \frac{Q'(w)}{\sqrt{w-t}}\,dw\right]\,dt.
\end{aligned}$$

To proceed further we interchange integrals and then integrate by parts to obtain

$$\begin{aligned}
C(R) &= -\frac{1}{2e|\beta|}\int\limits_{w=0}^\infty \int\limits_{t=0}^w \frac{dt}{\sqrt{(4\beta^2+t)(w-t)}}\,dQ(w)\\
&= \frac{1}{2e|\beta|}\int\limits_0^\infty Q(w)\,d\left[\int\limits_0^w \frac{dt}{\sqrt{(4\beta^2+t)(w-t)}}\right]\\
&= \frac{1}{2e|\beta|}\underbrace{\int\limits_0^\infty Q(w)\,\left[\frac{2|\beta|}{\sqrt{w}\,(w+4\beta^2)}\right]\,dw}_{\text{Let } w=\left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^2}
\end{aligned}$$

Finally, we obtain

$$
\begin{aligned}
C(R) &= \frac{1}{2e} \int\limits_{-\infty}^{\infty} g(u)\, \frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{\left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^2 + 4\beta^2}\, du \\
&= \frac{1}{4\pi e} \int\limits_{0}^{\infty} \frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{\left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^2 + 4\beta^2} \left[\int\limits_{-\infty}^{\infty} h(r) e^{iru}\, dr \right] du \\
&= \frac{1}{4\pi e} \int\limits_{-\infty}^{\infty} h(r) \left[\int\limits_{-\infty}^{\infty} \frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{\left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^2 + 4\beta^2}\, e^{iru}\, du \right] dr \\
&= \frac{1}{2e|\beta|} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{2\pi m r}{e}}}{1 + e^{-2\pi r}} \cdot h(r)\, dr
\end{aligned}
$$

□

5. The Selberg Trace Formula for $SL(2,\mathbb{R})$

We now state the complete version of the Selberg Trace Formula for $SL(2,\mathbb{R})$.

Theorem 5.1. *Let $h:\mathbb{C}\to\mathbb{C}$ (with Fourier transform g) satisfy*

(1) *$h(r)=h(-r)$ for all $r\in\mathbb{R}$.*
(2) *$h(r)$ is holomorphic in the strip $\left\{\mathrm{Im}(r)<\frac{1}{2}+\epsilon \;\middle|\; \text{for some } \epsilon>0\right\}$.*
(3) *$h(r)=\mathcal{O}\left((1+r^2)^{-1-\epsilon}\right)$ in the above strip.*

Let ϕ_0 be the constant function of Petersson norm one, and let $\phi_1,\phi_2,\ldots$ denote an orthonormal basis of Maass cusp forms which satisfy $\Delta\phi_i=\left(\frac{1}{4}+r_i^2\right)\phi_i$ for $i=1,2\ldots$

The Selberg trace formula is the following identity

$$\begin{aligned}\sum_{i=0}^{\infty} h(r_i) \;&=\; \frac{1}{12}\int_{-\infty}^{\infty} r\tanh(\pi r)\,h(r)\,dr\\ &+\underbrace{\sum_{R_0}}_{\text{primitive elliptic}}\ \sum_{m=1}^{e-1}\frac{1}{2e\sin\left(\frac{m\pi}{e}\right)}\int_{-\infty}^{\infty}\frac{e^{-\frac{2\pi m r}{e}}}{1+e^{-2\pi r}}\cdot h(r)\,dr\\ &+\underbrace{\sum_{P_0}}_{\text{primitive hyperbolic}}\ \sum_{m=1}^{\infty}\frac{\log N(P_0)}{\left(N(P_0)^{\frac{m}{2}}-N(P_0)^{-\frac{m}{2}}\right)}\cdot g\big(m\log N(P_0)\big)\\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\Gamma'}{\Gamma}(1+ir)+\mathrm{Re}\left(\frac{M'}{M}(1/2+ir)\right)\right]dr\\ &+\frac{1-M(1/2)}{4}-g(0)\log 2.\end{aligned}$$

Here

$$M(s)=\frac{\sqrt{\pi}\,\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Proof. This follows immediately from propositions 4.1, 4.2, 4.10, 4.14 under the assumption that $h(r)\ll e^{-5|r|}$ is even and holomorphic in a strip $|\mathrm{Im}(r)|\le\frac{1}{2}+\epsilon$ for $\epsilon>0$. We now show that the range in which the trace formula holds can be extended by allowing the test function

h to satisfy

$$h(r) = \mathcal{O}\left((1 + r^2)^{-1-\epsilon}\right).$$

We define

$$\omega(r) := \frac{\Gamma'}{\Gamma}(1 + ir) + \mathrm{Re}\left(\frac{M'}{M}(1/2 + ir)\right).$$

and move the last integral on the right side of the Selberg trace formula in theorem 5.1 to the left side (spectral side) which then becomes

$$\sum_{j=1}^{\infty} h(r_j) \; + \; \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\omega(r)\,dr.$$

We now choose $h(r) = e^{-\frac{r^2}{R^2}}$ for $R \to \infty$, which satisfies the condition $h(r) \ll e^{-5|r|}$ and is even and holomorphic in the strip $|\mathrm{Im}(r)| \leq \frac{1}{2} + \epsilon$. It follows that

$$\sum_{|r_j|\leq R} 1 \; + \; \frac{1}{2\pi}\int_{-R}^{R} \omega(r)\,dr \; = \; \mathcal{O}\left(R^2\right)$$

as $R \to \infty$. Quoting from Selberg's Gottingen lectures ([Sel89], page 668) with some modifications in the choice of variables and citations to previous equations:

This implies that all series and integrals occurring in theorem 5.1 converge absolutely if $h(r)$ only satisfies the usual conditions mentioned above. Also for a class of functions which satisfies these conditions uniformly, convergence of series and integrals is seen to be uniform. Considering now for a fixed $h(r)$ the class $h(r)e^{-\epsilon r^2}$ with $0 \leq \epsilon \leq 1$, these constitute such a class, we have that for $\epsilon > 0$ that

$$h(r)e^{-\epsilon r^2} \ll e^{-5|r|}$$

is satisfied and so proposition 4.2 is valid. Making $\epsilon \to 0$ we obtain theorem 5.1.

There is one final point in the proof of theorem 5.1 that needs to be clarified. It is necessary to make sure that the infinite sum over primitive hyperbolic conjugacy classes converges absolutely.

Let $P_0 = \eta \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \eta^{-1}$ be a primitive hyperbolic matrix with trace $t = \rho + \rho^{-1}$ and norm $N(P_0) = \rho^2$. Now $t > 2$, and

$$\rho = \frac{t + \sqrt{t^2 - 4}}{2}, \qquad N(P_0) = \left(\frac{t + \sqrt{t^2 - 4}}{2}\right)^2.$$

We will show in §6 that the number of distinct primitive hyperbolic conjugacy classes P_0 with the same trace t is exactly the class number of the real quadratic field $\mathbb{Q}\left(\sqrt{t^2-4}\right)$. Since it is known that this class number is bounded by $\mathcal{O}\left((t^2-4)^{\frac{1}{2}+\epsilon}\right) = \mathcal{O}\left(t^{1+\epsilon}\right)$, it follows that

$$\begin{aligned}\sum_{P_0}\sum_{m=1}^{\infty} & \frac{\log N(P_0)}{\big(N(P_0)^{\frac{m}{2}} - N(P_0)^{-\frac{m}{2}}\big)} \cdot g\big(m\log N(P_0)\big) \\ & \ll \sum_{t=3}^{\infty} t^{1+\epsilon} \sum_{m=1}^{\infty} \frac{\log t}{t^{2m}+t^{-2m}} \cdot g(m\log t)\end{aligned}$$

which converges absolutely because of the rapid decay of the function $g(u)$ as $u \to \infty$. $\square$

6. The Kuznetsov Trace Formula for $SL(2,\mathbb{R})$

There are two fundamental ways to express a group Γ as a disjoint union of interesting subsets of Γ. The first way breaks Γ into conjugacy classes

$$\Gamma \;=\; \bigcup_{\gamma}\,[\gamma], \qquad [\gamma] \;=\; \{\sigma\gamma\sigma^{-1} \mid \sigma \in \Gamma\}.$$

This was the basis for the Selberg trace formula with $\Gamma = SL(2,\mathbb{Z})$. The second basic way is with double cosets. Let H, K be subgroups of Γ acting by left and right multiplication, respectively. Then we have the decomposition

$$\Gamma = \bigcup_{\gamma\in H\backslash\Gamma/K} H\gamma K.$$

The double coset decomposition is the basis for the Kuznetsov trace formula.

Definition 6.1. (Poincaré Series for $SL(2,\mathbb{Z})$) *Let $p : \mathbb{R} \to \mathbb{C}$ be a smooth function satisfying*

$$p(y) \ll \begin{cases} y^{1+\epsilon} & \text{if } 0 < y \le 1,\\ y^{-C} & \text{if } 1 < y,\end{cases}$$

for some $\epsilon > 0$ and some fixed constant $C > 1$. Let

$$B := \left\{ \begin{pmatrix} 1 & m\\ 0 & 1\end{pmatrix} \;\middle|\; m \in \mathbb{Z}\right\}$$

denote the Borel subgroup of $\Gamma = SL(2,\mathbb{Z})$. Then for $z \in \mathfrak{h}$, we define the Poincaré series $\mathcal{P}_m(*, p) : \mathfrak{h} \to \mathbb{C}$ by

$$\boxed{\mathcal{P}_m(z,p) = \sum_{\gamma\in B\backslash\Gamma} p\big(m\cdot \mathrm{Im}(\gamma z)\big)\; e^{2\pi i m\cdot \mathrm{Re}(\gamma z)}.}$$

Let $\mathcal{P}_m(z,p)$, $\mathcal{Q}_n(z,q)$ be two Poincaré series with $m, n \ge 1$ and smooth functions $p, q : \mathbb{R} \to \mathbb{C}$ as in definition 6.1 . The Kuznetsov trace formula is obtained by computing the Petersson inner product

$$\Big\langle \mathcal{P}_m(*,\,p),\; \mathcal{Q}_n(*,\,q)\Big\rangle = \int\limits_{\Gamma\backslash\mathfrak{h}} \mathcal{P}_m(z,p)\,\overline{\mathcal{Q}_n(z,q)}\;\frac{dxdy}{y^2}$$

in two different ways.

- **First way, Spectral Side:** *Take the spectral expansion of $\mathcal{P}_m(*,p)$ and unravel $\mathcal{Q}_n(*,q)$ in the inner product.*
- **Second way, Geometric Side:** *Compute the Fourier expansion of $\mathcal{P}_m(*,p)$ by double coset decomposition and unravel $\mathcal{Q}_n(*,q)$.*

7. The $SL(2,\mathbb{R})$ Kuznetsov Trace Formula (Spectral Side)

We first need to show that $\mathcal{P}_m(*,p) \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. Since $p(y) \ll y^{1+\epsilon}$ it easily follows from the growth properties of p that

$$\begin{aligned}
|\mathcal{P}_m(z,p)| &\ll \sum_{\gamma\in B\backslash\Gamma} \mathrm{Im}\big(\gamma z\big)^{1+\epsilon}\\
&\ll \; p(y) \; + \sum_{\substack{\gamma\in B\backslash\Gamma\\ \gamma\neq I_2}} \mathrm{Im}(\gamma z)^{1+\epsilon}\\
&\ll \; 1.
\end{aligned}$$

where $I_2 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. We have thus proved that $\mathcal{P}_m(*,p) \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$.

By the Selberg spectral decomposition given in theorem 3.31 we see that

$$\mathcal{P}_m(z,p) = \sum_{j=0}^{\infty} \Big\langle \mathcal{P}_m(*,p),\ \phi_j\Big\rangle \frac{\phi_j(z)}{\langle \phi_j,\phi_j\rangle} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \Big\langle \mathcal{P}_m(*,p),\ E(*,1/2+ir)\Big\rangle\, E(z,1/2+ir)\, dr.$$

It follows that the inner product of two Poincaré series is given by

(7.1)

$$\Big\langle \mathcal{P}_m(*,\,p),\ \mathcal{Q}_n(*,\,q)\Big\rangle = \sum_{j=0}^{\infty} \frac{\big\langle \mathcal{P}_m(*,p),\ \phi_j\big\rangle\,\big\langle \phi_j,\ \mathcal{Q}_n(*,q)\big\rangle}{\langle \phi_j,\phi_j\rangle} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \Big\langle \mathcal{P}_m(*,p),\ E(*,1/2+ir)\Big\rangle\Big\langle E(*,1/2+ir),\ \mathcal{Q}_n(*,q)\Big\rangle\, dr.$$

To proceed further we require formulae for the inner product of the Poincaré series $\mathcal{P}_m$ with a Maass form or Eisenstein series.

Let ϕ_0 be the constant function of Petersson norm one, and let $\phi_1,\phi_2,\ldots$ denote an orthonormal basis of Maass cusp forms which satisfy $\Delta\phi_i = \left(\frac14 + r_i^2\right)\phi_i$ for $i=1,2\ldots$ Each Maass form ϕ_j with $j>0$ has a Fourier expansion of the form

$$\phi_j(z) = \sum_{\ell\neq 0} A_j(\ell)\sqrt{y}\,K_{ir_j}(2\pi|\ell|y)e^{2\pi i\ell x},$$

where

$$K_{ir}(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(u+\frac{1}{u}\right)} u^{ir}\, \frac{du}{u}$$

is the K-Bessel function.

Lemma 7.2. (Inner product of $\mathcal{P}_m$ with a Maass form ϕ_j)

$$\boxed{\Big\langle \mathcal{P}_m(*,p),\ \phi_j \Big\rangle = \begin{cases} 0 & \text{if } j=0, \\ A_j(m)m^{\frac12}\int\limits_0^\infty p(y)K_{ir_j}(2\pi y)\,\frac{dy}{y^{\frac32}} & \text{if } j>0. \end{cases}}$$

Proof. We compute

$$\begin{aligned}
\Big\langle \mathcal{P}_m(*,p),\ \phi_j \Big\rangle &= \int\limits_{\Gamma\backslash\mathfrak{h}} \sum_{\gamma\in B\backslash\Gamma} p\big(m\cdot \operatorname{Im}(\gamma z)\big)\, e^{2\pi i m \operatorname{Re}(\gamma z)} \cdot \overline{\phi_j(z)}\ \frac{dxdy}{y^2} \\
&= \int\limits_{B\backslash\mathfrak{h}} p\big(my\big)\, e^{2\pi i m x}\, \overline{\phi_j(z)}\ \frac{dxdy}{y^2} \\
&= \int\limits_0^\infty \underbrace{\left[\int\limits_0^1 e^{2\pi i m x}\overline{\phi_j(x+iy)}\, dx\right]}_{\text{This integral } = 0 \text{ if } j=0.} \cdot\, p(my)\ \frac{dy}{y^2} \\
&= \overline{A_j(m)}\underbrace{\int\limits_0^\infty \sqrt{y}\, K_{ir_j}(2\pi m y)\cdot p(my)\ \frac{dy}{y^2}}_{\text{Here we assume } j\neq 0.}
\end{aligned}$$

Every Maass form ϕ_j for $SL(2,\mathbb{Z})$ is self dual so $A_j(m) = \overline{A_j(m)}$. $\square$

Next, we consider the inner product of $\mathcal{P}_m$ and $E(*,s)$. The Fourier expansion of the Eisenstein series is given in theorem 3.4. Let

$$A(m,s) := \frac{2\pi^s \sigma_{1-2s}(m) m^{s-\frac12}}{\Gamma(s)\zeta(2s)}$$

denote the m^{th} arithmetic Fourier coefficient of $E(z,s)$.

Lemma 7.3. (Inner product of $\mathcal{P}_m$ with an Eisenstein series)

$$\boxed{\Big\langle \mathcal{P}_m(*,p),\ E(*,s) \Big\rangle = \overline{A(m,s)}\, m^{\frac12}\int\limits_0^\infty p(y)\, K_{ir_j}(2\pi y)\ \frac{dy}{y^{\frac32}}.}$$

Proof. The proof is the same as the proof for a Maass form in lemma 7.2 . $\square$

Proposition 7.4. (Kuznetsov trace formula, Spectral Side) *Let $\mathcal{P}_m(z,p)$, $\mathcal{Q}_n(z,q)$ be Poincaré series. Define the Bessel transforms:*

$$p^{\#}(ir) := \int_0^\infty p(y)K_{ir}(2\pi y)\,\frac{dy}{y^{\frac{3}{2}}}, \qquad q^{\#}(ir) := \int_0^\infty q(y)K_{ir}(2\pi y)\,\frac{dy}{y^{\frac{3}{2}}}.$$

Then the spectral side of the Kuznetsov trace formula is

$$\boxed{\begin{aligned}\Big\langle \mathcal{P}_m(*,\,p),\ \mathcal{Q}_n(*,\,q)\Big\rangle &= \sqrt{mn}\sum_{j=1}^\infty A_j(m)A_j(n)\cdot\frac{p^{\#}(ir_j)\overline{q^{\#}(ir_j)}}{\langle\phi_j,\ \phi_j\rangle}\\ &+\frac{\sqrt{mn}}{4\pi}\int\limits_{-\infty}^{\infty} A\left(m,\frac{1}{2}+ir\right)\overline{A\left(n,\frac{1}{2}+ir\right)}\cdot p^{\#}(ir)\,\overline{q^{\#}(ir)}\,dr.\end{aligned}}$$

Proof. This follows immediately from (7.1) and lemmas 7.2, 7.3. $\square$

8. The $SL(2,\mathbb{R})$ Kuznetsov Trace Formula (Geometric Side)

To compute the geometric side of the Kuznetsov tace formula we first need to rewrite the Poincaré series $P_m(*,\,p)$ as a sum over double cosets $B\backslash\Gamma/B$. We require the following lemma.

Lemma 8.1. (Double coset decomposition of $\Gamma = SL(2,\mathbb{Z})$) *Let $B = \left\{\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \mid m\in\mathbb{Z}\right\}\subset\Gamma$. Then*

$$\Gamma = B\ \cup\ \left(\bigcup_{c>0}\ \bigcup_{d\ (\mathrm{mod}\ c)} B\begin{pmatrix} * & *\\ c & d\end{pmatrix}B\right).$$

Proof. Let $c\ge 1$. Then we have the matrix identity

$$\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & *\\ c & d\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a+cm & *\\ c & d+cn\end{pmatrix}.$$

It follows that the couble coset $B\left(\begin{smallmatrix} * & *\\ c & d\end{smallmatrix}\right)B$ is determined uniquely by c and $d\,(\mathrm{mod}\ c)$ and this double coset is independent of the top row of $\left(\begin{smallmatrix} * & *\\ c & d\end{smallmatrix}\right)$. $\square$

It follows from lemma 8.1 that the Poincaré series $\mathcal{P}_m(z,p)$ can be rewritten in the following form.

$$\mathcal{P}_m(z,p) = p\big(my\big)e^{2\pi imx}\ +\ \sum_{c=1}^\infty\sum_{d\in\mathbb{Z}}{}' \, p\left(\frac{my}{|cz+d|^2}\right)e^{2\pi im\cdot\mathrm{Re}\left(\frac{az+b}{cz+d}\right)},$$

where the prime on the sum denotes that the sum is over $(d,c)=1$. Further a,b is chosen so that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. One checks that the Poincaré series is independent of the choice of a,b.

The geometric side of the Kuznetsov trace formula involves classical Kloosterman sums which we now define.

Definition 8.2. (Kloosterman Sum) *Let $m,n,c\in\mathbb{Z}$ with $c\geq 1$. The Kloosterman sum is defined by*

$$S(m,n;c) := \sum_{1\leq a\leq c}{}' \, e^{2\pi i\cdot \frac{am+dn}{c}}$$

where the prime on the sum denotes that $(a,c)=1$. Further d is chosen so that $ad\equiv 1 \pmod{c}$.

Proposition 8.3. Kuznetsov tace formula, Geometric Side) *Let $\mathcal{P}_m(z,p)$, $\mathcal{Q}_n(z,q)$ be Poincaré series. The geometric side of the Kuznetsov trace formula is*

$$\boxed{\begin{aligned} \Big\langle \mathcal{P}_m(*,\,p),\ \mathcal{Q}_n(*,\,q)\Big\rangle \;&=\; \delta_{m,n}\int_0^\infty p(my)\,\overline{q(ny)}\,\frac{dy}{y^2} \\ +\int_0^\infty\int_{-\infty}^\infty p\left(\frac{my}{c^2y\,(x^2+1)}\right)&\overline{q(ny)}\sum_{c=1}^\infty S(m,n;c)\,e^{-2\pi i x\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\,\frac{dx\,dy}{y}. \end{aligned}}$$

Proof. The proof of proposition 8.3 requires the Fourier expansion of the Poincaré series given in the following lemma.

Lemma 8.4. (Fourier expansion of $\mathcal{P}_m(z,p)$) *We have*

$$\begin{aligned} \int_0^1 \mathcal{P}_m(z,p)e^{-2\pi i n x}\,dx \;&=\; \delta_{m,n}\cdot p(ny) \\ &+y\cdot\sum_{c=1}^\infty S(m,n;c)\int_{-\infty}^\infty p\left(\frac{my}{c^2y\,(x^2+1)}\right)e^{-2\pi i x\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\,dx. \end{aligned}$$

where $\delta_{m,n}=\begin{cases}1 & m=n,\\ 0 & m\neq n,\end{cases}$ is Kronecker's symbol.

Proof. We compute.

$$\int_0^1 \mathcal{P}_m(z,p)e^{-2\pi inx}\,dx = \int_0^1 p\big(my\big)e^{2\pi i(m-n)x}\,dx$$

$$+\int_0^1 \left[\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}}{}' \, p\left(\frac{my}{|cz+d|^2}\right)e^{2\pi im\cdot\mathrm{Re}\left(\frac{az+b}{cz+d}\right)}\right]e^{-2\pi inx}\,dx$$

$$= \delta_{m,n}\cdot p(my) \;+\mathcal{I}_p(z).$$

Now

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}.$$

Next let $d = \ell c + r$. We see that $\mathcal{I}_p(z)$ is equal to

$$\underbrace{\int_0^1 \left[\sum_{c=1}^{\infty}\sum_{\ell=-\infty}^{\infty}\sum_{\substack{r=1\\(r,c)=1}}^{c} p\left(\frac{my}{|cz+\ell c+r|^2}\right)e^{2\pi im\cdot\mathrm{Re}\left(\frac{a}{c}-\frac{1}{c(cz+\ell c+r}\right)}\right]e^{-2\pi inx}\,dx}_{\text{Make the change of variables } x\to x-\ell-\frac{r}{c}}$$

$$= \sum_{c=1}^{\infty}\sum_{\ell=-\infty}^{\infty}\sum_{\substack{r=1\\(r,c)=1}}^{c}\int_{-\ell-\frac{r}{c}}^{-\ell-\frac{r}{c}+1} p\left(\frac{my}{|cz|^2}\right)e^{2\pi im\cdot\mathrm{Re}\left(\frac{a}{c}-\frac{1}{c^2z}\right)}\,e^{-2\pi in\left(x-\ell-\frac{r}{c}\right)}\,dx$$

$$= \sum_{c=1}^{\infty}S(m,n;c)\int_{-\infty}^{\infty}\underbrace{p\left(\frac{my}{c^2\left(x^2+y^2\right)}\right)e^{2\pi im\cdot\mathrm{Re}\left(\frac{-1}{c^2z}\right)e^{-2\pi inx}}}_{\text{Let } x\mapsto xy.}\,dx$$

$$= y\cdot\sum_{c=1}^{\infty}S(m,n;c)\int_{-\infty}^{\infty}p\left(\frac{my}{c^2y^2\left(x^2+1\right)}\right)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\,dx$$

$\square$

The proof of proposition 8.3 follows from the above lemma together with the fact that by unraveling $\mathcal{Q}_n$

$$\Big\langle \mathcal{P}_m(*,\,p),\;\mathcal{Q}_n(*,\,q)\Big\rangle = \int_0^{\infty}\left[\int_0^1 \mathcal{P}_m(z,p)e^{-2\pi inx}\,dx\right]\overline{q(ny)}\,\frac{dy}{y^2}.$$

$\square$

9. **The Kuznetsov Trace Formula for** $SL(2,\mathbb{R})$

We now state the complete version of the Kuznetsov Trace Formula for $SL(2,\mathbb{R})$.

Theorem 9.1. *Let $p, q : \mathbb{R} \to \mathbb{C}$ be a smooth functions satisfying*

$$p(y), q(y) \ll \begin{cases} y^{1+\epsilon} & \text{if } 0 < y \le 1, \\ y^{-C} & \text{if } 1 < y, \end{cases}$$

for some $\epsilon > 0$ and some $C > 1$. Define the Bessel transforms

$$p^{\#}(ir) := \int_0^\infty p(y) K_{ir}(2\pi y)\, \frac{dy}{y^{\frac{3}{2}}}, \quad q^{\#}(ir) := \int_0^\infty q(y) K_{ir}(2\pi y)\, \frac{dy}{y^{\frac{3}{2}}}.$$

Let $\phi_j(z) = \sum\limits_{\ell \ne 0} A_j(\ell) \sqrt{y}\, K_{ir_j}(2\pi|\ell|y) e^{2\pi i \ell x}$, $(j = 1, 2, \ldots)$ denote an orthonormal basis of Maass cusp forms where $\Delta \phi_i = \left(\frac{1}{4} + r_i^2\right) \phi_i$ for $i \ge 1$. Let $A(m,s) := \frac{2\pi^s \sigma_{1-2s}(m) m^{s-\frac{1}{2}}}{\Gamma(s)\zeta(2s)}$ denote the m^{th} arithmetic Fourier coefficient of $E(z,s)$.

The Kuznetsov trace formula is the following identity:

$$\begin{aligned} &\sqrt{mn} \sum_{j=1}^{\infty} A_j(m) A_j(n) \cdot \frac{p^{\#}(ir_j)\, \overline{q^{\#}(ir_j)}}{\langle \phi_j,\ \phi_j \rangle} \\ &\quad + \frac{\sqrt{mn}}{4\pi} \int\limits_{-\infty}^{\infty} A\left(m, 1/2 + ir\right) \overline{A\left(n, 1/2 + ir\right)} \cdot p^{\#}(ir)\, \overline{q^{\#}(ir)}\, dr \\ &\quad = \underbrace{\delta_{m,n} \int\limits_0^\infty p(my)\, \overline{q(ny)}\, \frac{dy}{y^2}}_{\text{Main Term}} \quad + \quad \sum_{c=1}^{\infty} S(m,n;c) \int\limits_0^\infty \int\limits_{-\infty}^{\infty} \\ &\qquad \cdot p\left(\frac{m}{c^2 y^2 \left(x^2+1\right)}\right) \overline{q(ny)} \cdot e^{-2\pi i x\left(\frac{m}{c^2 y (x^2+1)} + ny\right)}\, \frac{dx dy}{y}. \end{aligned}$$

9.1. **Kontorovich-Lebedev Transform.**

The Kuznetsov trace formula involves a test function $p : \mathbb{R} \to \mathbb{C}$ and the transform

$$p^{\#}(ir) := \int_0^\infty p(y) K_{ir}(2\pi y)\, \frac{dy}{y^{\frac{3}{2}}}.$$

This type of transform is the well known Kontorovich-Lebedev transform (see [Leb72]) which was originally developed for applications in physics. The inverse transform (for a proof see [GKS11]) is given by

$$p(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} p^{\#}(ir)\sqrt{y}\, K_{ir}(2\pi y)\, \frac{dr}{\Gamma(ir)\Gamma(-ir)}. \tag{9.2}$$

We also note the Mellin transform pair given by

$$\int_0^\infty \sqrt{y}\, K_{ir}(2\pi y)\, y^s\, \frac{dy}{y} = \frac{\Gamma\left(\frac{1}{4} + \frac{s+ir}{2}\right)\Gamma\left(\frac{1}{4} + \frac{s-ir}{2}\right)}{4\pi^{\frac{1}{2}+s}},$$

$$\sqrt{y}\, K_{ir}(2\pi y) = \frac{1}{2\pi i} \int_{s=\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{1}{4} + \frac{s+ir}{2}\right)\Gamma\left(\frac{1}{4} + \frac{s-ir}{2}\right)}{4\pi^{\frac{1}{2}+s}}\, y^{-s}\, ds,$$

for any $\sigma > -\frac{1}{2}$.

It follows form (9.2) and the above Mellin transform that

$$\boxed{p(y) = \frac{1}{8\pi^2 i} \int_{-\infty}^{\infty} \int_{\epsilon-i\infty}^{\epsilon+i\infty} p^{\#}(ir)\, \frac{\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)}{\Gamma(ir)\Gamma(-ir)} \pi^{-s} y^{\frac{1}{2}-s}\, ds\, dr.} \tag{9.3}$$

9.2. **Explicit choice of the $p_{T,R}$ test function.** We now present an explicit choice of test function for the Kuznetsov trace formula for $SL(2,\mathbb{R})$ that is useful for many applications and generalizes to higher rank groups. We let $T \to \infty+$ be a large real variable and let R be a sufficiently large (to be determined later) fixed positive real number. For applications it is convenient to define the test function on the spectral side to be given by

$$\boxed{p^{\#}_{T,R}(\alpha) := e^{\alpha^2/T^2} \cdot \Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right).} \tag{9.4}$$

The main reason for choosing this type of test function is that if we assume, for example, that $p = q$ in the Kuznetsov trace formula given in theorem 9.1 then the contribution of the Maass forms on the spectral side will be

$$\sum_{j=1}^{\infty} A_j(m)A_j(n) \frac{|p^{\#}_{T,R}(ir_j)|^2}{\langle \phi_j, \phi_j \rangle} = \sum_{j=1}^{\infty} A_j(m)A_j(n) \frac{e^{-2r_j^2/T^2} \left|\Gamma\left(\frac{2+R+ir_j}{4}\right)\right|^2}{\langle \phi_j, \phi_j \rangle}$$

which will turn out to naturally count sums of weighted products of Fourier coefficients $A_j(m)A_j(n)$ of Maass forms with Lapace eigenvalue

$\lambda_j \ll T^2$. This is due to the fact that $\lambda_j = \frac{1}{4} + r_j^2$ so the function $e^{-2r_j^2/T^2}$ is essentially supported on $\lambda_j \ll T^2$. The constant R is needed later to allow us to shift the contour in α of a certain integral without crossing any poles of the Gamma function $\Gamma\left(\frac{2+R\pm\alpha}{4}\right)$.

9.3. Bounds for the $p_{T,R}$ test function.

Proposition 9.5. *Let $\epsilon > 0$ and $R > 4$ with $1 \leq a < \frac{R}{4}$. Then for $y > 0$ and $T \to +\infty$ we have*

$$\boxed{p_{T,R}(y) \ \ll\ y^{\frac{1}{2}+2a}\, T^{\epsilon+\frac{R}{2}+\frac{3}{2}-a}.}$$

Proof. We recall the identity (9.3) which states that $p_{T,R}(y)$ is equal to

(9.6)
$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\pi^{-s}y^{\frac{1}{2}-s}\,ds\right]}_{=\ \mathcal{I}(y,r)} \cdot \frac{p^{\#}_{T,R}(ir)\,dr}{\Gamma(ir)\,\Gamma(-ir)}.$$

We may shift the line of integration in the above integral $\mathcal{I}(y,r)$ to the line $\mathrm{Re}(s) = -2a$ where $a > 0$ and $a \notin \mathbb{Z}$. Since $\Gamma\left(\frac{s+ir}{2}\right)$ has simple poles at $s = -2\ell - ir$ for $\ell \in \mathbb{Z}$ and $\ell \geq 0$ it follows from Cauchy's residue theorem that

$$\begin{aligned}
\mathcal{I}(y,r) &= \frac{1}{2\pi i}\int_{-2a-i\infty}^{-2a+i\infty} \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\pi^{-s}y^{\frac{1}{2}-s}\,ds \\
&\quad + \sum_{0\leq\ell\leq a}\left(\operatorname*{Res}_{s=-2\ell+ir} + \operatorname*{Res}_{s=-2\ell-ir}\right)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\pi^{-s}y^{\frac{1}{2}-s} \\
&= \frac{1}{2\pi i}\int_{-2a-i\infty}^{-2a+i\infty} \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\pi^{-s}y^{\frac{1}{2}-s}\,ds \\
&\quad + \sum_{0\leq\ell\leq a}(-1)^{\ell}\Big(2\pi^{2\ell-ir}y^{\frac{1}{2}+2\ell-ir}\,\Gamma(-\ell+ir) \ +\ 2\pi^{2\ell+ir}y^{\frac{1}{2}+2\ell+ir}\,\Gamma(-\ell-ir)\Big).
\end{aligned}$$

We plug this expression into (9.6) and make the change of variable $\alpha = ir$. It follows that

$$\begin{aligned}
p_{T,R}(y) = \frac{-1}{8\pi^2} \int\limits_{\mathrm{Re}(\alpha)=0} \int\limits_{\mathrm{Re}(s)=-2a} & \frac{\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s-\alpha}{2}\right)}{\Gamma(\alpha)\Gamma(-\alpha)} \frac{y^{\frac{1}{2}-s}}{\pi^s} \cdot p^{\#}_{T,R}(\alpha)\, ds\, d\alpha \\
- \frac{1}{\pi i} \int\limits_{\mathrm{Re}(\alpha)=0} \sum_{0\le \ell\le a} (-1)^\ell & \Bigg(\frac{\pi^{2\ell-\alpha} y^{\frac{1}{2}+2\ell-\alpha}\,\Gamma(-\ell+\alpha)}{\Gamma(\alpha)\Gamma(-\alpha)} \\
& + \frac{\pi^{2\ell+\alpha} y^{\frac{1}{2}+2\ell+\alpha}\,\Gamma(-\ell-\alpha)}{\Gamma(\alpha)\Gamma(-\alpha)} \Bigg) \cdot p^{\#}_{T,R}(\alpha)\, d\alpha
\end{aligned}$$

$$\begin{aligned}
= \frac{-1}{8\pi^2} \int\limits_{\{0\}} \int\limits_{\{-2a\}} & \frac{y^{\frac{1}{2}-s} e^{\frac{\alpha^2}{T^2}}}{\pi^s} \cdot \frac{\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right)}{\Gamma(\alpha)\Gamma(-\alpha)}\; ds d\alpha \\
- \frac{1}{\pi i} \int\limits_{\mathrm{Re}(\alpha)=0} \sum_{0\le \ell\le a} (-1)^\ell & \Bigg(\frac{\pi^{2\ell-\alpha} y^{\frac{1}{2}+2\ell-\alpha}\;\Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right)}{(\alpha-1)(\alpha-2)\cdots(\alpha-\ell)\cdot\Gamma(-\alpha)} \\
& + \frac{\pi^{2\ell+\alpha} y^{\frac{1}{2}+2\ell+\alpha}\;\Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right)}{(-\alpha-1)(-\alpha-2)\cdots(-\alpha-\ell)\cdot\Gamma(\alpha)} \Bigg) \cdot e^{\frac{\alpha^2}{T^2}}\, d\alpha
\end{aligned}$$

$$= \mathcal{I}^{(1)}_{T,R}(y) + \mathcal{I}^{(2)}_{T,R}(y) + \mathcal{I}^{(3)}_{T,R}(y).$$

We will now bound the first of the three integrals $\mathcal{I}^{(1)}_{T,R}(y)$. To do so we use Stirling's bound (for $\sigma, t \in \mathbb{R}$ with σ fixed and $|t| \to \infty$) given by

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}\; \underbrace{|t|^{\sigma-\frac{1}{2}}}_{\substack{\text{polynomial}\\ \text{term}}} \cdot \underbrace{e^{-\frac{\pi}{2}|t|}}_{\substack{\text{exponential}\\ \text{term}}}. \tag{9.7}$$

Here $|t|^{\sigma-\frac{1}{2}}$ is called the polynomial factor because it has polynomial growth or decay in $|t|$ as $|t| \to \infty$ and $e^{-\frac{\pi}{2}|t|}$ is called the exponential term because it has exponential decay in $|t|$ as $|t| \to \infty$.

Thus, setting $s = -2a + i\xi$ and $\alpha = ir$, we see that the first term for $p_{T,R}(y)$ is bounded by

$$\mathcal{I}^{(1)}_{T,R}(y) \ll \int\limits_{\mathrm{Re}(\alpha)=0} \int\limits_{\mathrm{Re}(s)=-2a} e^{\frac{\alpha^2}{T^2}} \frac{\Gamma(\frac{2+R+2\alpha}{4})\Gamma(\frac{2+R-2\alpha}{4})\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s-\alpha}{2}\right)}{\Gamma(\alpha)\Gamma(-\alpha)} \cdot y^{\frac{1}{2}-s}\, ds\, d\alpha$$

$$\ll y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \int\limits_{\xi=-\infty}^{\infty} \frac{\big(1+|r|\big)^{\frac{R}{2}+1}}{\big(1+|\xi+r|\big)^{\frac{1+2a}{2}}\big(1+|\xi-r|\big)^{\frac{1+2a}{2}}} \cdot e^{-\frac{\pi}{4}\left(|\xi+\tau|+|\xi-\tau|-2|\tau|\right)}\, d\xi\, d\tau.$$

We rewrite this as

$$\boxed{\mathcal{I}^{(1)}_{T,R}(y) \ \ll\ y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \int\limits_{\xi=-\infty}^{\infty} \mathcal{P}\cdot\exp\left(-\frac{\pi}{4}\,\mathcal{E}\right)\, d\xi\, d\tau}$$

where

$$\mathcal{P} := \frac{\big(1+|r|\big)^{\frac{R}{2}+1}}{\big(1+|\xi+r|\big)^{\frac{1+2a}{2}}\big(1+|\xi-r|\big)^{\frac{1+2a}{2}}}$$

is the polynomal term and

$$\mathcal{E} := \big(|\xi+r|+|\xi-r|-2|r|\big)$$

is the exponential term. It is easy to see that $\mathcal{E} \ge 0$ for all ξ, r.

Definition 9.8. (Exponential zero set of $\mathcal{E}$) *The exponential zero set of $\mathcal{E}$ is the set of all $(\xi, r) \in \mathbb{R}^2$ such that $\mathcal{E} = 0$.*

Lemma 9.9. *The exponential zero set of $\mathcal{E} := \big(|\xi+r|+|\xi-r|-2|r|\big)$ is given by the set of (ξ, r) satisfying $\boxed{-r \le \xi \le r.}$*

One observes that there is exponential decay in the region of integration of the integral $I^{(1)}_{T,R}(y)$ that is outside the exponential zero set. It follows that

$$\mathcal{I}^{(1)}_{T,R}(y) \ \ll\ y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \int\limits_{\xi=-r}^{r} \frac{\big(1+r\big)^{\frac{R}{2}+1}}{\big(1+\xi+r\big)^{\frac{1+2a}{2}}\big(1-\xi+r\big)^{\frac{1+2a}{2}}}\, d\xi\, dr.$$

To complete the proof we require the following lemma.

Lemma 9.10. *Suppose that C, $C' \ge 0$ (with $C, C' \ne 1$) and $A > 0$. Then*

$$\int\limits_{x=0}^{A} \frac{dx}{(1+x)^{\frac{1+C}{2}}(1+A-x)^{\frac{1+C'}{2}}} \ll A^{-\frac{1}{2}\min\left\{1+C,\, 1+C',\, C+C'\right\}}.$$

Proof. Exercise for the reader. □

So using Lemma 9.10, we obtain

$$\begin{aligned}
\mathcal{I}_{T,R}^{(1)}(y) &\ll y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \underbrace{\left[\int\limits_{\xi=-r}^{r} \frac{\big(1+r\big)^{\frac{R}{2}+1}}{\big(1+\xi+r\big)^{\frac{1+2a}{2}}\big(1-\xi+r\big)^{\frac{1+2a}{2}}}\, d\xi\right]}_{\text{Make the change of variable } x=\xi+r.} dr \\
&\ll y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \big(1+r\big)^{\frac{R}{2}+1} \int\limits_{x=0}^{2r} \frac{dx}{(1+x)^{\frac{1+2a}{2}}(1+2r-x)^{\frac{1+2a}{2}}}\, dr \\
&\ll y^{\frac{1}{2}+2a} \int\limits_{r=0}^{T^{1+\epsilon}} \big(1+r\big)^{\frac{R}{2}+1} (2r)^{-\frac{1}{2}\min\left\{1+2a,\, 4a\right\}}\, dr \\
&\ll y^{\frac{1}{2}+2a}\, T^{1+\epsilon}\cdot T^{\frac{R}{2}+1}\cdot T^{-\frac{1}{2}-a} \\
&\ll y^{\frac{1}{2}+2a}\, T^{\epsilon+\frac{R}{2}+\frac{3}{2}-a}
\end{aligned}$$

as claimed.

We now bound the remaining two integrals: $\mathcal{I}_{T,R}^{(2)}(y)$, $\mathcal{I}_{T,R}^{(3)}(y)$. We have

$$\mathcal{I}_{T,R}^{(2)}(y) = \sum_{0\le \ell\le a} \int\limits_{\mathrm{Re}(\alpha)=0} e^{\frac{\alpha^2}{T^2}} \cdot \frac{\pi^{2\ell-\alpha} y^{\frac{1}{2}+2\ell-\alpha}\, \Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right)}{\pi i\,(1-\alpha)(2-\alpha)\cdots(\ell-\alpha)\cdot\Gamma(-\alpha)}\, d\alpha$$

Since we have assumed $1 < a < \frac{R}{2}$ we may shift the line of integration to $\mathrm{Re}(\alpha) = 2\ell - a$ in the above integral without crossing any poles. It then follows from Stirling's asymptotic formula that for $\alpha = 2\ell - 2a + ir$

and $T \to \infty$ we have the bound

$$\begin{aligned}
\mathcal{I}^{(2)}_{T,R}(y) &= \sum_{0\le \ell\le a} \int\limits_{\mathrm{Re}(\alpha)=2\ell-2a} e^{\frac{\alpha^2}{T^2}} \cdot \frac{\pi^{2\ell-\alpha} y^{\frac12+2\ell-\alpha}\, \Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right)}{\pi i\,(1-\alpha)(2-\alpha)\cdots(\ell-\alpha)\cdot\Gamma(-\alpha)}\, d\alpha \\
&\ll y^{\frac12+a}\cdot \sum_{0\le\ell\le a} \int\limits_{r=0}^{T^{1+\epsilon}} \frac{\left|\Gamma\left(\frac{2+R+4\ell-2a+2ir}{4}\right)\Gamma\left(\frac{2+R-4\ell+2a-2ir)}{4}\right)\right|}{(1+|r|)^{\ell}\cdot\big|\cdot\Gamma(2a-2\ell-ir)\big|}\, dr \\
&\ll y^{\frac12+a}\cdot \sum_{0\le\ell\le a} \int\limits_{r=0}^{T^{1+\epsilon}} \frac{(1+r)^{\frac{R}{2}}}{(1+r)^{\ell}\cdot(1+r)^{2a-2\ell-\frac12}}\, dr \\
&\ll y^{\frac12+a}\, T^{\epsilon+\frac{R}{2}+\frac32-a}
\end{aligned}$$

Thus we get the same bound as for $I^{(1)}_{T,R}(y)$. The estimation of $I^{(3)}_{T,R}(y)$ is similar and gives the same bound. This completes the proof of proposition 9.5.

□

9.4. **Bounding the Geometric Side of the KTF.**

We consider the Kuznetsov trace formula with test functions $p = q = p_{T,R}$ with $p_{T,R}$ given by (9.4). In this case, the geometric side is

(9.11)
$$\mathcal{G} = \sum_{c=1}^{\infty} S(m,n;c) \int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} p_{T,R}\left(\frac{m}{c^2y^2\,(x^2+1)}\right)\overline{p_{T,R}(ny)} \cdot \exp\left(-2\pi i x\left(\frac{m}{c^2y(x^2+1)} + ny\right)\right)\,\frac{dxdy}{y}.$$

We will now use the $p_{T,R}$ bound in proposition 9.5:

$$\boxed{p_{T,R}(y) \ll y^{\frac12+2a}\, T^{\frac{R}{2}+\frac32-a}, \qquad (\text{for } 1\le a< R/4),} \tag{9.12}$$

together with the Kloosterman bound

$$\boxed{S(m,n;c) \ll c^{\frac12+\epsilon}.} \tag{9.13}$$

The idea is to break the y-integral in (9.11) as follows

$$\int\limits_{y=0}^{\infty} = \int\limits_{y=0}^{1} + \int\limits_{y=1}^{\infty}$$

and then to use different $p_{T,R}$ bounds for $p_{T,R}\left(\frac{m}{c^2y^2(x^2+1)}\right)$ and $p_{T,R}(ny)$ which appear in (9.11).

In particular, we will use

$$p_{T,R}(y) \ll y^{\frac{1}{2}+2a}\, T^{\frac{R}{2}+\frac{3}{2}-a}$$

to estimate $p_{T,R}\left(\frac{m}{c^2y^2(x^2+1)}\right)$ and we will use

$$p_{T,R}(y) \ll y^{\frac{1}{2}+2b}\, T^{\frac{R}{2}+\frac{3}{2}-b}$$

to estimate $p_{T,R}(ny)$.

Accordingly, we first consider the integral with $0 \le y \le 1$. Then after applying the above Kloosterman bounds we have to estimate

$$\sum_{c=1}^{\infty} c^{\frac{1}{2}+\epsilon} \int\limits_{y=0}^{1} \int\limits_{x=-\infty}^{\infty} \left[\left(\frac{m}{c^2y^2\,(x^2+1)}\right)^{\frac{1}{2}+2a} T^{\frac{R}{2}+\frac{3}{2}-a}\right] \cdot \left[(ny)^{\frac{1}{2}+2b}\, T^{\frac{R}{2}+\frac{3}{2}-b}\right] \frac{dxdy}{y}.$$

Here we take $a = 1$ and $b = \frac{R}{4} - 1$, which guarantees that the c-sum and the x and y integrals converge. We then obtain the bound $\mathcal{O}\left(T^{R+3-\frac{R}{4}}\right) = \mathcal{O}\left(T^{\frac{3R}{4}+3}\right)$.

Next, we consider $1 \le y \le \infty$. In this case, we must bound

$$\sum_{c=1}^{\infty} c^{\frac{1}{2}+\epsilon} \int\limits_{y=1}^{\infty} \int\limits_{x=-\infty}^{\infty} \left[\left(\frac{m}{c^2y^2\,(x^2+1)}\right)^{\frac{1}{2}+2a} T^{\frac{R}{2}+\frac{3}{2}-a}\right] \cdot \left[(ny)^{\frac{1}{2}+2b}\, T^{\frac{R}{2}+\frac{3}{2}-b}\right] \frac{dxdy}{y}.$$

Here we take $a = \frac{R}{4} - 1$ and $b = 1$, which guarantees that the c-sum and the x and y integrals converge. We again obtain the bound $\mathcal{O}\left(T^{\frac{3R}{4}+3}\right)$.

We have now proved the following bound.

Proposition 9.14. (Bounding the geometric side of KTF) *Let*

$$\mathcal{G} = \sum_{c=1}^{\infty} S(m,n;c) \int_0^{\infty}\int_{-\infty}^{\infty} p_{T,R}\left(\frac{m}{c^2y^2\,(x^2+1)}\right)\overline{p_{T,R}(ny)} \cdot \exp\left(-2\pi i x\left(\frac{m}{c^2y(x^2+1)} + ny\right)\right)\,\frac{dxdy}{y}.$$

donote the geometric side of the Kuznetsov trace formula. Then

$$\boxed{\mathcal{G} = \mathcal{O}\left(T^{\frac{3R}{4}+3}\right), \qquad (T\to\infty).}$$

9.5. **Computation of the Main Term $\mathcal{M}$ in the KTF.**

The main term $\mathcal{M}$ only contributes when $m = n$ and is given by

$$\mathcal{M} = m\int_0^{\infty} |p_{T,R}(y)|^2\,\frac{dy}{y^2}$$

To evaluate $\mathcal{M}$ we shall need the Plancherel formula which we now derive. For $\alpha\in\mathbb{C}$ and $y>0$, let $W_\alpha(y) := \sqrt{y}\,K_\alpha(2\pi y)$ denote the $GL(2,\mathbb{R})$ Whittaker function. We have the Kuznetsov-Lebedev transform pair:

$$p^{\#}(\alpha) = \int_0^{\infty} p(y)\cdot W_\alpha(y)\,\frac{dy}{y^2}$$

$$p(y) = \frac{1}{\pi i}\int_{\mathrm{Re}(\alpha)=0} p^{\#}(\alpha)\,\overline{W_\alpha(y)}\,\frac{d\alpha}{\Gamma(\alpha)\Gamma(-\alpha)}.$$

As a consequence we can now derive the Plancherel formula for the $GL(2)$ Whittaker transform

Proposition 9.15. (Plancherel Formula for $GL(2)$) *Let $p:[0,\infty]\to\mathbb{C}$ be a smooth function satisfying*

$$p(y) \ll \begin{cases} y^{1+\epsilon} & \text{if } 0<y\le 1,\\ y^{-C} & \text{if } 1<y.\end{cases}$$

Then

$$\boxed{\int_0^{\infty} |p(y)|^2\,\frac{dy}{y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left|p^{\#}(ir)\right|^2\frac{dr}{|\Gamma(ir)|^2}.}$$

Proof.

$$\int_0^\infty |p(y)|^2 \, \frac{dy}{y^2} = \int_0^\infty p(y) \cdot \left[\frac{1}{\pi i} \int\limits_{\mathrm{Re}(\alpha)=0} \overline{p^\#(\alpha)} \, W_\alpha(y) \, \frac{d\alpha}{\Gamma(\alpha)\Gamma(-\alpha)} \right] \frac{dy}{y^2}$$

$$= \frac{1}{\pi i} \int\limits_{\mathrm{Re}(\alpha)=0} \overline{p^\#(\alpha)} \int_0^\infty p(y) \, W_\alpha(y) \, \frac{dy}{y^2} \, \frac{d\alpha}{\Gamma(\alpha)\Gamma(-\alpha)}$$

$$= \frac{1}{\pi i} \int\limits_{\mathrm{Re}(\alpha)=0} \left| p^\#(\alpha) \right|^2 \, \frac{d\alpha}{\Gamma(\alpha)\Gamma(-\alpha)}.$$

□

We are now in a position to obtain an asymptotic formula for the main term in the KTF as $T \to +\infty$ when the test functions p, q are chosen to be the same and equal to the $p_{T,R}$ function given in (9.4).

Applying the Plancherel formula of proposition 9.15, the main term $\mathcal{M}$ is given by

$$\mathcal{M} = m \int_0^\infty |p_{T,R}(y)|^2 \, \frac{dy}{y^2}$$

$$= \frac{2m}{\pi} \int_0^\infty e^{\frac{-2r^2}{T^2}} \cdot \frac{\left| \Gamma\left(\frac{2+R+2ir}{4}\right) \Gamma\left(\frac{2+R-2ir}{4}\right) \right|^2}{\Gamma(ir)\Gamma(-ir)} \, dr.$$

It follows from Stirling's asymptotic formula

$$|\Gamma(\sigma + ir)|^2 = 2\pi \cdot |r|^{2\sigma-1} \, e^{-\pi|r|}, \qquad \big(\sigma \text{ fixed}, \ |r| \to +\infty\big)$$

that

$$\mathcal{M} \sim \frac{2m}{\pi} \int_0^\infty e^{\frac{-2r^2}{T^2}} \cdot (1 + |r|)^{R+1} \, dr$$

$$\sim T^{R+2} \cdot \frac{2m}{\pi} \int_0^\infty e^{-2r^2} \cdot \big(T^{-1} + |r|\big)^{R+1} \, dr$$

$$\sim c_0 \cdot T^{R+2}$$

for some constant $c_0 > 0$. With more care one can prove

$$\mathcal{M} \sim c_0 T^{R+2} + c_1 T^{R+1} + c_2 T^R + \cdots$$

for fixed real constants $c_1, c_2, c_3, \ldots$

9.6. **Explicit KTF for $GL(2,\mathbb{R})$ with Error Term.**

Theorem 9.16. *Let $R > 4$ be fixed and let*

$$p^{\#}_{T,R}(\alpha) := e^{\alpha^2/T^2} \cdot \Gamma\left(\frac{2+R+2\alpha}{4}\right)\Gamma\left(\frac{2+R-2\alpha}{4}\right).$$

Let $\phi_j(z) = \sum\limits_{\ell\neq 0} A_j(\ell)\sqrt{y}\, K_{ir_j}(2\pi|\ell|y)e^{2\pi i\ell x}$, $(j = 1, 2, \dots)$ denote an orthonormal basis of Maass cusp forms where $\Delta\phi_i = \left(\frac{1}{4} + r_i^2\right)\phi_i$ for $i \geq 1$. Let

$$A(m,s) := \frac{2\pi^s \sigma_{1-2s}(m) m^{s-\frac{1}{2}}}{\Gamma(s)\zeta(2s)}$$

denote the m^{th} arithmetic Fourier coefficient of $E(z,s)$.

Then we have

$$\boxed{\begin{aligned} &\sqrt{mn}\,\sum_{j=1}^{\infty} A_j(m)A_j(n)\cdot\frac{\left|p^{\#}(ir_j)\right|^2}{\langle\phi_j,\ \phi_j\rangle} \\ &\qquad + \frac{\sqrt{mn}}{4\pi}\int\limits_{-\infty}^{\infty} A\left(m, 1/2+ir\right)\,\overline{A\left(n,1/2+ir\right)}\cdot\left|p^{\#}(ir)\right|^2\, dr \\ &\qquad = \delta_{m,n}\cdot\left(c_0T^{R+2} + c_1T^{R+1} + c_2T^R + \cdots\right) + \mathcal{O}\left(T^{\frac{3R}{4}+3}\right), \end{aligned}}$$

where $c_0 > 0$ and $c_1, c_2, \dots$ are fixed real constants.

9.7. **Bounding the Eisenstein term in the KTF.**

The Eisenstein term is given by

$$\begin{aligned} &\frac{\sqrt{mn}}{4\pi}\int\limits_{-\infty}^{\infty} A\left(m,1/2+ir\right)\,\overline{A\left(n,1/2+ir\right)}\cdot\left|p^{\#}(ir)\right|^2\, dr \\ &\qquad \ll \int\limits_{-\infty}^{\infty} \frac{e^{\frac{-2r^2}{T^2}}\left|\Gamma\left(\frac{2+R+2ir}{4}\right)\Gamma\left(\frac{2+R-2ir}{4}\right)\right|^2}{\left|\Gamma(1/2+ir)\zeta(1+2ir)\right|^2}\, dr \\ &\qquad \ll \int\limits_{0}^{T^{1+\epsilon}} \frac{(1+|r|)^R}{|\zeta(1+2ir)|}\, dr. \end{aligned}$$

Here we have used Stirling's asymptotic formula for the Gamma funciton. To complete the estimation we require the prime number theorem which says that $\zeta(1+2ir) \neq 0$ for any $r \in \mathbb{R}$. A more quantitative version is given by

$$\zeta(1+2ir)^{-1} \ll \log r.$$

We have thus shown that the Eisenstein term in the Kuznetsov trace formula is bounded by

$$\boxed{\frac{\sqrt{mn}}{4\pi} \int\limits_{-\infty}^{\infty} A\left(m, 1/2+ir\right) \overline{A\left(n, 1/2+ir\right)} \cdot \left|p^{\#}(ir)\right|^2 \, dr \;\ll\; T^{R+1+\epsilon}.}$$

10. **The Kuznetsov Trace Formula for** $GL(3,\mathbb{R})$

Let $\mathfrak{h}^3 := GL(3,\mathbb{R})/(O(3,\mathbb{R})\cdot\mathbb{R}^\times)$ and let U_3 be the subgroup of $GL(3)$ consisting of upper triangular unipotent matrices. Every coset representative in $\mathfrak{h}^3$ can be written as $g = xy$ with the specific parametrization

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \tag{10.1}$$

(where $x_{12}, x_{13}, x_{23} \in \mathbb{R}$, $y_1, y_2 > 0$) to assign coordinates on $\mathfrak{h}^3$.

Let $\mathcal{D}^3$ denote the invariant differential operators on $\mathfrak{h}^3$, namely all polynomials (with complex coefficients) in the variables

$$\left\{ \frac{\partial}{\partial x_{12}}, \quad \frac{\partial}{\partial x_{13}}, \quad \frac{\partial}{\partial x_{23}}, \quad \frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial y_2} \right\}$$

which are invariant under all $GL(3,\mathbb{R})$ transformations.

Definition 10.2. (Langlands parameters) *We define the Langlands parameters for $GL(3)$ to be complex numbers $\{\alpha_1,\alpha_2,\alpha_3\}$ which satisfy $\alpha_1+\alpha_2+\alpha_3 = 0$. By abuse of notation we often refer to the Langlands parameters as a vector $\alpha = (\alpha_1,\alpha_2,\alpha_3)$.*

Remark. These parameters are used to classify automorphic representations at the archimedean place.

An important role is played by the eigenfunctions $I(*,\alpha)$.

Definition 10.3. (The eigenfunction $I(*,\alpha)$) *Let $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ denote Langlands parameters. We define a power function on $xy \in \mathfrak{h}^3$ by*

$$\boxed{I(xy,\alpha) := y_1^{1-\alpha_3}\, y_2^{1+\alpha_1}.} \tag{10.4}$$

The function $I(*,\alpha)$ is an eigenfunction of $\mathcal{D}^3$.

Lemma 10.5. *The Laplace eigenvalue of $I(*,\alpha)$ is $1 - \frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{2}$.*

Proof. The Laplacian Δ on $\mathfrak{h}^3$ is the second order differential operator given by

$$\Delta = -y_1^2\frac{\partial^2}{\partial y_1^2} - y_2^2\frac{\partial^2}{\partial y_2^2} + y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} - y_1^2\left(x_{1,2}^2+y_2^2\right)\frac{\partial^2}{\partial x_{1,3}^2}$$
$$- y_1^2\frac{\partial^2}{\partial x_{2,3}^2} - y_2^2\frac{\partial^2}{\partial x_{1,2}^2} - 2y_1^2x_{1,2}\frac{\partial^2}{\partial x_{2,3}\partial x_{1,3}}.$$

The lemma can then be easily proved with a brute force computation. □

In order to develop the Kunetsov trace formula on $SL(3,\mathbb{R})$ we need to introduce the relevant Poincaré series. Recall from definition 6.1 that the Poincaré series for $SL(2,\mathbb{R})$ is given by

$$\sum_{\gamma\in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} p\big(m\cdot \mathrm{Im}(\gamma z)\big)\; e^{2\pi i m\cdot \mathrm{Re}(\gamma z)}.$$

We need analogues of the test function p and the additive character $e^{2\pi i m\cdot \mathrm{Re}(z)}$ when we move to $SL(3,\mathbb{R})$.

Definition 10.6. (Additive character) *Let $M=(m_1,m_2)\in\mathbb{Z}^2$. We shall define an additive character $\psi_M:\mathfrak{h}^3\to\mathbb{C}$ as follows. First of all for $xy\in\mathfrak{h}^3$ (with x,y as in (10.1)) we require $\psi_M(xy)=\psi_M(x)$. This guarantees that ψ_M is a character of the non-abelian unitary group $U_3(\mathbb{R})$.*

Further, for $x=\left(\begin{smallmatrix}1 & x_{1,2} & x_{1,3}\\ 0 & 1 & x_{2,3}\\ 0 & 0 & 1\end{smallmatrix}\right)\in U_3(\mathbb{R})$, we define

$$\boxed{\psi_M(x):=e^{2\pi i\left(m_1\,x_{1,2}+m_2\,x_{2,3}\right)},}$$

which satisfies $\psi_M(x\cdot x')=\psi_M(x)\psi_M(x')$ for all $x,x'\in U_3(\mathbb{R})$.

Definition 10.7. (Test function p) *Let $p:\mathfrak{h}^3\to\mathbb{C}$ be a smooth function satisfying $p(xy)=p(y)$ for all $x\in U_3(\mathbb{R})$ and all $y=\left(\begin{smallmatrix}y_1y_2 & 0 & 0\\ 0 & y_1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$ with $y_1,y_2>0$. We further assume that $p(y)$ satisfies*

$$p(y)\ll y_1^{B_1}y_2^{B_2},$$

for $B_i=\begin{cases}1+b & \text{if } 0<y_i\le 1\\ -1-b & \text{if } 1<y_i\end{cases}$ for sufficiently large $b>1$ and $i=1,2$.

We may now define the Poincaré series.

Definition 10.8. (Poincaré Series) *Let $M=(m_1,m_2)\in\mathbb{Z}^2$ with $m_1m_2\neq 0$. Let $p:\mathfrak{h}^3\to\mathbb{C}$ be a smooth test function as in definition 10.7 and define $M^*:=\left(\begin{smallmatrix}m_1m_2 & 0 & 0\\ 0 & m_1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$. Then for $g\in\mathfrak{h}^3$, we define the Poincaré series $\mathcal{P}_M(*,p):\mathfrak{h}^3\to\mathbb{C}$ by*

$$\boxed{\mathcal{P}_M(g,p):=\sum_{\gamma\in U_3(\mathbb{Z})\backslash SL(3,\mathbb{Z})} p\big(M^*\cdot\gamma g\big)\;\psi_M(\gamma g).}$$

Recall that a locally compact group G is termed Hausdorff provided every pair of distinct elements in G lie in disjoint open sets. The general linear group $GL(n,\mathbb{R})$ is a locally compact Hausdorff topological group.

Theorem 10.9. *Let G be a locally compact Hausdorff topological group, and let H be a compact closed subgroup of G. Let μ be a Haar measure on G, and let ν be a Haar measure on H, normalized so that $\int_H d\nu(h) = 1$. Then there exists a unique (up to scalar multiple) quotient measure $\tilde{\mu}$ on G/H. Furthermore*

$$\int_G f(g)\,d\mu(g) = \int_{G/H} \left(\int_H f(gh)\,d\nu(h) \right) d\tilde{\mu}(gH),$$

for all integrable functions $f : G \to \mathbb{C}$.

Proof. See Halmos ????? $\square$

Definition 10.10. (Left invariant measure on $\mathfrak{h}^3$) *Let $g = xy \in \mathfrak{h}^3$ with*

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}$$

The left invariant measure $d^\times g$ on $\mathfrak{h}^3$ is given explicitly by

$$d^\times g \;=\; dxdy \;=\; dx_{1,2}dx_{1,3}dx_{2,3}\,\frac{dy_1 dy_2}{(y_1 y_2)^2}.$$

Here $d^\times g$ satisfies $d^\times(\alpha g) = d^\times g$ for all $\alpha \in GL(3, \mathbb{R})$.

Proof. The group $GL(3, \mathbb{R})$ is generated by $U_3(\mathbb{R})$, diagonal matrices in $GL(3, \mathbb{R})$ and the The Weyl group of $GL(3, \mathbb{R})$ is given by

$$W_3 = \left\{ \left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&0\\0&0&1\\1&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&1\\1&0&0\\0&1&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right) \right\}.$$

It is enough to show that $d^\times g$ is invariant under these three subgroups.

It is clear that $d^\times(xg) = d^\times g$ if x is an upper triangular matrix with ones on the diagonal. This is because the measures $dx_{i,j}$ (with $1 \le i < j \le 3$) are all invariant under translation. One also checks that $d^\times(ag) = d^\times g$ for all diagonal matrices $a \in GL(3, \mathbb{R})$.

It remains to check the invariance of $d^\times g$ under the Weyl group W_3. Now, if $w \in W_3$ and

$$a = \begin{pmatrix} a_3 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \in GL(3, \mathbb{R})$$

is a diagonal matrix, then waw^{-1} is again a diagonal matrix whose diagonal entries are a permutation of $\{a_1, a_2, a_3\}$. The Weyl group is generated by the two transpositions ω_1, ω_2 which interchange (transpose) a_1 and a_{i+1} when a is conjugated by ω_i. With a brute force calculation one checks that $d^\times(\omega_i g) = d^\times g$ for $i = 1, 2$.

□

With this measure, one may show that the volume is

$$\int\limits_{SL(3,\mathbb{Z})\backslash\mathfrak{h}^3} d^\times g \;=\; \frac{3\zeta(3)}{2\pi}.$$

For $M = (m_1, m_2)$ and $N = (n_1, n_2)$ let $\mathcal{P}_M(g,p)$, $\mathcal{Q}_N(g,q)$ (with $m_1m_2n_1n_2 \neq 0$) be two Poincaré series with smooth functions $p, q : \mathbb{R} \to \mathbb{C}$ as in definition 10.8. The Kuznetsov trace formula is obtained by computing the Petersson inner product

$$\Big\langle \mathcal{P}_M(*,\,p),\;\mathcal{Q}_N(*,\,q)\Big\rangle = \int\limits_{SL(3,\mathbb{Z})\backslash\mathfrak{h}^3} \mathcal{P}_m(g,p)\,\overline{\mathcal{Q}_n(g,q)}\;\,d^\times g$$

in two different ways.

- **First way, Spectral Side:** *Take the spectral expansion of* $\mathcal{P}_M(*,p)$ *and unravel* $\mathcal{Q}_N(*,q)$ *in the inner product.*
- **Second way, Geometric Side:** *Compute the Fourier expansion of* $\mathcal{P}_M(*,p)$ *by double coset decomposition and unravel* $\mathcal{Q}_N(*,q)$.

10.1. Whittaker functions and Maass forms on $\mathfrak{h}^3$.

We now define the canonically normalized Whittaker function on $GL(3,\mathbb{R})$ which appears in the Fourier expansion of automorphic forms for $SL(3,\mathbb{Z})$.

Definition 10.11. *Let* $w_3 := \left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0\end{smallmatrix}\right)$ *and* $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{C}^3$. *Then for* $g \in GL(3,\mathbb{R})$, *the function*

$$W_\alpha^{\pm 1}(g) := \prod_{1\le j<k\le 3} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \cdot \int\limits_{U_3(\mathbb{R})} I(w_3ug,\alpha)\,\overline{\psi_{1,1,\pm 1}(u)}\;du$$

defined as an absolutely convergent integral for $\mathrm{Re}(\alpha_1-\alpha_2) > 0$ *and* $\mathrm{Re}(\alpha_2-\alpha_3) > 0$ *extends to a holomorphic function on the set of all* $\big\{\alpha \in \mathbb{C}^3 \;\big|\; \alpha_1+\alpha_2+\alpha_3 = 0\big\}$.

Remark. The integral is Jacquet's integral representation. The product of Gamma factors is included so that $W_\alpha^\pm$ is invariant under all permutations of $\alpha_1,\alpha_2,\alpha_3$. Moreover, even though Jacquet's integral often vanishes identically as a function of α, this normalization never does. If g is a diagonal matrix in $GL(n,\mathbb{R})$ then the value of $W_\alpha^\pm(g)$ is independent of sign, so we drop the $\pm$. We also drop the $\pm$ if the sign is $+1$.

Jacquet's Whittaker function is characterized (up to scalar multiples depending on α) by the following properties:

- $\delta W^{\pm}_{\alpha}(g) = \lambda_{\delta}(\alpha) \cdot W^{\pm}_{\alpha}(g), \qquad \big(\text{for all } \delta \in \mathcal{D}^3, \quad g \in GL(3,\mathbb{R})\big),$
- $W^{\pm}_{\alpha}(ugzk) = \psi(u) W^{\pm}_{\alpha}(g), \qquad \big(\text{for all } u \in U_3(\mathbb{R}),\ g \in GL(3,\mathbb{R}),$
  $z \in \mathbb{R}^{\star},\ k \in O_3(\mathbb{R})\big),$
- $(y_1y_2y_3)^N W^{\pm}_{\alpha}\big(\mathrm{diag}(y)\big) = \mathcal{O}(1), \quad \big(\text{for any } N > 0 \text{ and } y_i \geq 1\big),$
- $W^{\pm}_{\alpha}$ is entire in α,
- $W^{\pm}_{\alpha} = W^{\pm}_{\alpha'}$ where α' is any permutation of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Definition 10.12. (Maass cusp form for $SL(3,\mathbb{Z})$) *A Maass cusp form $\phi : SL(3,\mathbb{Z})\backslash\mathfrak{h}^3 \to \mathbb{C}$ for $SL(3,\mathbb{Z})$ is a joint eigenfunction of $\mathcal{D}^3$ which satisfies a bound of the form*

$$|\phi(xy)| \ll_N (y_1y_2y_3)^{-N}$$

in the range $y_1, y_2, y_3 \geq 1$, for any $N > 0$. If the eigenvalues of ϕ agree with those of $I(*, \alpha)$ then we term α the Langlands parameters of ϕ. The Maass form is termed a Hecke eigenform if it is a simultaneous eigenfunction of all the Hecke operators.

Theorem 10.13. (Fourier expansion of Maass forms) *Let $\phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form for $SL(3,\mathbb{Z})$ with Langlands parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$. Then for $g \in GL(3,\mathbb{R})$*

$$\phi(g) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A_{\phi}(M)}{m_1|m_2|} W^{\mathrm{sgn}(m_2)}_{\alpha}\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right),$$

where $M = (m_1, m_2)$ and $M^* = \begin{pmatrix} |m_1m_2| & & \\ & |m_1| & \\ & & 1 \end{pmatrix}$ with $m_1m_2 \neq 0$..

11. The $SL(3,\mathbb{R})$ Kuznetsov Trace Formula (Spectral Side)

By the Selberg spectral decomposition for $GL(3,\mathbb{R})$ first proved by Langlands ??? we have

$$\boxed{\mathcal{P}_M(g,p) = \sum_{j=0}^{\infty} \Big\langle \mathcal{P}_M(*,p),\ \phi_j \Big\rangle \frac{\phi_j(z)}{\langle \phi_j, \phi_j \rangle} + \Big\{\text{Eisenstein contribution}\Big\}.}$$

Lemma 11.1. (Inner product of $\mathcal{P}_M$ with a Maass form ϕ_j) *Let $M = (m_1, m_2)$ with $m_1 m_2 \neq 0$. Then*

$$\boxed{\Big\langle \mathcal{P}_M(*, p),\ \phi_j \Big\rangle \ = \ \begin{cases} 0 & \text{if } j = 0, \\ \frac{\overline{A_j(M)}}{m_1 m_2} \int\limits_0^\infty \int\limits_0^\infty \overline{W_{\alpha^{(j)}}\,(M^* y)} \cdot p(M^* y)\ \frac{dy_1 dy_2}{(y_1 y_2)^3} & \text{if } j > 0. \end{cases}}$$

Proof. Let $\Gamma = SL(3, \mathbb{Z})$. We compute

$$\begin{aligned}
\Big\langle \mathcal{P}_M(*, p),\ \phi_j \Big\rangle &= \int\limits_{\Gamma\backslash\mathfrak{h}^3} \sum_{\gamma \in U_3(\mathbb{Z})\backslash\Gamma} p\big(M^* \cdot \gamma g\big)\ \psi_M(\gamma g) \cdot \overline{\phi_j(g)}\ d^\times g \\
&= \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} p\big(M^* y\big)\, \psi_M(x)\, \overline{\phi_j(g)}\ d^\times g \\
&= \int\limits_0^\infty \int\limits_0^\infty \underbrace{\left[\ \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \psi_M(x) \overline{\phi_j(xy)}\ dx \right]}_{\text{This integral } = 0 \text{ if } j = 0.} \cdot\, p(M^* y)\ \frac{dy_1 dy_2}{(y_1 y_2)^3} \\
&= \frac{\overline{A_j(M)}}{m_1 m_2} \underbrace{\int\limits_0^\infty \int\limits_0^\infty \overline{W_{\alpha^{(j)}}\,(M^* y)} \cdot p(M^* y)\ \frac{dy_1 dy_2}{(y_1 y_2)^3}}_{\text{Here we assume } j \neq 0.} \\
&= m_1 m_2\, \overline{A_j(M)} \cdot p^{\#}\left(\alpha^{(j)}\right)
\end{aligned}$$

□

$$\begin{aligned}
\Big\langle \mathcal{P}_M(*,\, p),\ \mathcal{Q}_N(*,\, q) \Big\rangle &= \sum_{j=1}^\infty \frac{\big\langle \mathcal{P}_M(*, p),\ \phi_j \big\rangle \big\langle \phi_j,\ \mathcal{Q}_N(*, q) \big\rangle}{\big\langle \phi_j, \phi_j \big\rangle} \\
&\qquad + \Big\{ \text{Eisenstein Contribution} \Big\} \\
&= m_1 m_2 n_1 n_2 \sum_{j=1}^\infty \overline{A_j(M)} A_j(N)\ \frac{p^{\#}\left(\alpha^{(j)}\right) \overline{q^{\#}\left(\alpha^{(j)}\right)}}{\big\langle \phi_j, \phi_j \big\rangle} \\
&\qquad + \Big\{ \text{Eisenstein Contribution} \Big\}.
\end{aligned}$$

11.1. **Eisenstein contribution.**

There are two types of Eisenstein series for $GL(3,\mathbb{R})$, corresponding to whether one takes a minimal or maximal parabolic subgroup. Let

$$\mathcal{B} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \text{and} \qquad \mathcal{P}_{2,1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

denote the Borel and a specific maximal parabolic, respectively. Any parabolic subgroup of $GL(3)$ is conjugate to a standard parabolic, (i.e., one containing $\mathcal{B}$), and the only other standard parabolics of G are $\mathcal{P}_{1,2}$ (defined analogously to $\mathcal{P}_{2,1}$, but instead with zero entries in the bottom two entries of the first column) and the full group G itself.

Definition 11.2. (Borel Eisenstein series) *The Borel Eisenstein series for* $G = GL(3,\mathbb{R})$ *is defined for* $\mathrm{Re}(\alpha_1 - \alpha_2)$ *and* $\mathrm{Re}(\alpha_2 - \alpha_3) > 1$ *by the absolutely convergent sum*

$$\boxed{E_{\mathcal{B}}(g,\alpha) := \sum_{\gamma \in U_3(\mathbb{Z})\backslash SL(3,\mathbb{Z})} I(\gamma g, \alpha), \qquad (g \in \mathfrak{h}^3),}$$

and for general α *by meromorphic continuation.*

Definition 11.3. (Induced Maass form Φ associated to $\mathcal{P}_{2,1}$) *Let*

$$\mathcal{P}_{2,1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}}_{N^{\mathcal{P}_{2,1}}} \cdot \underbrace{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}}_{M^{\mathcal{P}_{2,1}}}$$

Let $\phi : GL(2,\mathbb{R}) \to \mathbb{C}$ *be a Maass cusp form left-invariant under* $SL(2,\mathbb{Z})$. *Let* $K = O(3,\mathbb{R})$. *The Maass form* Φ *is then defined on* $GL(3,\mathbb{R}) = \mathcal{P}_{2,1}(\mathbb{R})K$ *by the formula*

$$\boxed{\Phi(nmk) := \phi\Big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \Big), \qquad (n \in N^{\mathcal{P}_{2,1}},\ m \in M^{\mathcal{P}_{2,1}},\ k \in K),}$$

where $m \in M^{\mathcal{P}_{2,1}}$ *has the form* $m = \left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & * \end{smallmatrix}\right)$.

Definition 11.4. (Langlands Eisenstein series twisted by Φ) *Consider the parabolic subgroup $\mathcal{P}_{2,1}$ with induced Maass form Φ as in the above definition. Let $s = (1/2 + s_1, -1 - 2s_1)$ with $s_1 \in \mathbb{C}$ and let $\alpha = (s_1 - 1/2, s_1 + 1/2, -2s_1)$. The Langlands Eisenstein series determined by this data is defined by*

$$E_{\mathcal{P}_{2,1},\Phi}(g,\alpha) := \sum_{\gamma \in (\mathcal{P}_{2,1}\cap\Gamma)\backslash\Gamma} \Phi(\gamma g)\cdot I(\gamma g,\alpha) \tag{11.5}$$

as an absolutely convergent sum for $\mathrm{Re}(s_1)$ *sufficiently large, and it extends to $s_1 \in \mathbb{C}$ by meromorphic continuation.*

In particular, if E is any one of the two Eisenstein series defined above, then for $M = (m_1, m_2)$ as above, we have

$$\int\limits_{U_3(\mathbb{Z})\backslash U_3(\mathbb{R})} E(ug,s)\overline{\psi_M(u)}\,du = \frac{A_E(M,\alpha)}{|m_1m_2|}W_\alpha(M^*g),$$

where W_α is a Whittaker function and $A_E(M,\alpha)$ is termed the M^{th} arithmetic Fourier coefficient of E. The *first coefficient* of E is defined to be $A_E((1,1),\alpha)$. Furthermore, we have

$$A_E(M,\alpha) = A_E((1,1),\alpha)\cdot\lambda_E(M,\alpha).$$

where $\lambda_E(M,\alpha)$ is the M^{th} Hecke eigenvalue of E and $\lambda_E((1,1),\alpha) = 1$.

Theorem 11.6.

(1) *Let $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{C}^3$ with $\alpha_1+\alpha_2+\alpha_3 = 0$. Then the first coefficient $A_{E_{\mathcal{B}}}((1,1),\alpha)$ of the $GL(3,\mathbb{R})$ Borel Eisenstein series is equal (up to a nonzero constant) to*

$$\Big(\zeta^*\big(1+\alpha_1-\alpha_2\big)\zeta^*\big(1+\alpha_2-\alpha_3\big)\zeta^*\big(1+\alpha_1-\alpha_3\big)\Big)^{-1}. \tag{11.7}$$

(2) *Let $s = \big(\frac{1}{2}+s_1,\ -1-2s_1\big)$, $\alpha = (s_1-1/2, s_1+1/2, -2s_1)$, and let ϕ be a Maass form for $SL(2,\mathbb{Z})$ with Petersson norm one. Then the first coefficient $A_{E_{\mathcal{P}_{2,1},\Phi}}((1,1),\alpha)$ of the $GL(3,\mathbb{R})$ Eisenstein series $E_{\mathcal{P}_{2,1},\Phi}(g,\alpha)$ induced from ϕ is equal to*

$$\Big(L^*(1,\mathrm{Ad}\ \phi)^{1/2}\cdot L^*(\phi, 1+3s_1)\Big)^{-1} \tag{11.8}$$

times an explicit non zero constant which is independent of ϕ.

We can now state the Eisenstein contribution to the Langlands spectral decomposition of the Poincaré series.

Proposition 11.9. *Let $\phi_j, j = 1, 2, \ldots$ denote an orthonormal basis of Maass cusp forms for $SL(3,\mathbb{Z})$. Then the Eisenstein contribution in the spectral decomposition of $\mathcal{P}_M(*, p)$ is given by*

$$\frac{1}{(4\pi i)^2} \int\limits_{\mathrm{Re}(\alpha_1)=\frac{1}{2}} \int\limits_{\mathrm{Re}(\alpha_2)=\frac{1}{2}} \Big\langle \mathcal{P}_M(*, p), E_{\mathcal{B}}(*, \alpha) \Big\rangle E_{\mathcal{B}}(g, \alpha)\, d\alpha_1 d\alpha_2$$
$$+ \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int\limits_{\mathrm{Re}(\alpha)=\frac{1}{2}} \Big\langle \mathcal{P}_M(*, p), E_j(*, \alpha) \Big\rangle E_j(g, \alpha)\, d\alpha.$$

We can now state the spectral side of the Kuznetsov trace formula for $GL(3,\mathbb{R})$.

Proposition 11.10. (Kuznetsov trace formula, Spectral Side) *Let $\mathcal{P}_M(z, p)$, $\mathcal{Q}_N(z, q)$ be Poincaré series. Define the Bessel transforms:*

$$p^{\#}(\alpha) := \int_0^\infty \int_0^\infty \overline{W_\alpha(y)} \cdot p(y)\, \frac{dy_1 dy_2}{(y_1 y_2)^3}$$
$$q^{\#}(\alpha) := \int_0^\infty \int_0^\infty \overline{W_\alpha(y)} \cdot q(y)\, \frac{dy_1 dy_2}{(y_1 y_2)^3}.$$

Then the spectral side of the Kuznetsov trace formula is

$$\boxed{\begin{aligned} \Big\langle \mathcal{P}_M(*,\, p),\ \mathcal{Q}_N(*,\, q) \Big\rangle &= m_1 m_2 n_1 n_2 \sum_{j=1}^{\infty} \overline{A_j(M)} A_j(N)\, \frac{p^{\#}\big(\alpha^{(j)}\big)\, \overline{q^{\#}\left(\alpha^{(j)}\right)}}{\big\langle \phi_j, \phi_j \big\rangle} \\ &+ \frac{1}{(4\pi i)^2} \int\limits_{\mathrm{Re}(\alpha_1)=\frac{1}{2}} \int\limits_{\mathrm{Re}(\alpha_2)=\frac{1}{2}} \frac{p^{\#}(\alpha)\, \overline{q^{\#}(\alpha)}}{\zeta^*\big(1+\alpha_1-\alpha_2\big)\zeta^*\big(1+\alpha_2-\alpha_3\big)\zeta^*\big(1+\alpha_1-\alpha_3\big)}\, d\alpha_1 d\alpha_2 \\ &+ \tfrac{1}{2\pi i} \sum_{j=0}^{\infty} \int\limits_{\mathrm{Re}(\alpha)=\frac{1}{2}} \frac{p^{\#}(\alpha)\, \overline{q^{\#}(\alpha)}}{L^*(1, \mathrm{Ad}\ \phi)^{1/2} \cdot L^*(\phi, 1+3s_1)}\, d\alpha. \end{aligned}}$$

Acknowledgments:

I would like to thank Kevin Kwan and David Soudry for carefully reading these notes and spotting many typos and errors which are now corrected.

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