http://dx.doi.org/10.1090/pspum/066.1/1703748

Proceedings of Symposia in Pure Mathematics
Volume **66.1**, 1999

Zeta Functions Formed With Modular Symbols

Dorian Goldfeld

Dedicated to Goro Shimura

ABSTRACT. A new class of automorphic functions is introduced where the automorphic relation involves a shift by a modular symbol. In the special case of Eisenstein series, this leads to new zeta functions whose Dirichlet coefficients are themselves modular symbols. On the basis of this new theory a number of conjectures are made concerning the distribution of moments of modular symbols.

1. Introduction and statement of results

For $N = 1, 2, 3\ldots$, let $\Gamma_0(N)$ denote the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, and $c \equiv 0 \pmod N$. Fix an even Dirichlet character χ to the modulus N and for a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ define $\chi(\gamma) = \chi(d)$ and $j(\gamma, z) = cz + d$. Let

$$\mathfrak{h} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

denote the upper half plane and define $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ to be the extended upper half–plane including the cusps. Consider the set of automorphic forms $\mathrm{M}_k(N, \chi)$ of weight k and character χ. Thus, if $f \in \mathrm{M}_k(N, \chi)$, then f is a function (not necessarily holomorphic in z) from h^* to $\mathbb{C}$ satisfying

$$f(\gamma z) = \chi(\gamma)\, j(\gamma, z)^k\, f(z), \quad \gamma \in \Gamma_0(N),\ z \in h^*,$$

which has polynomial growth at the cusps. This last condition simply means that if $s \in \mathbb{Q}$ is a cusp of $\Gamma_0(N)$ and $\rho \in \mathrm{SL}_2(\mathbb{R})$ satisfies $\rho s = \infty$, then $f(\rho^{-1}z)\cdot j(\rho^{-1}, z)^{-k}$ has a polynomial growth in $y = \mathrm{Im}(z)$ as $y \to \infty$. Similarly, let $\mathrm{S}_k(N)$ denote the space of holomorphic cusp forms in $\mathrm{M}_k(N, \chi_0)$ where χ_0 is the trivial character. There is a bilinear pairing

$$\mathrm{S}_2(N) \times \mathrm{H}_1(X_0(N), \mathbb{C}) \longrightarrow \mathbb{C}$$

1991 *Mathematics Subject Classification.* 11F67.

Key words and phrases. Modular symbols, automorphic forms, Eisenstein series, Fourier expansions.

Supported in part by NSF grant DMS9003907.

given by

$$(f,\gamma) \mapsto \langle\gamma, f\rangle = -2\pi i \int_\gamma f(z)\,dz.$$

The modular symbols $\langle\gamma, f\rangle$ have played a crucial role in the study of cusp forms and elliptic curves (see, [**C**], [**M**], [**MSD**]). Very little is known at present on the distribution properties of the values $\langle\gamma, f\rangle$ as γ ranges over $\Gamma_0(N)$. Using results of Shimura [**Sh**], one may write

$$\langle\gamma, f\rangle \;=\; \sum_{i=1}^{2n} m_i\Omega_i$$

where Ω_i are fixed complex numbers (periods of $J_0(N)$), $m_i \in \mathbb{Z}$, and n is the degree (over $\mathbb{Q}$) of the smallest field containing the Fourier coefficients of f. We have conjectured [**G89**] that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$ with $|c| \le N^2$, then

$$m_i \ll N^\kappa \quad (\text{for } i = 1,2,3,\dots 2n,\ N \to \infty)$$

for some fixed $\kappa > 0$. In the special case that f is associated to an elliptic curve over $\mathbb{Q}$, the above conjecture is equivalent (see [**G89**]) to the well known conjecture of Szpiro [**Sz**],

$$D \ll N^C \quad (\text{for some fixed constant } C > 6)$$

relating the conductor N and the discriminant D of the elliptic curve.

In relation to the problem of obtaining distribution theorems on the values $\langle\gamma, f\rangle$, it is natural to introduce the Epstein type zeta function

$$\sum_{\substack{c\equiv 0 \pmod N\\ (c,d)=1}} \frac{\langle\gamma, f\rangle}{(c^2+d^2)^s}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. This is a special case (set $z = i,\ \chi = \chi_0 =$ trivial character) of the generalized Eisenstein series

$$E^*(z,s,\chi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(\gamma)\,\langle\gamma, f\rangle\,\mathrm{Im}(\gamma z)^s,$$

where Γ_∞ denotes the group $\left\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right) \;\middle|\; n\in\mathbb{Z}\right\}$ which stabilizes infinity. It is easy to see that $E^*(z,s,\chi)$ satisfies the automorphic relation

$$E^*(\gamma z,s,\chi) = \bar\chi(\gamma)E^*(z,s,\chi) - \bar\chi(\gamma)\langle\gamma, f\rangle E(z,s,\chi)$$

where $E(z,s,\chi) = \sum\limits_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(\gamma)\mathrm{Im}(\gamma z)^s$ is the classical non–holomorphic weight zero Eisenstein series for $\Gamma_0(N)$.

Set $E^*(z,s) = E(z,s,\chi_0)$, $E(z,s) = E(z,s,\chi_0)$. Since $E(z,s)$ has a simple pole at $s=1$, the automorphic relation for E^* implies that $E^*(z,s)$ must have a pole at $s=1$. Let $R(z)$ denote the residue at $s=1$. Since $E^*(z,s)$ is an eigenfunction of the Laplacian it immediately follows that $R(z)$ must be harmonic. Further, $R(z)$ must satisfy the automorphic relation

$$R(\gamma z) = R(z) - \langle\gamma, f\rangle r_N$$

where

$$r_N = \frac{3}{\pi}\prod_{p\mid N}\left(1+\frac{1}{p}\right)^{-1}$$

is the residue at $s = 1$ of $E(z, s)$. In this paper, we compute the Fourier expansion of E^* at infinity and show that the constant term vanishes. It follows from this that

$$R(z) = r_N \cdot F(z)$$

where $F(z) = 2\pi i \int_z^{i\infty} f(w)\, dw$ is the anti-derivative of f.

The above calculations suggest that for arbitrary $M > 0$,

$$S_N(X) = S_N(X; f) = \sum_{\substack{c^2M^2+d^2 \le X \\ c\equiv 0 \pmod N}} \langle \gamma, f\rangle \sim R(iM) \cdot X$$

for $X \to \infty$. We have made some preliminary computations with the software package Mathematica in the special case $M = 1$. For $N = 11$ there is a Hecke cusp form of weight two associated to the elliptic curve $y^2 - y = x^3 - x^2 - 10x - 20$ of conductor 11 with periods $\Omega_1 = 1.269209304...$ and $\Omega_2 = (1.458816617...) \cdot i$. In this case, we have computed:

$$\begin{aligned} S_{11}(5000) &= 8\Omega_1, \\ S_{11}(20,000) &= 6\Omega_1, \\ S_{11}(100,000) &= 14\Omega_1. \end{aligned}$$

Similarly, for $N = 67$, the elliptic curve $y^2 + y = x^3 + x^2 - 12x - 21$ has periods given by $\Omega_1 = 1.273770037...$ and $\Omega_2 = (3.029968401...) \cdot i$. As before, we have computed:

$$\begin{aligned} S_{67}(5000) &= 10\Omega_1, \\ S_{67}(20,000) &= -25\Omega_1, \\ S_{67}(100,000) &= 39\Omega_1, \\ S_{67}(500,000) &= 35\Omega_1. \end{aligned}$$

In this paper, we compute the Fourier expansion of E^*. Further properties of E^*, such as the analytic continuation and functional equation, should be obtainable by the method of Selberg (see [**S56**]). It would also be of great interest to study even more general Eisenstein series

$$E_p^*(z, s, \chi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(\gamma) \cdot p\left(\langle\gamma, f\rangle, \overline{\langle\gamma, f\rangle}\right) \operatorname{Im}(\gamma z)^s,$$

where $p(x, y)$ is an arbitrary polynomial in two variables x, y with complex coefficients. For example, the constant term of the Fourier expansion of the Eisenstein series

$$E^{**}(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} |\langle\gamma, f\rangle|^2 \operatorname{Im}(\gamma z)^s$$

has a simple pole at $s = 1$. Let $R^*(z)$ denote the residue of E^{**} at $s = 1$. Then R^* satisfies the automorphic relation

$$R^*(\gamma z) = R^*(z) - \overline{\langle\gamma, f\rangle}\, R(z) - \langle\gamma, f\rangle\, \overline{R(z)} + |\langle\gamma, f\rangle|^2\, r_N.$$

This suggests the asymptotic formula

$$\sum_{c^2+d^2\le X} \left| \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle \right|^2 \sim R^*(i) \cdot X$$

for $X \to \infty$. The precise determination of $R^*(z)$ is a problem of considerable interest.

The meromorphic continuation and functional equations of other such generalized Eisenstein series will provide important information on the higher moment distributions of modular symbols. Since special values of L-functions associated to cusp forms of weight two can be expressed in terms of modular symbols, the analytic study of the proposed generalized Eisenstein series should ultimately give new information on the growth properties and value distribution of special values of L-functions.

The generalized Eisenstein series E^* is a special case of a new class of automorphic functions where the automorphic relation involves a shift by a modular symbol multiplied by a classical automorphic form. We give a rapid introduction to this new theory by specializing to the case of the congruence subgroups $\Gamma_0(N)$. The further study of this class of automorphic functions opens a new horizon in the analytic theory of modular symbols.

Acknowledgment: The author would like to thank Cormac O'Sullivan for several helpful conversations.

2. A new class of automorphic functions

Let M, N be positive integers with $M|N$, and let χ be Dirichlet character modulo N. Fix $f \in \mathrm{S}_2(M)$ and $G \in \mathrm{M}_k(N,\chi)$. We do not assume that G is holomorphic. We shall consider the $\mathbb{C}$–vector space of automorphic forms $G^* : \mathfrak{h}^* \to C$ which have polynomial growth at the cusps and satisfy the relation

$$G^*(\gamma z) = j(\gamma, z)^k \bigg(\chi(\gamma)\, G^*(z) + c \cdot \chi(\gamma) \langle \gamma, f\rangle\, G(z) \bigg)$$

for some fixed $c \in \mathbb{C}$, all $z \in \mathfrak{h}^*$ and all $\gamma \in \Gamma_0(N)$. Let $\mathrm{M}_k^*(f,G)$ denote the set of automorphic forms as defined above. It is clear that if $G_1^*, G_2^* \in \mathrm{M}_k^*(f,G)$, then there exist constants $c_1, c_2 \in \mathbb{C}$ (not both zero) such that

$$c_1 G_1^*(z) - c_2 G_2^*(z) \in \mathrm{M}_k(N,\chi).$$

The existence of automorphic forms in $\mathrm{M}_k^*(f,G)$ can be shown by explicitly constructing them as Poincaré series. We first consider the case where G is a holomorphic modular form of integer weight $k > 2$. Define the Poincaré series

$$P_n(z,k,\chi,f) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma) \langle \gamma, f \rangle j(\gamma, z)^{-k} e^{2\pi i n \gamma(z)}.$$

The absolute convergence of the above series is an immediate consequence of the following lemma:

LEMMA 1. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. *Then* $\langle \gamma, f \rangle \ll |c|^{1/2+\epsilon}$ *where the implied constant depends only on* ϵ *and* f.

PROOF. Let

$$f(z) = \sum_{n=1}^{\infty} a_n\, e^{2\pi i n z}$$

be the Fourier expansion of f where $a_n \ll n^{1/2+\epsilon}$. It follows by integration that

$$\begin{aligned}\langle\gamma, f\rangle &= -2\pi i \int_\tau^{\gamma\tau} f(z)\,dz \\ &= \sum_{n=1}^\infty \frac{a_n e^{2\pi i n\tau}}{n} - \sum_{n=1}^\infty \frac{a_n e^{2\pi i n\gamma\tau}}{n}\end{aligned}$$

is independent of τ. If $c = 0$ then $\langle\gamma, f\rangle = 0$. Otherwise, choose $\tau = -d/c + i/|c|$ so that $\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau) = 1/|c|$. Then

$$\langle\gamma, f\rangle \ \ll \sum_{n \ll |c|^{1+\epsilon}} \frac{1}{\sqrt{n}} \ \ll \ |c|^{1/2+\epsilon}.$$

This completes the proof of the lemma. □

By construction, we have

$$P_n(\gamma z, k, \chi, f) = j(\gamma, z)^k \Big(\chi(\gamma)\, P_n(z, k, \chi, f) - \chi(\gamma)\langle\gamma, f\rangle P_n(z, k, \chi)\Big)$$

where

$$P_n(z, k, \chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma) j(\gamma, z)^{-k} e^{2\pi i n\gamma(z)}.$$

Since every holomorphic modular form $G \in \mathrm{M}_k(N, \chi)$ can be constructed as a linear combination $\sum\limits_n c_n P_n(z, \chi)$ with $c_n \in \mathbb{C}$, the existence of automorphic forms in $\mathrm{M}_k^*(f, G)$ is established. The same argument extends to nonholomorphic forms and forms of weight $k < 2$ if one uses the nonholomorphic Poincaré series

$$P_n(z, k, s, \chi, f) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma)\langle\gamma, f\rangle j(\gamma, z)^{-k}\, \mathrm{Im}(\gamma z)^s\, e^{2\pi i n\gamma(z)}$$

instead of $P_n(z, k, \chi, f)$ as above.

An alternative method to construct $G^* \in \mathrm{M}_k^*(f, G)$ is to introduce the anti-derivative

$$F(z) = 2\pi i \int_z^{i\infty} f(w)\,dw = \sum_{n=1}^\infty \frac{a_n}{n} e^{2\pi i n z}$$

which satisfies the automorphic relation

$$F(\gamma z) = F(z) - \langle\gamma, f\rangle$$

for all $\gamma \in \Gamma_0(N)$. Then it is easily seen that if $G \in \mathrm{M}_k(N, \chi)$, then

$$G^*(z) = G(z) \cdot F(z) \in \ \mathrm{M}_k^*(f, G).$$

We shall say $G^* \in \mathrm{M}_k^*(f, G)$ is a holomorphic cusp form if for each cusp $s \in \mathbb{Q}$ and $\rho \in \mathrm{SL}(2, \mathbb{R})$ satisfying $\rho s = \infty$ we have $G^*(\rho^{-1}z) \cdot j(\rho^{-1}, z)^{-k}$ vanishes at $z = i\infty$. Let $\mathrm{S}^*(f, G)$ denote the $\mathbb{C}$–vector space of holomorphic cusp forms in $\mathrm{M}^*(f, G)$.

An immediate consequence of the above is the following proposition.

PROPOSITION 2. *Let M, N be positive integers with $M|N$, and let χ be a Dirichlet character modulo N. Fix $f \in S_2(M)$ and $G \in M_k(N, \chi)$. Then we have*

$$\mathrm{Dim}\left(\mathrm{S}^*(f, G)\right) = 1 + \mathrm{Dim}\left(S_k(N, \chi)\right).$$

It is also possible to define the space of holomorphic Eisenstein automorphic forms in $\mathrm{M}^*(f, G)$. Again, the dimensions exceed the dimensions of the classical holomorphic Eisenstein modular forms in $\mathrm{M}_k(N, \chi)$ by one.

3. Fourier expansions

Let $G^* \in \mathrm{M}^*(f, G)$. Fix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$. Then, since

$$\langle T, f\rangle = 0, \quad \chi(T) = 1, \quad j(T, z) = 1,$$

we see that

$$G^*(Tz) = G^*(z+1) = G^*(z)$$

and, therefore, for any fixed $y > 0$, G^* has a Fourier expansion of the form

$$G^*(z) = \sum_{n=-\infty}^{\infty} c(n, y) e^{2\pi i n x}$$

for $c(n, y) \in \mathbb{C}$. If G^* is holomorphic then, the Fourier expansion takes the simpler form

$$G^*(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}$$

with $c(n) \in \mathbb{C}$.

The Poincaré series

$$P_m(z, k, s, \chi, f) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma) \langle \gamma, f\rangle j(\gamma, z)^{-k} \operatorname{Im}(\gamma z)^s \, e^{2\pi i m \gamma(z)}$$

has a Fourier expansion

$$P_m(z, k, s, \chi, f) = \sum_{n=-\infty}^{\infty} A_n(k, s, \chi, f, m) e^{2\pi i n x}$$

where

$$\begin{aligned} &A_n(k, s, \chi, f, m) \\ &= \sum_{\substack{c \equiv 0 \pmod N \\ c \neq 0}} \frac{S(m, n, \chi, f; c)}{|c|^{2s} c^k} \cdot \int_{-\infty}^{\infty} e^{\frac{-2\pi i m}{c^2 y(x+i)}} \cdot \frac{y^{1-s-k}}{(x^2+1)^s (x+i)^k} e^{-2\pi i n x y} \, dx \end{aligned}$$

and

$$S(m, n, \chi, f; c) = \sum_{\substack{r=1 \\ (r, c)=1}} \bar{\chi}(r) e^{\frac{2\pi i (m r + n \bar{r})}{c}} \left\langle \begin{pmatrix} * & * \\ c & r \end{pmatrix}, f \right\rangle.$$

The sum $S(m, n, \chi, f; c)$ is a type of generalized Kloosterman sum which has an associated zeta function

$$Z(m, n, \chi, f; s) = \sum_{\substack{c \equiv 0 \pmod N \\ c \neq 0}} \frac{S(m, n, \chi, f; c)}{|c|^{2s}}.$$

These Kloosterman type zeta functions are generalizations of those first studied by Selberg [**S65**], and it would be of great interest to obtain their analytic continuation and polar structure. The zeta function $Z(0, 0, \chi, f; s)$ occurs in the constant term in the Fourier expansion of $E^*(z,s,\chi)$. If χ is a nonprincipal character to the modulus N, we have

$$S(0,0,\chi,f;c) = \sum_{\substack{r=1\\(r,c)=1}} \bar{\chi}(r) \left\langle \begin{pmatrix} * & * \\ c & r \end{pmatrix}, f \right\rangle$$
$$= \sum_{\substack{r=1\\(r,c)=1}} \bar{\chi}_c(r) \left\langle \begin{pmatrix} * & * \\ c & r \end{pmatrix}, f \right\rangle$$

where χ_c is a primitive character induced from χ, i.e., $\chi_c(n) = \chi(n)$ if $(n,c) = 1$ and otherwise $\chi_c(n) = 0$. Note that the conductor of χ_c must divide c. It follows that

$$S(0,0,\chi,f;c) = \tau(\chi_c)\, L_f(1,\bar{\chi}_c)$$

where

$$\tau(\chi_c) = \sum_{a=1}^{c} \chi_c(z) e^{\frac{2\pi i a}{c}}$$

is the Gauss sum and

$$L_f(s,\chi_c) = \sum_{n=1}^{\infty} \frac{a(n)\chi_c(n)}{n^s}$$

is the twisted L-series associated to

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

Thus, the zeta function $Z(0,0,\chi,f;s)$ can be written in the form

$$Z(0,0,\chi,f;s) = \sum_{\substack{c\equiv 0 \pmod N\\ c\neq 0}} \frac{\tau(\chi_c)\, L_f(1,\bar{\chi}_c)}{|c|^{2s}}.$$

On the other hand, if $\chi = \chi_0$ is the principal character (mod N), then

$$S(0,0,\chi,f;c) = \sum_{\substack{r=1\\(r,c)=1}}^{c} \left\langle \begin{pmatrix} * & * \\ c & r \end{pmatrix}, f \right\rangle = 0$$

because of the identity (choose $\tau = -r/c + i/|c|$ as in the proof of Lemma 1)

$$\left\langle \begin{pmatrix} * & * \\ c & r \end{pmatrix}, f \right\rangle = \sum_{n=1}^{\infty} \frac{A(n)}{n} \exp\left(\frac{-2\pi n}{|cN|}\right) \left[\exp\left(\frac{2\pi \in r}{cN}\right) - \exp\left(\frac{-2\pi \in \bar{r}}{cN}\right)\right]$$

where

$$r\bar{r} \equiv 1 \pmod c.$$

When we sum over r in the identity above, the sum with r and the sum with $\bar{r}$ cancel to give zero.

In a similar manner, we may obtain the Fourier expansion of the more general Eisenstein series:

$$E^{**}(z,s,\chi_0) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \chi_0(\gamma)\ |\ \langle \gamma, f\rangle\ |^2\ \mathrm{Im}(\gamma z)^s.$$

For example, the constant term of the Fourier expansion takes the form

$$\int_0^1 E^{**}(z,s,\chi_0)\,dx = \sqrt{\pi}y^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\,Z(s)$$

where

$$Z(s) = \lim_{w\to 1} \sum_{\substack{c\,\equiv\,0\,(N)\\ c\neq 0}} \frac{1}{c^{2s}} \sum_{d|c} \mu\left(\frac{c}{d}\right)\cdot d \sum_{m\,\equiv\,n\,(d)} \frac{a_m\bar{a}_n}{(mn)^w}.$$

The meromorphic continuation and polar structure of $Z(s)$ is not easy to obtain. One approach would be to use the method of Selberg [**S56**]. Preliminary calculations suggest that $Z(s)$ has a simple pole at $s=1$ with residue proportional to the Petersson inner product $\langle f,f\rangle$.

4. L-functions

Let

$$G^*(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi inz}$$

be in $\mathrm{M}^*(f,G)$. The L-function associated to G^* is defined to be

$$\mathrm{L}_{G^*}(s) = \sum_{n=1}^{\infty}\frac{c(n)}{n^s}.$$

We will show that the Dirichlet series for $\mathrm{L}_{G^*}(s)$ converges absolutely in a half plane, has an analytic continuation to the entire complex s–plane, and in certain cases has a functional equation of simple type.

LEMMA 3. *Let $G\in M_k(N,\chi)$, and let $G^*(z)=\sum_{n=0}^{\infty} c(n)e^{2\pi inz}\in M^*(f,G)$. Then if G is cuspidal we have*

$$c(n) \ll n^{k/2+\epsilon}.$$

Otherwise

$$c(n) \ll n^{k-1/2+\epsilon}.$$

PROOF. By the results of §2, every holomorphic $G^*\in \mathrm{M}^*(f,G)$ may be expressed in the form

$$G^*(z) = cF(z)G(z) + G_1(z) \qquad c\in\mathbb{C},$$

with $G_1(z)\in \mathrm{M}_k(N,\chi)$ also holomorphic. Recall that $F(z)=\sum_{n=1}^{\infty} a_ne^{2\pi inz}/n$ where the coefficients $a(n)$ satisfy the Ramanujan bound $a(n)\ll n^{1/2+\epsilon}$. If we write

$$G(z)=\sum_{n=0}^{\infty} b_ne^{2\pi inz}, \quad G_1(z)=\sum_{n=0}^{\infty} {b'}_ne^{2\pi inz},$$

then one easily sees that

$$c(n) = c\sum_{m=0}^{n-1} b_m\frac{a_{n-m}}{n-m} + {b'}_n.$$

If G,G_1 are holomorphic cusp forms of weight k, we have the estimate $b_n, {b'}_n \ll n^{(k-1)/2+\epsilon}$. Otherwise, $b_n, {b'}_n \ll n^{k-1+\epsilon}$. Plugging these bounds in the formula for $c(n)$ given above concludes the proof of the lemma. □

It immediately follows from lemma 3 that the Dirichlet series $\sum\limits_{n=1}^{\infty} c(n)n^{-s}$ converges absolutely in the half–plane $\mathrm{Re}(s) > k + 1/2$. Similar estimates can also be obtained in the case that $G^*(z)$ is not holomorphic.

In certain cases, the L-function $\mathrm{L}_{G^*}(s)$ will have a simple functional equation. Let

$$W_N = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$$

denote the involution on $\Gamma_0(N)$. Let $G \in \mathrm{M}_k(N, \chi)$, with k an even positive integer, satisfy

$$G(W_N z) = \epsilon \cdot N^{k/2} z^k G(z)$$

with $\epsilon = \pm 1$. If we assume that $\langle W_N, f\rangle = 0$ and G^* is of the form

$$G^*(z) = cG(z)F(z) + c'G(z)$$

with $c, c' \in \mathbb{C}$, then we see that G^* satisfies the automorphic relation

$$G^*(W_N z) = \epsilon \cdot N^{k/2} z^k G^*(z).$$

Riemann's method of analytic continuation yields the meromorphic continuation of $\mathrm{L}_{G^*}(s)$ to the entire complex s–plane and the functional equation

$$\psi(s) = (2\pi)^{-s}\Gamma(s)N^{s/2}\mathrm{L}_{G^*}(s) = \epsilon \cdot (-1)^{k/2}\psi(k - s).$$

It can be shown that $\psi(s)$ has at most simple poles at $s = 0$ and $s = k$ with residue a multiple of $c(0)$, the constant term in the Fourier expansion of $G^*(z)$.

If $< W_N, f\rangle \neq 0$, then the automorphic relation takes the form

$$G^*(W_N z) = \epsilon \cdot N^{k/2} z^k G^*(z) + \epsilon'\langle W_N, f\rangle G(z)$$

where ϵ' is a root of unity. In this case, Riemann's method again gives the meromorphic continuation of $\mathrm{L}_{G^*}(s)$. In the most general case, we may express an arbitrary element $G^* \in \mathrm{M}^*(f, G)$ in the form

$$G^*(z) = cG(z)F(z) + g(z)$$

with $g \in \mathrm{M}_k(N, \chi)$ and $c \in \mathbb{C}$. Since we know that $\mathrm{L}_g(s)$ has a meromorphic continuation and satisfies a functional equation, it follows from the above that $\mathrm{L}_{G^*}(s)$ also has a meromorphic continuation. We have thus proved.

PROPOSITION 4. *Fix $f \in S_2(N)$, $G \in M_2(N, \chi)$. Then for every*

$$G^*(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi i n z} \in M^*(f, G)$$

the L-function associated to G^*,

$$L_{G^*}(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

converges absolutely for $Re(s) > k + 1/2$ and has a meromorphic continuation to the entire complex plane with at most a simple pole at $s = k$ whose residue is proportional to $c(0)$. A similar result holds if G^* is not holomorphic.

5. One–cocycles

Let $G^* \in \mathrm{M}_k^*(f, G)$. For $z \in \mathfrak{h}^*$ and $\gamma \in \Gamma_0(N)$, we define the one–cocycle (see **[G95]**)

$$\langle \gamma, z \rangle_{G^*} = 2\pi i \int_z^{\gamma z} G^*(w)\, dw$$

which satisfies the cocycle relation

$$\langle \gamma\gamma', z \rangle_{G^*} = \langle \gamma, \gamma' z \rangle_{G^*} + \langle \gamma', z \rangle_{G^*}$$

for all $\gamma, \gamma' \in \Gamma_0(N)$, $z \in \mathfrak{h}^*$. In the special case that $k = 2$ and $G = f$, the above cocycles can be explicitly computed. This is illustrated in the following proposition.

PROPOSITION 5. *Let N be a positive integer. Fix $f \in S_2(N)$. Then for all $G^* \in S^*(f, f)$ and $\gamma \in \Gamma_0(N)$, we have*

$$\langle \gamma, z \rangle_{G^*} = -\langle \gamma, f \rangle F(z) + \frac{\langle \gamma, f \rangle^2}{2} - \langle \gamma, g \rangle,$$

for some $g \in S_2(N)$. Further, for every $G_1 \in S_2(N)$, $\gamma \in \Gamma_0(N)$,

$$G_1(z) \cdot \langle \gamma, z \rangle_{G^*} \in S^*(f, G_1).$$

For fixed $G_1 \in S_2(N)$, the elements $G_1(z) \cdot \langle \gamma, z \rangle_{G^*}$ generate a one–dimensional subspace of $S^*(f, G_1)$ as γ runs over $\Gamma_0(N)$.

PROOF. Recall that every $G^* \in S^*(f, f)$ can be expressed in the form

$$G^*(z) = f(z)F(z) + g(z)$$

with $g \in S_2(N)$ and

$$F(z) = 2\pi i \int_z^{i\infty} f(w)\, dw.$$

It follows that

$$\begin{aligned} \langle \gamma, z \rangle_{G^*} &= 2\pi i \int_z^{\gamma z} \Big(f(w)F(w) + g(w) \Big)\, dw \\ &= 2\pi i \int_z^{\gamma z} f(w)F(w)\, dw - \langle \gamma, g \rangle. \end{aligned}$$

By integration by parts,

$$2\pi i \int_z^{\gamma z} f(w)F(w)\, dw = F(\gamma z)^2 - F(z)^2 - 2\pi i \int_z^{\gamma z} f(w)F(w)\, dw.$$

Consequently,

$$2\pi i \int_z^{\gamma z} f(w)F(w)\, dw = -\langle \gamma, f \rangle F(z) + \frac{\langle \gamma, f \rangle^2}{2}.$$

Hence,

$$\langle \gamma, z \rangle_{G^*} = -\langle \gamma, f \rangle F(z) + \frac{\langle \gamma, f \rangle^2}{2} - \langle \gamma, g \rangle.$$

The transformation Formula

$$F(\gamma z) = F(z) - \langle \gamma, f \rangle$$

immediately implies that $G_1(z) \cdot \langle \gamma, z \rangle_{G^*} \in \mathrm{S}^*(f, G_1)$. The rest of the proposition easily follows.

References

[C] J. E. Cremona, Algorithms for Modular Elliptic Curves, Cambridge: Cambridge University Press (1992).

[G89] D. Goldfeld, *Modular elliptic curves and diophantine problems*, Proc. of the First Canadian Number Theory Assoc., Banff, Canada 1988, New York: de Gruyter (1989).

[G95] D. Goldfeld, *Special values of derivatives of L-series*, Number Theory, Fourth Conference of the Canadaian Number Theory Association, 1994, Dalhousie Univ., Halifax, Nova Scotia, Karl Dilcher Editor, Canadian Math. Soc. Conference Proc. **15**, Providence, RI: Amer. Math. Soc. (1995), 159–173.

[M] Ju. I. Manin, *Parabolic points and zeta-functions of modular curves,* Math. USSR Izvestija **6** (1972), 19–64.

[MSD] B. Mazur and P. Swinnerton–Dyer, *Arithmetic of Weil curves*, Invent. Math. **25** (1974), 1–61.

[S56] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. B **20** (1956), 47–87.

[S65] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Symp. Pure Math. (Cal. Tech. Pasadena, CA 1963), **VIII**, Providence, RI: Amer. Math. Soc. (1965), 1–15.

[Sh] G. Shimura, *On the factors of the jacobian variety of a modular function field,* J. Math. Soc. Japan **25** (1973), 523–544.

[Sz] L. Szpiro, *Seminaire sur les pinceaux de courbes de genre au moins deux,* Astérisque, exposé n^o 3, **86** (1981), 44–78.

Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: `goldfeld@columbia.edu`