A converse theorem for double Dirichlet series

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1 Introduction

The main purpose of converse theorems is to show that Dirichlet series with nice properties (analytic continuation, moderate growth, functional equation) are, in fact, Mellin transforms of automorphic functions. Converse theorems establish a one-to-one correspondence between "nice" Dirichlet series and automorphic functions.

The first such converse theorem was due to Hecke [10] who showed that if $N = 1, 2, 3$ or 4, and a Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges in some right half-plane, is EBV (entire and bounded in vertical strips) and satisfies a functional equation of the type

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s} \Gamma(s) D(s) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s) D(k-s), \qquad w = \pm(-1)^{k/2},$$

then the function $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is in fact a modular form of weight k and level N, i.e.,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \left(\text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\right).$$

Hecke's results were later generalized by Maass [12] to the case of non-holomorphic forms. Hecke's method fails for $N \geq 5$. It was Weil [16] who obtained a converse theorem for all N by assuming that, in addition, the twisted Dirichlet series $D(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$, for each primitive Dirichlet character χ, also satisfies the EBV condition and an appropriate functional equation. For some other converse theorems improving on aspects of Weil's theorem see [14], [4], [5], [6]. For general converse theorems on GL_n see [1], [2], [3].

Multiple Dirichlet series are Dirichlet series in several complex variables. It is a natural question to ask if a converse theorem exists for multiple Dirichlet series. In Theorem 3.1, we examine the case of certain vector valued double Dirichlet series. We show that if such a vector valued double Dirichlet series (and all its twists by Dirichlet characters) have "nice" properties and satisfy appropriate functional equations, then the vector valued double Dirichlet series is, in fact, the Mellin transform of a vector valued metaplectic Eisenstein series.

One may ask if there exist multiple Dirichlet series which are not vector valued and satisfy meromorphic continuation (with finitely many poles), moderate growth, and a finite group of functional equations. It would be of great interest to characterize such multiple Dirichlet series by a converse theorem. In Theorem 4.2 we obtain a converse theorem for what seems to be historically the first example of a scalar double Dirichlet series studied from the point of view of functional equations in two variables. That study was carried out by Siegel ([15]) and the multiple Dirichlet series is essentially the Mellin transform of a metaplectic Eisenstein series on $\Gamma_0(4)$. For $N > 4$, we have not been able to discover linear combinations of metaplectic Eisenstein series on $\Gamma_0(N)$ whose Mellin transforms satisfy suitable pairs of functional equations. Therefore, it has not been possible to formulate a scalar converse theorem in that case. The difficulty to find multiple Dirichet series coming from linear combinations of higher level metaplectic Eisenstein series may suggest the possible existence of scalar valued double Dirichlet series that do not correspond to automorphic objects we are currently familiar with.

2 L-functions of metaplectic Eisenstein series

2.1 Metaplectic Eisenstein series

Fix a positive integer N. Let $\Gamma = \Gamma_0(4N)$ denote the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1 with $a, b, c, d \in \mathbb{Z}$ and $4N|c$. Define

$$v(\gamma) = \left(\frac{c}{d}\right) \epsilon_d^{-1},$$

with

$$\epsilon_d = \begin{cases} 1, & d \equiv 1 (\operatorname{mod} 4), \\ i, & d \equiv 3 (\operatorname{mod} 4), \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the usual Kronecker symbol. We shall also adopt the notation that we may write M in the form $M = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix}$. The arguments of complex numbers are chosen to be in $(\pi, \pi]$. Then, for $f : \mathfrak{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we recall the slash operator: $f|\gamma$. It is defined by the formula

$$(f|\gamma)(z) = f(\gamma z) \frac{(c_\gamma z + d_\gamma)^{1/2}}{|c_\gamma z + d_\gamma|^{1/2}},$$

and satisfies the relation

$$f|\gamma|\delta = r(\gamma, \delta) \cdot f|(\gamma\delta), \qquad (\gamma, \delta \in \mathrm{SL}_2(\mathbb{R})),$$

where

$$r(M, N) = \frac{(c_M Nz + d_M)^{1/2}(c_N z + d_N)^{1/2}}{(c_{MN} z + d_{MN})^{1/2}}, \qquad (\text{for } M, N \in \mathrm{SL}_2(\mathbb{R})).$$

To compute $r(M, N)$ we will tacitly be using Theorem 16 of [11].

Lemma 2.1. *Let* $M = \left(\begin{smallmatrix} * & * \\ m_1 & m_2 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ *and* $MS = \left(\begin{smallmatrix} * & * \\ m_1' & m_2' \end{smallmatrix}\right)$. *Then* $r(M, S) = e^{\frac{\pi i}{4} w(M,S)}$, *with*

$$w(M, S) = \begin{cases} (\operatorname{sgn}(c) + \operatorname{sgn}(m_1) - \operatorname{sgn}(m_1') - \operatorname{sgn}(m_1 c m_1')), & m_1 c m_1' \neq 0, \\ (\operatorname{sgn}(c) - 1)(1 - \operatorname{sgn}(m_1)), & m_1 c \neq 0, m_1' = 0, \\ (\operatorname{sgn}(c) + 1)(1 - \operatorname{sgn}(m_2)), & m_1' c \neq 0, m_1 = 0, \\ (1 - \operatorname{sgn}(a))(1 + \operatorname{sgn}(m_1)), & m_1 m_1' \neq 0, c = 0, \\ (1 - \operatorname{sgn}(a))(1 - \operatorname{sgn}(m_2)), & m_1 = c = m_1' = 0. \end{cases}$$

Now, we fix a set $\{\mathfrak{a}_i, i = 1, \dots, m\}$ of inequivalent cusps of $\Gamma_0(4N)$ among which the first m^* are singular with respect to v (i.e. $v(\gamma_{\mathfrak{a}}) = 1$, if $\gamma_{\mathfrak{a}}$ is the generator of the stabilizer $\Gamma_{\mathfrak{a}}$ of $\mathfrak{a}$). We choose the $\mathfrak{a}$'s so that $\mathfrak{a}_1 = \infty$ and $\mathfrak{a}_{m^*} = 0$.

For each $\mathfrak{a}$ we fix a scaling matrix $\sigma_{\mathfrak{a}}$ such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. In particular, we select $\sigma_{\mathfrak{a}_1} = I$, $\sigma_{\mathfrak{a}_{m^*}} = W_{4N}$, where I is the identity matrix and W_{4N} is the Fricke involution $\left(\begin{smallmatrix} 0 & -1/(2\sqrt{N}) \\ 2\sqrt{N} & 0 \end{smallmatrix}\right)$.

For each of the cusps $\mathfrak{a}_i$ ($i = 1, \dots, m^*$) and $w \in \mathbb{C}$ with $\Re(w) > 1$, we define an Eisenstein series

$$E_i(z, w) = \sum_{\gamma \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma} \frac{\operatorname{Im}(\sigma_{\mathfrak{a}_i}^{-1}\gamma z)^w}{r(\sigma_{\mathfrak{a}_i}^{-1}, \gamma)v(\gamma)} \left(\frac{c_{\sigma_{\mathfrak{a}_i}^{-1}\gamma} z + d_{\sigma_{\mathfrak{a}_i}^{-1}\gamma}}{|c_{\sigma_{\mathfrak{a}_i}^{-1}\gamma} z + d_{\sigma_{\mathfrak{a}_i}^{-1}\gamma}|} \right)^{-1/2}.$$

This Eisenstein series has a meromorphic continuation to the w-plane ([13], Section 10) and, for all $\delta \in \Gamma$, it satisfies

$$E_i(\cdot, w)|\delta = v(\delta)E_i(\cdot, w).$$

For convenience, for every function f on $\mathfrak{H}$ we set

$$\check{f} := e^{\frac{\pi i}{4}} f|W_{4N}.$$

Thus, $\check{f}(iy) = f(i/(4Ny))$.

Next, if T denotes transpose, set

$$\mathbf{E}(z, w) = (E_1(z, w), \dots, E_{m^*}(z, w))^T$$

and

$$\check{\mathbf{E}}(z, w) = (\check{E}_1(z, w), \dots, \check{E}_{m^*}(z, w))^T.$$

Each E_i is an eigenfunction of the weight $1/2$ Laplacian

$$\Delta_{1/2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{iy}{2}\frac{\partial}{\partial x}$$

with eigenvalue $w(w-1)$ ([13], (10.10)). This implies that, if $z := x + iy$, then, for all $i, j \in \{1, \dots, m^*\}$, there are functions of w, a_n^{ij}, such that

$$E_i(\cdot, w)|\sigma_{\mathfrak{a}_j} = \delta_{ij}y^w + p_{ij}(w)y^{1-w} + \sum_{n \neq 0} a_n^{ij}(w) W_{\operatorname{sgn}(\frac{n}{4}),\, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}$$

where δ_{ij} is the Kronecker delta and $p_{ij}(w)$ the ij-th entry of the scattering matrix $\Phi(w)$. Here, $W_{\cdot,\cdot}$ is the classical Whittaker function with integral representation

$$W_{a,b}(z) = \frac{e^{-z/2}z^a}{\Gamma(1/2 - a + b)} \int_0^\infty u^{-a-1/2+b}(1 + z^{-1}u)^{a-1/2+b}e^{-u}du$$

(cf. [17], pg. 340).

If w and $1 - w$ are not poles of any of the E_i ($i = 1, \dots, m^*$), then, by [13], (10.19),

$$\mathbf{E}(z, 1 - w) = \Phi(1 - w)\mathbf{E}(z, w). \tag{1}$$

2.2 Twists

We first introduce in a general form the formalism of twists we will be using.

For every positive integer D (with $(D, 4N) = 1$), let χ be a Dirichlet character modulo D. For every function f on $\mathfrak{H}$ we define its twist (denoted $f(\cdot\,;\chi)$) by the formula

$$f(\cdot\,;\chi) = \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) f\big| \left(\begin{smallmatrix} 1 & m/D \\ 0 & 1 \end{smallmatrix}\right).$$

We shall be interested in functions $f(z, w)$ of two variables $z = x + iy \in \mathfrak{H}$, $w \in \mathbb{C}$, which have Fourier expansions of the form

$$f(z, w) = a(w)y^{1-w} + b(w)y^w + \sum_{n\neq 0} a_n(w) W_{\mathrm{sgn}(\frac{n}{4}),\, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}.$$

Then the twisted function $f(\cdot\,;\chi)$, in terms of z, is

$$f(z, w; \chi) = \tau_0(\chi)\Big(a(w)y^{1-w} + b(w)y^w\Big) + \sum_{n\neq 0} \tau_n(\chi) a_n(w) W_{\mathrm{sgn}(\frac{n}{4}),\, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}$$

where

$$\tau_n(\chi) = \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) e^{2\pi i m n/D}.$$

We have

$$f(\cdot\,;\chi)\Big| \begin{pmatrix} 0 & \frac{-1}{2D\sqrt{N}} \\ 2D\sqrt{N} & 0 \end{pmatrix}$$
$$= \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) r\Bigg(\left(\begin{smallmatrix} 1 & m/D \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 0 & \frac{-1}{2D\sqrt{N}} \\ 2D\sqrt{N} & 0 \end{pmatrix} \Bigg) f\Big| \begin{pmatrix} 2m\sqrt{N} & \frac{-1}{2D\sqrt{N}} \\ 2D\sqrt{N} & 0 \end{pmatrix}. \quad (2)$$

By repeated use of Lemma 2.1 we deduce that (2) equals

$$e^{-\pi i/4} \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) \check{f}\ \Big|\ \begin{pmatrix} 0 & \frac{1}{2\sqrt{N}} \\ -2\sqrt{N} & 0 \end{pmatrix} \begin{pmatrix} 2m\sqrt{N} & \frac{-1}{2D\sqrt{N}} \\ 2D\sqrt{N} & 0 \end{pmatrix} =$$
$$e^{-\pi i/4} \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) \check{f}\big| \left(\begin{smallmatrix} D & 0 \\ -4mN & 1/D \end{smallmatrix}\right).$$

For each pair of positive integers m, D with $(m, D) = 1$, we choose r, s such that $r, s > 0$ and $Ds - 4Nmr = 1$. Then, as m ranges over a reduced system of residues mod$D$, so does r too, so by the last equality we deduce that

$$f(\cdot\,;\chi)\Big| \begin{pmatrix} 0 & \frac{-1}{2D\sqrt{N}} \\ 2D\sqrt{N} & 0 \end{pmatrix} = e^{-\pi i/4} \sum_{\substack{m (\bmod D)\\(m,D)=1}} \chi(m) \check{f}\big| \left(\begin{smallmatrix} D & -r \\ -4mN & s \end{smallmatrix}\right) | \left(\begin{smallmatrix} 1 & r/D \\ 0 & 1 \end{smallmatrix}\right)$$
$$= e^{-\pi i/4}\overline{\chi(-4N)} \sum_{\substack{r (\bmod D)\\(r,D)=1}} \overline{\chi(r)} \check{f}\big| \left(\begin{smallmatrix} D & -r \\ -4mN & s \end{smallmatrix}\right) | \left(\begin{smallmatrix} 1 & r/D \\ 0 & 1 \end{smallmatrix}\right). \quad (3)$$

We also consider the Dirichlet character $\check{\chi} \pmod D$ given by

$$\check{\chi}(m) := \left(\frac{m}{D}\right)\overline{\chi(m)}.$$

Note that $\check{\chi}$ is a character since $(D, 4N) = 1$, D is odd and $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

In the next sections we will compare (3) with the function

$$\check{f}(\cdot\,;\check{\chi}) = \epsilon_D\left(\frac{4N}{D}\right)\sum_{\substack{r(\mathrm{mod}\,D)\\(r,D)=1}}\overline{\chi(r)}\left(\frac{4Nr}{D}\right)\epsilon_D^{-1}\check{f}\big|\left(\begin{smallmatrix}1 & r/D\\ 0 & 1\end{smallmatrix}\right). \tag{4}$$

2.3 L-functions

We now associate an L-function to the metaplectic Eisenstein series $\mathbf{E}$ (of section 2.1) to obtain a "perfect" double Dirichlet series.

Let $a_n^i(w)$ denote the n-th coefficient of the expansion at ∞ of $E_i(z, w)$. We define

$$L_i^{\pm}(s, w) = \sum_{\pm n>0}\frac{a_n^i(w)}{|n|^s}$$

for $\mathrm{Re}(s)$ large enough. More generally, for χ a Dirichlet character modulo D ($(D, 4N) = 1$), we set

$$L_i^{\pm}(s, w; \chi) = \sum_{\pm n>0}\frac{\tau_n(\chi)a_n^i(w)}{|n|^s}.$$

We also define the "completed" L-function associated to $L_i^{\pm}$:

$$\Lambda_i(s, w; \chi) := \int_0^{\infty}\Big(E_i(iy, w; \chi) - \tau_0(\chi)\Big(\delta_{i0}y^w + p_{i0}(w)y^{1-w}\Big)\Big)y^s\,\frac{dy}{y}.$$

With this notation we define

$$\mathbf{L}_E^{\pm}(s, w; \chi) = (L_1^{\pm}(s, w; \chi), \ldots, L_{m^*}^{\pm}(s, w; \chi))^T,$$

and

$$\boldsymbol{\Lambda}_E(s, w; \chi) = (\Lambda_1(s, w; \chi), \ldots, \Lambda_{m^*}(s, w; \chi))^T.$$

We also set $\check{\Lambda}_i$, $\check{\mathbf{L}}_E$ and $\check{\boldsymbol{\Lambda}}_E$ for the corresponding functions associated to $\check{\mathbf{E}}$.

By [11], (pgs 216, 219 (12)),

$$\Lambda_i(s, w; \chi) = (2\pi)^{-s}\Gamma(w + s)\Gamma(s - w + 1)\left(F^+(s, w)L_i^+(s, w; \chi) + F^-(s, w)L_i^-(s, w; \chi)\right),$$

where

$$F^+(s, w) = \frac{2^{1/4}F\left(w - \frac{1}{4},\ \frac{3}{4} - w,\ s + \frac{3}{4};\ \frac{1}{2}\right)}{\Gamma\left(s + \frac{3}{4}\right)},$$

and

$$F^-(s, w) = \frac{2^{-1/4}F\left(w + \frac{1}{4},\ \frac{5}{4} - w,\ s + \frac{5}{4};\ \frac{1}{2}\right)}{\Gamma\left(s + \frac{5}{4}\right)},$$

where $F(a,b,c;d)$ denotes the Gaussian hypergeometric function.

Before we can establish the analytic continuation and functional equation in s we need to evaluate the constant term $a_0(y,w;\chi)\check{}$ of the expansion of $E_i(\cdot,w;\chi)|W_{4ND^2}$ (at ∞). By (3),

$$E_i(\cdot,w;\chi)|W_{4ND^2} = e^{-\pi i/4}\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\sum_{\substack{r(\bmod D)\\(r,D)=1}}\check{\chi}(r)\check{E}_i|\left(\begin{smallmatrix}1 & r/D\\ 0 & 1\end{smallmatrix}\right).$$

Therefore,

$$a_0(y,w;\chi)\check{} = \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\Big(\delta_{im^*}y^w + p_{im^*}(w)y^{1-w}\Big).$$

We can then apply the standard Riemann trick to get

$$\Lambda_i(s,w;\chi) = \int_{\frac{1}{2D\sqrt{N}}}^{\infty}\Big(E_i(iy,w;\chi) - \tau_0(\chi)\Big(\delta_{i0}y^w + p_{i0}(w)y^{1-w}\Big)\Big)y^s\,\frac{dy}{y} +$$

$$\int_{\frac{1}{2D\sqrt{N}}}^{\infty}\left[e^{\frac{\pi i}{4}}\left(E_i(\cdot,w;\chi)|W_{4ND^2}\right)(iy) - \tau_0(\chi)\Big(\delta_{i0}\cdot(4ND^2y)^{-w} + p_{i0}(w)(4ND^2y)^{w-1}\Big)\right](4ND^2y)^{-s}\,\frac{dy}{y}$$

$$\begin{aligned}
= \int_{\frac{1}{2D\sqrt{N}}}^{\infty}\Bigg[&(E_i(iy,w;\chi) - \tau_0(\chi)(\delta_{i0}y^w + p_{i0}(w)y^{1-w}))y^s + \\
&\left(\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{E}_i(iy,w;\check{\chi}) - a_0(y,w;\chi)\check{}\right)(4ND^2y)^{-s}\Bigg]\frac{dy}{y} \\
+ \int_{\frac{1}{2D\sqrt{N}}}^{\infty}\Bigg(&-\tau_0(\chi)(\delta_{i0}\cdot(4ND^2y)^{-w} + p_{i0}(w)(4ND^2y)^{w-1}) + \\
&\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})(\delta_{im^*}y^w + p_{im^*}(w)y^{1-w})\Bigg)(4ND^2y)^{-s}\,\frac{dy}{y}.
\end{aligned} \tag{5}$$

By the exponential decay of $W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-1/2}(iy)$ as $y\to\infty$, the first integral of the last equality is clearly convergent giving an entire function of s. An elementary computation implies that the last integral of (5) is

$$\begin{aligned}
-(2D\sqrt{N})^{-s-w}&\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\delta_{im^*}}{w-s} + \frac{\tau_0(\chi)\delta_{i0}}{w+s}\right) + \\
&(2\sqrt{N}D)^{-s+w-1}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})p_{im^*}(w)}{w+s-1} + \frac{\tau_0(\chi)p_{i0}(w)}{w-s-1}\right).
\end{aligned}$$

This implies that

(i) $\mathbf{\Lambda}_E(s,w;\chi)$ is meromorphic on $\mathbb{C}^2$.

(ii)

$$\Lambda_i(s,w;\chi) \;+\; (2D\sqrt{N})^{-s-w}\left(\frac{\overline{\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\delta_{im^*}}}{w-s} \;+\; \frac{\tau_0(\chi)\delta_{i0}}{w+s}\right)$$
$$- (2\sqrt{N}D)^{-s+w-1}\left(\frac{\overline{\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})p_{im^*}(w)}}{w+s-1} \;+\; \frac{\tau_0(\chi)p_{i0}(w)}{w-s-1}\right)$$

is EBV (entire and bounded in vertical strips).

(iii)

$$(4ND^2)^s\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D\mathbf{\Lambda}_E(s,w;\chi) \;=\; \check{\mathbf{\Lambda}}_E(-s,w;\check{\chi}).$$

(iv) If w and $1-w$ are not poles of $\Phi(w)$, then

$$\mathbf{\Lambda}_E(s,1-w;\chi) \;=\; \Phi(1-w)\mathbf{\Lambda}_E(s,w;\chi). \tag{6}$$

The functional equations (iii) and (iv) are deduced from (5) and (1) respectively.

3 The converse theorem

We maintain the notation of sections 2.2 and 2.3.

Theorem 3.1. *Fix positive integers D, N, with $(D,N)=1$ and $1 \le D < (4N)^2$. Let $(a^i_{n,\ell})_{n,\ell,i}$ and $(\check{a}^i_{n,\ell})_{n,\ell,i}$ (with $\ell, n \in \mathbb{Z}$, $\ell \ge 1$, $i = 1,\ldots,m^*$) be sequences of complex numbers which are assumed to have polynomial growth in $|n|$ and ℓ as $|n|,\ell \to \infty$. For $s,w \in \mathbb{C}$ (with sufficiently large real parts) and each Dirichet character χ (mod D), define double Dirichlet series:*

$$L_i^{\pm}(s,w;\chi) = \sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{a^i_{n,\ell}\,\tau_n(\chi)}{\ell^w\,|n|^s}, \qquad \check{L}_i^{\pm}(s,w;\chi) = \sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{\check{a}^i_{n,\ell}\,\tau_n(\chi)}{\ell^w\,|n|^s}.$$

Next define vector valued double Dirichlet series:

$$\mathbf{L}^{\pm}(s,w;\chi) = (L_1^{\pm}(s,w;\chi),\ldots,L_{m^*}^{\pm}(s,w;\chi))^T, \check{\mathbf{L}}^{\pm}(s,w;\chi) = (\check{L}_1^{\pm}(s,w;\chi),\ldots,\check{L}_{m^*}^{\pm}(s,w;\chi))^T.$$

We shall assume that $\mathbf{L}^{\pm}(s,w;\chi)$ and $\check{\mathbf{L}}^{\pm}(s,w;\chi)$ satisfy assumptions (9), (10), (11) (listed below). We set

$$a^i_n(w) = \sum_{m=1}^{\infty}\frac{a^i_{n,m}}{m^w} \qquad \text{and} \qquad \check{a}^i_n(w) = \sum_{m=1}^{\infty}\frac{\check{a}^i_{n,m}}{m^w},$$

and assume that, for each fixed i, w (with Re(w) large enough), the functions: $|a^i_n(w)|$, $|\check{a}^i_n(w)| = O(|n|^C)$ $(C>0)$, as $n \to \infty$. Suppose also that $a_i(w), b_i(w), \check{a}_i(w), \check{b}_i(w)$, $(i = 1,\ldots,m^*)$ are meromorphic functions on $\mathbb{C}$ which are holomorphic for Re(w) large enough.

Then, for

$$\mathbf{f}(z,w) = (f_1(z,w),\ldots,f_{m^*}(z,w))^T,$$

where

$$f_i(z,w) = a_i(w)y^{1-w} + b_i(w)y^w + \sum_{n\neq 0} a_n^i(w) W_{\mathrm{sgn}\left(\frac{n}{4}\right),\, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}, \qquad (i=1,2,\ldots,m^*),$$

we have

$$\mathbf{f}(z,w) = A(w)\mathbf{E}(z,w),$$

where $A(w)$ is a matrix of functions and $\mathbf{E}(z,w)$ is the matrix of Eisenstein series given in §2.1. If $A(w)$ is meromorphic, then, for each w and $1-w$ which are not poles of $\Phi(w)$ and $A(w)$, we have

$$\Phi(1-w)A(w)\Phi(w) = A(1-w). \tag{7}$$

Assumptions: Set

$$\boldsymbol{\Lambda}(s,w;\chi) = (2\pi)^{-s}\Gamma(w+s)\Gamma(s-w+1)\Big(F^+(s,w)\mathbf{L}^+(s,w;\chi) + F^-(s,w)\mathbf{L}^-(s,w;\chi)\Big),$$

$$\check{\boldsymbol{\Lambda}}(s,w;\chi) = (2\pi)^{-s}\Gamma(w+s)\Gamma(s-w+1)\Big(F^+(s,w)\check{\mathbf{L}}^+(s,w;\chi) + F^-(s,w)\check{\mathbf{L}}^-(s,w;\chi)\Big),$$

and

$$\tilde{\boldsymbol{\Lambda}}(s,w;\chi) = (\tilde{\Lambda}_1(s,w;\chi),\ldots,\tilde{\Lambda}_{m^*}(s,w;\chi))^T = -\frac{i}{4}\boldsymbol{\Lambda}(s,w;\chi) + \frac{i}{(2\pi)^s}\Gamma(w+s+1)\Gamma(s-w+2)\Big(F^+(s+1,w)\mathbf{L}^+(s,w;\chi) - F^-(s+1,w)\mathbf{L}^-(s,w;\chi)\Big),$$

$$\tilde{\check{\boldsymbol{\Lambda}}}(s,w;\chi) = (\tilde{\check{\Lambda}}_1(s,w;\chi),\ldots,\tilde{\check{\Lambda}}_{m^*}(s,w;\chi))^T = -\frac{i}{4}\check{\boldsymbol{\Lambda}}(s,w;\chi) + \frac{i}{(2\pi)^s}\Gamma(w+s+1)\Gamma(s-w+2)\Big(F^+(s+1,w)\check{\mathbf{L}}^+(s,w;\chi) - F^-(s+1,w)\check{\mathbf{L}}^-(s,w;\chi)\Big).$$

Suppose that

$$\Lambda_i(s,w;\chi) + (2\sqrt{N}D)^{-s-w}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\check{b}_i(w)}{w-s} + \frac{\tau_0(\chi)b_i(w)}{w+s}\right) - (2\sqrt{N}D)^{-s+w-1}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\check{a}_i(w)}{w+s-1} + \frac{\tau_0(\chi)a_i(w)}{w-s-1}\right) \tag{8}$$

and

$$\tilde{\Lambda}_i(s,w;\chi) + \frac{i}{4(2\sqrt{N}D)^{s+w}}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\check{b}_i(w)}{w-s} - \frac{\tau_0(\chi)b_i(w)}{w+s}\right) - \frac{i}{4(2\sqrt{N}D)^{s-w+1}}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\check{a}_i(w)}{w+s-1} - \frac{\tau_0(\chi)b_i(w)}{w-s-1}\right) \tag{9}$$

are EBV for every w (with $\mathrm{Re}(w)$ large enough) and for every χ.

Further, assume that, for every Dirichlet character χ,

$$(4ND^2)^s\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D\mathbf{\Lambda}(s,w;\chi)=\check{\mathbf{\Lambda}}(-s,w;\check{\chi}),$$

and that

$$(4ND^2)^s\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D\tilde{\mathbf{\Lambda}}(s,w;\chi)=-\tilde{\check{\mathbf{\Lambda}}}(-s,w;\check{\chi}),$$

and

$$\mathbf{\Lambda}(s,1-w;\chi)=\Phi(1-w)\mathbf{\Lambda}(s,w;\chi). \tag{10}$$

Proof of Theorem 3.1.

• We first prove that, for every w (with $\mathrm{Re}(w)$ large enough), $f_i(\cdot,w)$ is invariant under the action of $\Gamma_0(4N)$.

For every w with $\mathrm{Re}(w)$ large enough, $i=1,\dots,m^*$, every character χ mod D and every $y>0$ define,

$$F_i(y,w;\chi)=\sum_{n\neq 0}a_n^i(w)\tau_n(\chi)W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-\frac{1}{2}}(4\pi|n|y),$$

$$\check{F}_i(y,w;\chi)=\sum_{n\neq 0}\check{a}_n^i(w)\tau_n(\chi)W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-\frac{1}{2}}(4\pi|n|y),$$

and

$$\tilde{F}_i(y,w;\chi)=2\pi iy\sum_{n\neq 0}na_n^i(w)\tau_n(\chi)W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-\frac{1}{2}}(4\pi|n|y)-\tfrac{i}{4}F_i(y,w;\chi),$$

$$\tilde{\check{F}}_i(y,w;\chi)=2\pi iy\sum_{n\neq 0}n\check{a}_n^i(w)\tau_n(\chi)W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-\frac{1}{2}}(4\pi|n|y)-\tfrac{i}{4}\check{F}_i(y,w;\chi).$$

Since for every w (with $\mathrm{Re}(w)$ large enough), $|a_n^i(w)|$, $|\check{a}_n^i(w)|=O(|n|^C)$, in the Mellin transforms of $F_i(y,w;\chi)$, $\tilde{F}_i(y,w;\chi)$ we can exchange summation and integration to get, for $\mathrm{Re}(s)$ large enough

$$\int_0^\infty F_i(y,w;\chi)y^s\frac{dy}{y}=\Lambda_i(s,w;\chi)\qquad\text{and}\quad\int_0^\infty\tilde{F}_i(y,w;\chi)y^s\frac{dy}{y}=\tilde{\Lambda}_i(s,w;\chi).$$

For each w (with $\mathrm{Re}(w)$ large enough), $(2\pi)^{-s}\Gamma(w+s)\Gamma(s-w+1)F^{\pm}(s,w)=O(e^{-c|\mathrm{Im}(s)|})$ as $|\mathrm{Im}(s)|\to\infty$ $(c>0)$ ([11], pg. 221), so we can apply Mellin inversion to get

$$F_i(y,w;\chi)=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Lambda_i(s,w;\chi)y^{-s}ds\qquad\text{and}\ \tilde{F}_i(y,w;\chi)=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\tilde{\Lambda}_i(s,w;\chi)y^{-s}ds \tag{11}$$

for σ_0 large enough and a line of integration to the right of the poles of Λ_i, $\tilde{\Lambda}_i$. By the same estimate for $F^{\pm}$, the standard Phràgmen-Lindelöf argument applies. We can, therefore, move the line of integration from σ_0 to $\sigma_1 = -\sigma_0$ to get

$$F_i(y,w;\chi) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda_i(s,w;\chi)y^{-s}ds + \sum_{s_0 \text{ pole}}\mathrm{Res}_{s=s_0}\Lambda_i(s,w;\chi)y^{-s}$$
$$= \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda_i(s,w;\chi)y^{-s}ds +$$
$$\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\Big(\check{b}_i(w)(4ND^2y)^{-w}$$
$$+\check{a}_i(w)(4ND^2y)^{w-1}\Big) - \tau_0(\chi)\Big(b_i(w)y^w + a_i(w)y^{1-w}\Big). \quad (12)$$

By the functional equation of Λ_i, the integral equals

$$\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}(4ND^2)^{-s}\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{\Lambda}_i(-s,w;\check{\chi})y^{-s}ds =$$
$$\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\check{\Lambda}_i(s,w;\chi)(4ND^2y)^s ds. \quad (13)$$

However, if

$$\check{f}_i(z,w;\check{\chi}) := \tau_0(\check{\chi})\Big(\check{a}_i(w)y^{1-w} + \check{b}_i(w)y^w\Big) + \sum_{n\neq 0}\check{a}_n^i(w;\check{\chi})W_{\mathrm{sgn}\left(\frac{n}{4}\right),\,w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x},$$

we have

$$f_i(iy,w;\chi) = F_i(y,w;\chi) + \tau_0(\chi)\Big(b_i(w)y^w + a_i(w)y^{1-w}\Big),$$

and

$$\check{f}_i(iy,w;\check{\chi}) = \check{F}_i(y,w;\check{\chi}) + \tau_0(\check{\chi})\Big(\check{b}_i(w)y^w + \check{a}_i(w)y^{1-w}\Big).$$

Therefore, (12), (13) imply that

$$f_i(iy,w;\chi) = \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{f}_i\left(\frac{i}{4ND^2y},\ w;\ \check{\chi}\right). \quad (14)$$

We will show that this transformation law can be extended to the entire upper half-plane. We will need the following lemma.

Lemma 3.2. *Let $z = x+iy$ be in the upper half-plane. Then for every w (with $Re(w)$ large enough)*

$$\frac{\partial}{\partial x}\left(i^{1/2}f_i\left(\frac{-1}{4ND^2z},\ w;\ \chi\right)\left(\frac{z}{|z|}\right)^{-1/2} - \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{f}_i(z,w;\check{\chi})\right)\Bigg|_{x=0} = 0. \quad (15)$$

Proof. Set

$$G_i(y,\chi) = y\frac{\partial f_i}{\partial x}(iy,w;\chi) - \frac{i}{4}f_i(iy,w;\chi) \ \text{ and } \check{G}_i(y,\check{\chi}) = y\frac{\partial \check{f}_i}{\partial x}(iy,w;\check{\chi}) - \frac{i}{4}\check{f}_i(iy,w;\check{\chi}).$$

We first observe that (15) holds if and only if

$$G_i(1/4ND^2y,\chi) = -\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{G}_i(y,\check{\chi}).$$

Indeed, the left-hand side of (15) equals

$$\begin{aligned}
&\frac{i^{1/2}}{4ND^2}\frac{\partial f_i}{\partial x}\left(\frac{-1}{4ND^2(x+iy)}, w; \chi\right)\frac{1}{(x+iy)^2}\left(\frac{x+iy}{(x^2+y^2)^2}\right)^{-1/2} \\
&\qquad\qquad - \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\frac{\partial \check{f}_i}{\partial x}(x+iy, w; \check{\chi}) \\
&- i^{1/2}\frac{(x^2+y^2)^{1/2} - x(x^2+y^2)^{-1/2}(x+iy)}{(x^2+y^2)}\left(\frac{x+iy}{(x^2+y^2)^{1/2}}\right)^{-3/2} f_i\left(\frac{-1}{4ND^2(x+iy)}, w; \chi\right),
\end{aligned} \tag{16}$$

and its value at $x = 0$ is

$$\begin{aligned}
&-\frac{\partial f_i}{\partial x}\left(\frac{-1}{4ND^2iy},\ w;\ \chi\right)\frac{1}{4ND^2y^2} + \frac{i}{2y}f_i\left(\frac{i}{4ND^2y},\ w;\ \chi\right) - \\
&\qquad\qquad \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\frac{\partial \check{f}_i}{\partial x}(iy, w; \check{\chi}).
\end{aligned}$$

By (14), this is 0 if and only if

$$\begin{aligned}
&\frac{\partial f_i}{\partial x}\left(\frac{-1}{4ND^2iy},\ w;\ \chi\right)\frac{1}{4ND^2y} - \frac{i}{4}f_i\left(\frac{i}{4ND^2y},\ w;\ \chi\right) = \\
&\qquad\qquad -\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\Big(\frac{\partial}{\partial x}\check{f}_i(iy, w; \check{\chi})y - \frac{i}{4}\check{f}_i(iy, w; \check{\chi})\Big),
\end{aligned}$$

as desired.

We also have

$$\begin{aligned}
G_i(y,\chi) &= \tilde{F}_i(y, w; \chi) - \tfrac{i}{4}\tau_0(\chi)\Big(a_i(w)y^{1-w} + b_i(w)y^w\Big), \\
\check{G}_i(y,\check{\chi}) &= \check{\tilde{F}}_i(y, w; \check{\chi}) - \tfrac{i}{4}\tau_0(\check{\chi})\Big(\check{a}_i(w)y^{1-w} + \check{b}_i(w)y^w\Big).
\end{aligned}$$

Therefore, to prove (15), it suffices to prove

$$\begin{aligned}
&\tilde{F}_i\left(\frac{1}{4ND^2y},\ w;\ \chi\right) - \frac{i}{4}\tau_0(\chi)\Big(a_i(w)(4ND^2y)^{w-1} + b_i(w)(4ND^2y)^{-w}\Big) = \\
&\qquad -\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\Big(\check{\tilde{F}}_i(y, w; \check{\chi}) - \frac{i}{4}\tau_0(\check{\chi})\Big(\check{a}_i(w)y^{1-w} + \check{b}_i(w)y^w\Big)\Big).
\end{aligned} \tag{17}$$

To verify this we move the line of integration in (11) as before to get

$$\tilde{F}_i(y,w;\chi) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \tilde{\Lambda}_i(s,w;\chi)y^{-s}ds$$
$$+\frac{i}{4}\Bigg(\tau_0(\check{\chi})\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\Big(\check{b}_i(w)(4ND^2y)^{-w}+\check{a}_i(w)(4ND^2y)^{w-1}\Big)$$
$$+\tau_0(\chi)(b_i(w)y^w+a_i(w)y^{1-w})\Bigg). \quad (18)$$

By the functional equation of $\tilde{\Lambda}$, the first term on the RHS equals

$$-\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\check{\tilde{\Lambda}}_i(s,w)((4ND^2y)^{-1})^{-s}ds.$$

This implies (17) and thus (15). □

It then follows from this lemma, together with (14) and Lemma 13.5.2 of [9], that

$$f_i\left(\frac{-1}{4ND^2z},\, w;\, \chi\right) = i^{-1/2}\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{f}_i(z,w;\check{\chi})\left(\frac{z}{|z|}\right)^{1/2}. \quad (19)$$

Together with (3) and (4), (19) implies that

$$\sum_{\substack{r(\operatorname{mod} D)\\(r,D)=1}}\overline{\chi(r)}\check{f}_i|\left(\begin{smallmatrix} D & -r\\ -4mN & s\end{smallmatrix}\right)|\left(\begin{smallmatrix}1 & r/D\\ 0 & 1\end{smallmatrix}\right) = \sum_{\substack{r(\operatorname{mod} D)\\(r,D)=1}}\overline{\chi(r)}\left(\frac{4Nr}{D}\right)\epsilon_D^{-1}\check{f}_i|\left(\begin{smallmatrix}1 & r/D\\ 0 & 1\end{smallmatrix}\right). \quad (20)$$

Character summation then implies that

$$\check{f}_i\Big|\begin{pmatrix} D & -r\\ -4mN & s\end{pmatrix} = \left(\frac{4Nr}{D}\right)\epsilon_D^{-1}\check{f}_i, \quad (21)$$

or, with Lemma 2.1,

$$f_i\Big|\begin{pmatrix} s & m\\ 4Nr & D\end{pmatrix} = \left(\frac{4Nr}{D}\right)\epsilon_D^{-1}f_i. \quad (22)$$

However, the matrices on the left-hand side of (22) generate Γ:

Lemma 3.3. *([14]) Let $r\in\mathbf{Z}_+$. For D ranging in a set of congruence classes modulo $4Nr$ ($(D,4Nr)=1$) choose $\left(\begin{smallmatrix} s & m\\ 4Nr & D\end{smallmatrix}\right)\in\Gamma$. Denote the set of all such matrices by S_r. Then Γ is generated by*

$$\bigcup_{r=1}^{4N}S_r\cup\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\}\cup\{\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\}.$$

Therefore, f_i is Γ-invariant. The reason all matrices described in the lemma are captured by (22) is that the summands of $f(\cdot,\chi)$ are independent of the choice of representative of congruence class mod D. So, if $r\in\{1,\ldots,4N\}$ and $D\in\{1,\ldots,4Nr\}$ are such that $\left(\begin{smallmatrix} s & m\\ 4Nr & D\end{smallmatrix}\right)\in\Gamma$, then we can consider a restricted system of residues modulo D containing m. Using this specific representative in the definition of $f(\cdot,\chi)$ and $\tau_n(\chi)$ for every χ, we deduce (22) for $\left(\begin{smallmatrix} s & m \\ 4Nr & D \end{smallmatrix}\right)$.

• We will now prove that each f_i has a moderate growth at the cusps. The moderate growth at infinity is automatic from the Whittaker expansion, so we can focus on cusps other than infinity. We first note ([17], pg. 343) that, if $\mathrm{Re}(w) > \mathrm{sgn}\left(\frac{n}{4}\right) + 1$ and $|z| > 1$ and $|\arg z| \leq \pi - \alpha$ then

$$W_{\mathrm{sgn}\left(\frac{n}{4}\right),\, w-\frac{1}{2}}(z) = e^{-z/2} z^{\mathrm{sgn}\left(\frac{n}{4}\right)} \left(1 + O\left(z^{-1}\right)\right)$$

with the constant depending on α only and tending to infinity as $\alpha \to 0$. Therefore, for a fixed i, w with $\mathrm{Re}(w)$ large enough and $y < 1$,

$$\begin{gathered} |f_i(z,w) - (a_i(w)y^{1-w} + b_i(w)y^w)| \leq \sum_{n \neq 0} |a_n^i(w)| \cdot \left|W_{\mathrm{sgn}\left(\frac{n}{4}\right),\, w-\frac{1}{2}}(4\pi|n|y)\right| = \\ \sum_{|n| \leq \frac{1}{4\pi y}} |a_n^i(w)| \cdot \left|W_{\mathrm{sgn}\left(\frac{n}{4}\right),\, w-\frac{1}{2}}(4\pi|n|y)\right| + \sum_{|n| > \frac{1}{4\pi y}} |a_n^i(w)| e^{-2\pi|n|y} (4\pi|n|y)^{\mathrm{sgn}\left(\frac{n}{4}\right)} \left(1 + O\left(y^{-1}\right)\right). \end{gathered} \tag{23}$$

If A is the upper bound for $\left|W_{\mathrm{sgn}\left(\frac{n}{4}\right),\, w-\frac{1}{2}}(z)\right|$ for $|z| < 1$, the first sum is

$$<< A \sum_{|n| \leq \frac{1}{4\pi y}} |n|^C \leq (4\pi y)^{-C-1}.$$

Since $a_n^i(w) = O_w\left(|n|^C\right)$, the second sum is $\leq$

$$\left(1 + O\left(y^{-1}\right)\right) \max\nolimits_n (e^{-2\pi|n|y} (2\pi|n|y)^{C+2+\mathrm{sgn}\left(\frac{n}{4}\right)} y^{-C-2}) = O_w(y^{-C-3}).$$

Therefore, for $y < 1$, $f_i(z,w) << y^{C_1}$, uniformly in x for some C_1 depending on w. Since for every $\gamma \in SL_2(\mathbb{R})$, $\mathrm{Im}(\gamma z) = O(1/\mathrm{Im}(z))$ uniformly on $|\mathrm{Re}(z)| \leq 1/2$, this implies that for every w (with $\mathrm{Re}(w)$ large enough) and for every scaling matrix σ,

$$f_i(\sigma z, w) << y^C$$

as $y \to \infty$, uniformly in x, for some C depending only on w, as desired.

• By ([13]), Satz 10.1 (3), the Γ-invariance, and the moderate growth at the cusps just proved, it follows that, for $\mathrm{Re}(w)$ large enough $f_i(\cdot,w)$ is the sum of a weight 1/2 Maass cusp form $g_i(\cdot,w)$ and a linear combination of the Eisenstein series $E_j(\cdot,w)$ $(j = 1,\ldots,m^*)$. The cusp form g_i must in fact vanish for $\mathrm{Re}(w)$ large enough. Otherwise, it is an eigenfunction of the Laplacian with eigen-value $w(w-1)$ because it is a linear combination of f_i and E_j $(j = 1,\ldots,m^*)$. This is a contradiction because the discrete spectrum of $\Delta_{1/2}$ lies in $(-\infty,-3/16]$ ([13], Satz 5.5), but, for $\mathrm{Re}(w)$ large enough, $w(w-1)$ cannot be a real number $\leq -3/16$. Therefore, for $\mathrm{Re}(w)$ large enough there are functions $l_j^i : \mathbb{C} \to \mathbb{C}$ such that,

$$f_i(z,w) = \sum_j l_j^i(w) E_j(z,w) \quad \text{or}$$

$$\mathbf{f}(z,w) = A(w)\mathbf{E}(z,w), \tag{24}$$

where $A(w) := (l_j^i(w))_{i,j=1}^{m^*}$.

Now, if A is meromorphic, (24) gives the meromorphic continuation of $\mathbf{f}(z,w)$ to the whole complex w-plane. Therefore, with (10) and (6), (24) implies that for w such that w and $1-w$ are not poles of $A(w)$ and $\Phi(w)$(and thus $\mathbf{E}(z,w)$), we have

$$A(1-w)\mathbf{\Lambda}_E(z,1-w) = \mathbf{\Lambda}(z,1-w) = \Phi(1-w)\mathbf{\Lambda}(z,w) = \\ \Phi(1-w)A(w)\mathbf{\Lambda}_E(z,w) = \Phi(1-w)A(w)\Phi(w)\mathbf{\Lambda}_E(z,1-w). \quad (25)$$

After taking an inverse Mellin transform on both sides it follows that

$$A(1-w)\mathbf{E}'(z,1-w) = \Phi(1-w)A(w)\Phi(w)\mathbf{E}'(z,1-w),$$

where the primes indicate that the constant terms have been subtracted. The functions of z, $E_i(z,1-w)$ are linearly independent and this implies that the entries of $\mathbf{E}'(z,1-w)$ are linearly independent too (otherwise $ay^w + by^{1-w}$ would have to be modular of weight 1/2 for some a,b.) From this we deduce (7).

4 Scalar multiple Dirichlet series

The converse Theorem 3.1 assumes that a vector valued double Dirichlet series satisfies certain assumptions, the most important of which are the existence of infinitely many twisted functional equations. It is then shown that the vector valued double Dirichlet series is, in fact, the standard vector of metaplectic Eisenstein series. A natural question to ask is whether there should be a scalar version of the converse theorem involving scalar double Dirichlet series. This will happen if each $f_i (i = 1,2,\dots,m^*)$ of Theorem 3.1 turns out to be a multiple of E_i. A priori, there seems to be no reason for this because for all $A(w)$ for which (7) holds, $A(w)\mathbf{E}(z,w)$ satisfies the conditions of the theorem.

However, in this section we will show that for levels $N = 1,2$, and slightly more restrictive conditions, our converse theorem can yield information about the components of $\mathbf{f}$.

Theorem 4.1. *Let $N = 1,2$. Let $(a^i_{n,\ell})_{n,\ell,i}$ and $(\check{a}^i_{n,\ell})_{n,\ell,i}$, with $\ell, n \in \mathbb{Z}$, $\ell \geq 1$, $i = 1,\dots,m^*$ be sequences of complex numbers which are assumed to have polynomial growth in $|n|$ and ℓ as $|n|,\ell \to \infty$. Here, as before, m^* stands for the number of cusps of $\Gamma_0(N)$ with respect to which v is singular. Let $a_i(w), b_i(w), \check{a}_i(w), \check{b}_i(w)$, $(i = 1,\dots,m^*)$ be meromorphic functions of $w \in \mathbb{C}$ which are holomorphic for $Re(w)$ large enough and satisfy the assumptions of Theorem 3.1. In addition, suppose that $b(w)$, (resp. $\check{b}(w)$) are holomorphic functions satisfying*

$$b_i(w) = \delta_{i1}b(w) \qquad \textit{(resp. } \check{b}_i(w) = \delta_{im^*}\check{b}(w)\textit{)} \tag{26}$$

and

$$b(w) = b(1-w) \qquad \textit{(resp. } \check{b}(w) = \check{b}(1-w)\textit{).}$$

Then the functions

$$f_i(z,w) = a_i(w)y^{1-w} + b_i(w)y^w + \sum_{n\neq 0} a^i_n(w) W_{sgn(\frac{n}{4}),\, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x},$$

($i=1,\dots,m^*$) which were previously defined in Theorem 3.1 satisfy

$$f_i(z,w) = b(w)E_i(z,w),$$

for all w such that w and $1-w$ are not poles of $a_i(w)$ and $\Phi(w)$. Here $E_i(z,w)$ is the metaplectic Eisenstein series of $\Gamma_0(4N)$ as defined in §2.1.

Proof of Theorem 4.1. For $N=1$, we observe that $\Gamma_0(4)$ has only two singular cusps. Therefore, a comparison of the coefficients of y^w in (24) combined with (26), implies that $l_1^1(w)=b(w)$ and $l_1^2(w)=l_2^1(w)=0$ which gives the result.

When $N=2$, we have $m^*=3$. Because of (26), a comparison of the coefficients of y^w in (24) implies that $l_1^i(w)=\delta_{i1}b(w)$ and $l_{m^*}^i(w)=\delta_{im^*}\check{b}(w)$.

We will now show that $l_2^1(w)=0$. Set $\mathbf{b}(w)=(b_1(w),b_2(w),b_3(w))^T$ and likewise for $\check{\mathbf{b}}$, $\mathbf{a}$ and $\check{\mathbf{a}}$. Then, by (8) and (10) we deduce that

$$\begin{aligned}
&\Bigg[(2\sqrt{N}D)^{-s-1+w}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})}{1-w-s}\check{\mathbf{b}}(1-w)+\frac{\tau_0(\chi)}{1-w+s}\mathbf{b}(1-w)\right)- \\
&\qquad (2\sqrt{N}D)^{-s-w}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})}{-w+s}\check{\mathbf{a}}(1-w)+\frac{\tau_0(\chi)}{-w-s}\mathbf{a}(1-w)\right)\Bigg]- \\
&\qquad \Phi(1-w)\Bigg[(2\sqrt{N}D)^{-s-w}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})}{w-s}\check{\mathbf{b}}(w)+\frac{\tau_0(\chi)}{w+s}\mathbf{b}(w)\right) \\
&\qquad\quad -(2\sqrt{N}D)^{-s-1+w}\left(\frac{\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})}{-1+w+s}\check{\mathbf{a}}(w)+\frac{\tau_0(\chi)}{-1+w-s}\mathbf{a}(w)\right)\Bigg] \qquad (27)
\end{aligned}$$

must be entire. This implies that for all $w\neq\frac{1}{2}$ for which $\Phi(1-w)$ $\mathbf{a}(1-w)$ are defined,

$$\mathbf{a}(1-w)=\Phi(1-w)\mathbf{b}(w)=\Phi(1-w)\mathbf{b}(1-w)$$

otherwise (27) would have a pole at $s=-w$. (The possible pole at $s=-w$ would not be cancelled by other poles in (27) unless $w=1/2$). With (26), this implies that $a_i(w)=b(w)p_{i1}(w)$ for all $w\neq\frac{1}{2}$ for which $p_{i1}(w)$ is defined (and, by continuity, for $w=1/2$ too).

Likewise, we infer that $\check{\mathbf{a}}(w)=\Phi(w)\check{\mathbf{b}}(w)$ and then (26) again, shows that $\check{a}_i(w)=b(w)p_{im^*}(w)$.

Now a comparison of the coeffients of y^{1-w} in (24) immediately implies that $l_2^2(w)=b(w)$ and thus, $f_2(z,w)=b(w)E_2(z,w)$. Next, together with the already proved $l_1^1(w)=b(w)$ and $l_3^1(w)=0$, a comparison of the coefficients of y^{1-w} gives $l_2^1=0$ and thus, $f_1(z,w)=b(w)E_1(z,w)$. Likewise, $f_3(z,w)=b(w)E_3(z,w)$.

In the case of $\Gamma_0(4)$, we can prove a genuine scalar converse theorem. Since this theorem is, in a sense, a converse theorem for Siegel's multiple Dirichlet series ([15]), we slightly modify our notation to conform to the formalism of ([15]). We set

$$j_{\frac{1}{2}}(\gamma,z)=v(\gamma)(cz+d)^{1/2}.$$

For every $\gamma, \delta \in \Gamma$ and $z \in \mathfrak{H}$ we have

$$j_{\frac{1}{2}}(\gamma\delta, z) = j_{\frac{1}{2}}(\gamma, \delta z) j_{\frac{1}{2}}(\delta, z).$$

The group $\Gamma_0(4)$ now acts on functions f on $\mathfrak{H}$ by

$$(f|_{\frac{1}{2}}\gamma)(z) := f(\gamma z) j_{\frac{1}{2}}(\gamma, z)^{-1}, \qquad \gamma \in \Gamma_0(4).$$

Further, we will expand eigenfunctions of $\Delta_{1/2}$ in terms of the functions $y^s K_n(s,y) e^{2\pi i n x}$ where

$$K_n(s,y) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n x}}{(x^2+y^2)^s (x+iy)^{1/2}} dx.$$

This is equivalent to the expansions in terms of $W_{\mathrm{sgn}\frac{n}{4}, w-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}$ used in earlier sections because of (69) of [7] (where though it should be taken into account that $W_{a,s}$ is denoted by $W_{a,s+1/2}$) Finally we set

$$G^+(s,t) = \frac{F((s+t)/2, (s-t+1)/2, (s+1)/2; 1/2)}{\Gamma((t+1)/2)\Gamma((s+1)/2)}$$

and

$$G^-(s,t) = \frac{F((s+t)/2, (s-t+1)/2, (s+2)/2; 1/2)}{\Gamma(t/2)\Gamma((s+2)/2)}.$$

With this notation, we obtain the following theorem.

Theorem 4.2. *Let $(a_{n,m})_{n \in \mathbb{Z}, m \geq 1}$ be a sequence of complex numbers of polynomial growth in $|n|, m$ as $|n|, m \to \infty$. For w with Re(w) large enough, set*

$$a_n(w) = \sum_{m=1}^{\infty} \frac{a_{n,m}}{m^w}$$

and assume that, for each fixed w, $a_n(w) = O(|n|^C)$ $(C > 0)$, as $n \to \infty$. For each w with Re(w) large enough, consider the pair of functions $L_w^+(s)$ $L_w^-(s)$ represented by

$$L_w^+(s) = \sum_{n>0} \frac{a_n(w)}{n^{\frac{s-w+1}{2}}}, \qquad L_w^-(s) = \sum_{n<0} \frac{a_n(w)}{(-n)^{\frac{s-w+1}{2}}}$$

for Re(s) large enough and set

$$\Lambda_w(s) = \frac{e^{-\pi i/4}\Gamma((s-w+1)/2)\Gamma((s+w)/2)}{2^{s-1/2}\pi^{(s-w-1)/2}} \Big(G^+(s,w) L_w^+(s) + G^-(s,w) L_w^-(s) \Big)$$

and

$$\tilde{\Lambda}_w(s) = i \frac{e^{-\pi i/4}\Gamma(\frac{s-w+3}{2})\Gamma(\frac{s+w+2}{2})}{2^{s+\frac{1}{2}}\pi^{\frac{s-w-1}{2}}} \Big(G^+(s+2,w) L_w^+(s) - G^-(s+2,w) L_w^-(s) \Big) - \frac{i}{4}\Lambda_w(s).$$

Let $a : \mathbb{C} \to \mathbb{C}$ be a meromorphic function which is holomorphic for Re(w) large enough and let $b : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying $b(s) = b(1-s)$. Assume that all the above functions satisfy the assumptions (29), (30), (31), (32) listed below. Then, for

$$f(z,w) = a(w)y^{(1-w)/2} + b(w)y^{w/2} + \sum_{n\neq 0} a_n(w)y^{w/2}K_n(w/2,y)e^{2\pi inx},$$

we have

$$f(z,w) = -(2e^{-\pi i/2})^{-1/2}b(w)z^{-1/2}E\left(-\frac{1}{4z},\frac{w}{2}\right) + b(w)E\left(z,\frac{w}{2}\right),$$

for each $w \in \mathbb{C}$ for which w, $1-w$ are not poles of $a(w)$. Here

$$E(z,w) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} \frac{\mathrm{Im}(\gamma z)^s}{j_{\frac{1}{2}}(\gamma,z)}.$$

Assumptions: We shall assume that the functions

$$\Lambda_w(s) - a(w)2^{(1-s+w)/2}\left(\frac{1}{2-s-w}+\frac{1}{w-s-1}\right) - b(w)2^{(2-s-w)/2}\left(\frac{1}{w-s+1}-\frac{1}{w+s}\right) \tag{28}$$

and

$$\tilde{\Lambda}_w(s) + a(w)i2^{(w-s-3)/2}\left(\frac{1}{w-s-1}-\frac{1}{2-w-s}\right) + b(w)i2^{(-2-s-w)/2}\left(\frac{1}{s-w-1}-\frac{1}{w+s}\right) \tag{29}$$

are EBV for every w.

Further, assume that

$$\Lambda_w(s) = -2^{-s+1/2}\Lambda_w(1-s) \qquad \text{and } \tilde{\Lambda}_w(s) = 2^{-s+1/2}\tilde{\Lambda}_w(1-s) \tag{30}$$

and

$$G(w)\Lambda_w(s) = G(1-w)\Lambda_{1-w}(s) \tag{31}$$

for $G(w) = (2^w-1)\xi(2w)$ and $\xi(s) := \zeta(s)\Gamma(s/2)\pi^{-s/2}$.

Proof of Theorem 4.2. For every w with Re(w) large enough and every $y > 0$ define,

$$F_w(y) = \sum_{n\neq 0} a_n(w)y^{w/2}K_n(w/2,y),$$

and

$$\tilde{F}_w(y) = 2\pi i\sum_{n\neq 0} na_n(w)y^{w/2+1}K_n(w/2,y) - \frac{i}{4}F_w(y).$$

As in the proof of Th. 3.1, we see that, for s with Re(s) large enough, we have

$$\int_0^\infty y^{s/2}F_w(y)\frac{dy}{y} = \Lambda_w(s) \qquad \text{and } \int_0^\infty y^{s/2}\tilde{F}_w(y)\frac{dy}{y} = \tilde{\Lambda}_w(s),$$

and that in the inverse Mellin transform of $\Lambda_w(2s)$ we can move the line of integration from σ_0 to $\sigma_1 = 1/2 - \sigma_0$ to get

$$F_w(y) = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \Lambda_w(2s) y^{-s} ds + \sum_{s_0 \text{ pole}} \mathrm{Res}_{s=s_0} \Lambda_w(2s) y^{-s} = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \Lambda_w(2s) y^{-s} ds$$
$$- a(w) y^{(w-2)/2} 2^{w-3/2} - b(w) y^{-(w+1)/2} 2^{-w-1/2} - a(w) y^{-(w-1)/2} - b(w) y^{w/2}. \quad (32)$$

This, together with the functional equation of Λ_w and

$$f(iy, w) = F_w(y) + b(w) y^{w/2} + a(w) y^{(1-w)/2}$$

implies that

$$f(iy, w) = -2^{1/2}(4y)^{-1/2} f(i/(4y), w). \quad (33)$$

We show that this holds on the entire upper half-plane and not just on the positive imaginary axis. We require the following lemma

Lemma 4.3. *For every $w \in \mathbb{C}$ with Re(w) large enough, we have*

$$\frac{\partial}{\partial x}\left(f(-1/4z, w) + (2z/i)^{1/2} f(z, w)\right)\Big|_{x=0} = 0. \quad (34)$$

Proof. Set

$$G(y) = y\frac{\partial f}{\partial x}(iy, w) - \frac{i}{4} f(iy, w).$$

As in Lemma (15), we observe that (34) holds if and only if

$$G(1/4y) = (2y)^{1/2} G(y).$$

We also have

$$G(y) = \tilde{F}_w(y) - \frac{i}{4}\left(a(w) y^{(1-w)/2} + b(w) y^{w/2}\right).$$

Therefore, to prove (34), it suffices to prove

$$\tilde{F}_w(\frac{1}{4y}) - \frac{i}{4}\left(a(w)(4y)^{\frac{w-1}{2}} + b(w)(4y)^{-\frac{w}{2}}\right) = (2y)^{\frac{1}{2}}\left(\tilde{F}_w(y) - \frac{i}{4}(a(w) y^{\frac{-w+1}{2}} + b(w) y^{\frac{w}{2}})\right). \quad (35)$$

To verify this we move the line of integration in the inverse Mellin transform of $\tilde{\Lambda}_w(2s)$ as before to get

$$\tilde{F}_w(y) = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \tilde{\Lambda}_w(2s) y^{-s} ds$$
$$+ a(w) i 2^{-2} y^{(1-w)/2} - a(w) i 2^{w-7/2} y^{w/2-1} - b(w) i 2^{-w-5/2} y^{-(w+1)/2} + b(w) i 2^{-2} y^{w/2}. \quad (36)$$

As before, by an application of the functional equation of $\tilde{\Lambda}_w$ we deduce (35) and thus (34). $\square$

This lemma, together with (33) and Lemma 13.5.2 of [9], implies that

$$f(-1/(4z), w) = -(2i)^{-1/2}(4z)^{1/2} f(z, w). \quad (37)$$

However,
$$\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1/4 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right).$$
An easy computation and (37), with the aid of Lemma 2.1 then imply that, for all w with $\mathrm{Re}(w)$ large enough, $f(-, w)$ is invariant under $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ in terms of the weight 1/2 action. Since $\Gamma_0(4)$ is generated by $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and the translations, this proves that, for all w with $\mathrm{Re}(w)$ large enough, $f(z, w)$ satisfies the weight 1/2 transformation law for $\Gamma_0(4)$.

To check the growth of $f(z, w)$ at the cusps in terms of z, we first note that, by construction, it has a moderate growth at infinity. For the other cusps, because of (69) of [7], the argument we used to prove moderate growth in Theorem 3.1 applies here too to deduce moderate growth for $\mathrm{Re}(w)$ large enough.

By (28) and (31) we deduce that

$$2^{(1-s+w)/2}\left(\frac{1}{2-s-w} - \frac{1}{1+s-w}\right)\Big(a(w)G(w) - b(1-w)G(1-w)\Big) \\ + 2^{(2-s-w)/2}\left(\frac{1}{w-s+1} - \frac{1}{s+w}\right)\Big(b(w)G(w) - a(1-w)G(1-w)\Big) \quad (38)$$

must be entire. This implies that for all w with $\mathrm{Re}(w)$ large enough, and such that $1-w$ is not a pole of $a(w)$

$$a(w)G(w) = b(1-w)G(1-w) = b(w)G(1-w),$$

otherwise (38) would have a pole at $s = 2-w$. Thus, the constant term of $f(z, w)$ at infinity is
$$b(w)\left(y^{\frac{w}{2}} + \frac{G(1-w)}{G(w)}y^{\frac{1-w}{2}}\right). \quad (39)$$

Now, by ([13]), Satz 10.1 (3), the Γ-invariance and the moderate growth at the cusps we proved, $f(\cdot, w)$ is the sum of a weight 1/2 Maass cusp form g and a linear combination of the Eisenstein series of weight 1/2 at the cusps of $\Gamma_0(4)$ that are singular in terms of v. One easily sees that the singular cusps are 0 and ∞. Also, using the same argument we used in the proof of (24), we see that g must vanish and thus

$$f(z, w) = \alpha(w)z^{-1/2}E(-1/(4z), w/2) + \beta(w)E(z, w/2) \quad (40)$$

for some functions α and β. However, the constant terms at infinity of $E(z, w/2)$ and $z^{-1/2}E(-\frac{1}{4z}, w/2)$ are

$$y^{w/2} + \frac{2^{-2w}}{1-2^{-2w}}\frac{\xi(2w-1)}{\xi(2w)}y^{\frac{1}{2}-w/2} \quad \text{and} \quad \frac{e^{-\pi i/4}(1-2^{1-2w})}{2^{w-1/2}(1-2^{-2w})}\frac{\xi(2w-1)}{\xi(2w)}y^{\frac{1}{2}-w/2},$$

respectively (cf. [8]). Therefore, upon comparison of the coefficients of $y^{w/2}$ on both sides of (40) we deduce that $\beta(w) = b(w)$. An elementary computation then implies that $\alpha(w) = -(2e^{-\pi i/2})^{-1/2}b(w)$, which implies the result.

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