Notes for Learning Seminar on Conformal Field Theory (Fall 2019)

Henry Liu

October 1, 2019


Abstract
These are my live-texed notes for the Fall 2019 student learning seminar on conformal field theory. Let me know when you find errors or typos. I’m sure there are plenty.

Contents

1 Kostya (Sep 24): Intro to CFT
2 Davis (Oct 01): The local structure of CFT

1 Kostya (Sep 24): Intro to CFT

This will be an informal talk from a physics perspective. We’ll try to make some mathematical connections, but there will be very few mathematical statements. The main purpose is to give a feeling of how CFT formalism appeared and why.

We’ll follow the paper by Belavin, Polyakov and Zamolodchikov from ’84. This paper actually arise as an attempt to understand an earlier paper in ’81 by Polyakov about string theory. String theory is a theory on Riemann surfaces \((\Sigma, g)\), with or without boundary, genus, and marked points. It feels as few of the features of the surfaces as possible, because we want it to embed into spacetime \(\mathbb{R}^{1,3}\). The theory should be:

1. covariant, i.e. independent of all coordinate transformations;
2. conformal, i.e. the Lagrangian should only depend on the conformal class of the metric \(g\).

For example, for a quantum particle, there is a worldline and a map from the worldline to \(\mathbb{R}^{1,3}\). In principle when we write a Lagrangian for the particle, it depends on the metric of the worldline. But in string theory we don’t want this dependence.

The partition function \(Z\) of string theory looks like

\[
Z = \int_{\text{all } (\Sigma, g)} DX Dg \exp \left( \frac{1}{\hbar} \left[ S_{\text{matter}}[X] + S_{\text{gravity}}[g] \right] \right).
\]

In general, we think of \(DX\) as a measure on something like \(\Gamma(E)\) where \(E\) is some bundle over the moduli of some \((\Sigma, g)\). When we consider conformal matter, gauge fixing for diffeomorphisms turns this action into

\[
Z = \int_{\text{conformal classes}} Dg \exp \left( \frac{1}{\hbar} \left[ S_{\text{matter}}[X] + S_{\text{gravity}}[g] + S_{\text{gauge fixing}} \right] \right).
\]

In a particular case, called critical strings, the gravity term \(S_{\text{gravity}}[g]\) is trivial and therefore we get a theory that only depends on the topology of \(\Sigma\). In the critical case, the central charge of matter is
c_{\text{matter}} = 26; in the supersymmetric version $c_{\text{matter}} = 15$. But in general we can consider arbitrary central charge, giving different gravity theories. One such example is Liouville gravity. Classically it coincides with the Liouville equation, which is also conformal. The most useful result in this direction appeared in the ’84 paper, which showed that CFTs could actually be solved. However the technique did not work for Liouville theory, which was worked out about a decade later.

What is a CFT? The first thing to discuss we should discuss, which is very particular to CFTs, is the state-operator correspondence. First forget about conformal structure and think only about topological theories. For such 2d theories there are Atiyah–Segal axioms, which basically say that such a theory is a functor

$$F : \text{Cob} \rightarrow \text{Vect}$$

from the category of cobordisms to the category of vector spaces. Boundaries are always disjoint unions of circles $S^1$, and

$$F(S^1) = \mathcal{H} = (\text{state space of the theory}).$$

For a cobordism represented by a 2d manifold $\Sigma$ with boundary, we get

$$F(\Sigma) = v \in F(\partial \Sigma).$$

For example, if $\Sigma$ is a cylinder, its boundary is two circles with different orientation, i.e.

$$\partial \Sigma = S^1 + (S^1)^{\text{op}}.$$ 

Part of the axioms say that orientation is encoded by dualizing, so

$$F(\text{cylinder}) \in \mathcal{H} \otimes \mathcal{H}^\vee$$

which we can view as an operator. This is actually the identity operator of the theory.

A state is something living on the boundary. So it is a vector in the Hilbert space $\mathcal{H}$. For example, taking the cylinder, we can plug in two states to get a correlation number of the theory. Another nice way to think of a state is, again, in the path integral formalism. For a non-topological theory, think of $\mathcal{H}$ as $L^2(S^1)$ in some sense. Say $\Sigma$ is a disk $D^2$. Then we can take expectation values

$$Z = \int DX \exp \frac{1}{\hbar} S \in \mathbb{C}$$

where the action is an integral over a disk

$$S = \int_\Sigma d^2 \xi \mathcal{L}[X, \partial X].$$

The point is that this integral is not well-defined unless we specify boundary conditions on the disk. A boundary condition is e.g.

$$X(e^{i\varphi}) = X_0(e^{i\varphi}).$$

A (local) operator are differential polynomials (or slight generalization) of the basic fields of the theory. (Usually operators in the CFT context are always local.) An example of a local operator would be something like $\partial X : X^3$. We can also think about non-local operators, e.g. $\langle X(z_0) X(z_1) \rangle$. They depend not on one point, but on multiple.

At the classical level, an example is as follows. Let $X(z) \in \mathbb{C}$ be a section of the trivial line bundle. After quantization, in the Hamiltonian formalism, it will be an operator. Then we can compute correlation functions such as

$$\langle X(z_0) \rangle = \int DX \cdot X(z_0) \exp \frac{1}{\hbar} S.$$

This defines an element of $\mathcal{H}^\vee$ associated to the same boundary of the disk, because the action still requires a boundary condition on the disk. Hence local operators define states. In fact any state in CFT can be obtained in this way.
In general, think of the operator-state correspondence as follows. Consider a cylinder, infinitely long in
the negative direction.

At the negative end, plug in a state $|\phi\rangle$. It produces a state $e^{itH}|\phi\rangle$ on the other $S^1$. But since the theory
is conformal, we can convert this picture into a disk

The boundary on the lhs of the cylinder is mapped to a puncture at the origin, and the boundary on the rhs
becomes the boundary $S^1$. Then the object at the puncture is a local object $A(0)$, which is a local operator
at 0. This is in general called \textbf{radial quantization}.

An important consequence is the following. Take a sphere with three punctures. Put two states $|\phi\rangle$ and
$|\psi\rangle$ in two of the punctures. Then we get an element $|v\rangle \in \mathcal{H}$ associated to the third puncture. Apply the
state-operator correspondence. Pick a basis $\{\phi_\alpha\}_{\alpha \in I}$ for $\mathcal{H}$ and decompose

$$|v\rangle = \sum C_\gamma^\alpha \phi_\alpha(z)\phi_\gamma(w)$$

for some coefficients $C$. Hence we get

$$\phi_\alpha(z)\phi_\beta(z) = \sum \gamma C_\gamma^\alpha \phi_\gamma(z)\phi_\beta(w).$$

This gives an \textbf{operator algebra} structure to $\mathcal{H}$. In general, this expression can be put into any correlation
function, as

$$\langle A(z_1, \ldots, z_N)\phi_\alpha(z)\phi_\beta(w) \rangle = \langle \sum \gamma C_\gamma^\alpha \phi_\gamma(w) A(z_1, \ldots, z_N) \rangle.$$ 

Associativity of such a product structure imposes constraints on $C_\gamma^\alpha \beta$, and allows us to solve for them. This
is the \textbf{bootstrap} approach to solving CFTs.

A very important object to work with in CFT is the \textbf{stress-energy tensor}. It is the generator of time
transformations. If we think about a cylinder, pick some slice $t = 0$ in it. The Hamiltonian is

$$H(t) = \int_{t=t_0} dt T_{0,0}(\xi, \bar{\xi}).$$

In classical field theory, the Hamiltonian generates time translations, i.e.

$$\{H(t), X\} = \partial_t X.$$ 

In CFT we can get much more. The theory being conformal is equivalent to the stress-energy tensor $T_{ab}$
being traceless, i.e. $T_{a}^{a} = 0$. Noether’s theorem gives $\partial_a T_{a}^{a} = 0$. In complex coordinates, these two properties
become:

$$T_{zz} = \tr T = 0, \quad \bar{T}_{zz} = 0, \quad \partial T_{zz} = 0.$$ 

Due to this, we write the holomorphic and anti-holomorphic functions

$$T(z) = T_{zz}(z), \quad \bar{T}(\bar{z}) = T_{zz}(\bar{z}).$$
They are rank-2 tensors, so they transform like

\[ T(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^2 T(\xi). \]

After quantization, \([H(t), X] = \partial_t X\), so

\[ \int_{t=t_0} T(z), X \] = \partial_z X.

But more generally we can use any complex transformation \( \epsilon(z) = \sum_n \epsilon_n z^n \), to get

\[ \int dx \epsilon(z) T(z), X \] = \delta_{\epsilon} X.

In radial quantization, this commutator becomes just

\[ \partial \epsilon X = \int dt \epsilon(z) T(z) X(z_0) |0\rangle \]

where \(|0\rangle\) is the state inserted at 0. This is called a Ward identity.

A field \( \phi \) is primary if it transforms as

\[ \phi_\alpha(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^{\Delta_\alpha} \left( \frac{\partial \xi}{\partial \zeta} \right)^{\Delta_\alpha} \phi_\alpha(\xi), \]

i.e. it is a “tensor” of rank \((\Delta_\alpha, \overline{\Delta}_\alpha)\), called the conformal dimension, which can be non-integers. For such fields, the operator product expansion is

\[ T(z) \phi_\alpha(w, \overline{w}) = \frac{\Delta_\alpha}{(z-w)^2} \phi_\alpha + \frac{\partial_w \phi_\alpha}{z-w}. \]

However it turns out that \( T \) is not a primary field. (In the classical theory it of course is, because it is a rank-2 tensor. But in the quantum theory it is not.) It transforms as

\[ T(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^2 T(\xi) + \frac{c}{12} \{ \xi, z \} \]

where the second (correction) term is the Schwartz derivative and \( c \) is a constant called the central charge. One can try to understand this term physically in many ways, but it is believed to be the most possible such expression. Using this transformation law with the Ward identities, we get

\[ T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \]

Out of this, we can get the Virasoro algebra as follows. Formally decompose

\[ T(w) = \sum \frac{L_n}{z^{n+2}}. \]

(The +2 here is because of the conformal dimension 2.) Then

\[ \int dz z^{n+1} T(z) = L_n. \]

Applying this to the operator product expansion, we get Virasoro relations

\[ [L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m}. \]
Note that if we define vector fields $\ell_n := z^{n+1}\partial_z$, we get just the first term in this commutator:

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}.$$ 

This comes from the relation between the classical stress-energy tensor and the Hamiltonian. Note also that this means $\{L_{-1}, L_0, L_1\}$ span an SL$_2$ subalgebra.

Using the OPE $T(z)\phi_0(0)$, one can show that the state operator correspondence $\phi(z) \rightarrow |\phi\rangle = \phi(0)|0\rangle$ sends the operator $\phi$ to a highest-weight vector for the Virasoro algebra. In other words,

$$L_n|\phi\rangle = 0, \quad n > 0$$

and the highest weight is exactly the conformal weight $\Delta$. Generically,

$$L_{-n_1} \cdots L_{-n_k}|\phi\rangle, \quad n_1 < \cdots < n_k$$

are independent. By the state-operator correspondence, they can be represented by some contour integrals. These are descendants of the primary field $\phi(0)$.

To sum up in representation-theoretic language, the entire Hilbert space $\mathcal{H}$ decomposes into a sum of irreducible highest weight representations of the Virasoro. Primary fields are highest weight vectors, and descendants are generated by primary fields. Correlation functions for all descendants are computed through the ones for primary fields, precisely because we know the OPE of the stress-energy tensor with any field. So to study CFTs, we only care about primary fields.

If in the decomposition $\mathcal{H} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{V}_{\alpha}$ the indexing set $\mathcal{A}$ is finite, then we have a finite number of primaries and the CFT is called rational. Also, when

$$\text{Vir} \subset U(\mathfrak{g})$$

for some $\mathfrak{g}$, e.g. Heisenberg, loop/affine algebra (like WZW theory), then we can ask about reps of this bigger symmetry algebra $U(\mathfrak{g})$. Examples include minimal models, WZW theories, KS models.

Another consequence of the Ward identities, which leads to differential equations for conformal blocks, comes from considering

$$\langle T(z)\phi_{\alpha_1}(z_1)\cdots\phi_{\alpha_n}(z_n) \rangle = \sum_{i=1}^n \frac{\Delta_i}{(z - z_i)^2} + \frac{\partial/\partial z_i}{z - z_i} \times \langle \phi_{\alpha_1}(z_1)\cdots\phi_{\alpha_n}(z_n) \rangle.$$

What is a conformal block? Recall the product $\phi_1(z)\phi_2(w) = \sum C^\alpha_{12}(z)\phi_\alpha(0)$. Decompose $\phi_\alpha$ into primary/descendants to get

$$\phi_1(z)\phi_2(z) = \sum_{\alpha} C^\alpha_{12} \sum_{\{k\}} \beta^{\alpha,\{k\}}_{12} z^{\Delta_1 - \Delta_2 + \sum k_i} L_{-n_1} \cdots L_{-n_k} \phi^\alpha(0).$$

Apart from the structure constants $\beta^{\alpha,\{k\}}_{12}$, everything else is determined by conformal symmetry, e.g.

$$\langle \phi_\alpha(z) \rangle = 0 \quad \text{if } \phi_\alpha \neq \text{id}$$

$$\langle \phi_\alpha(z)\phi_\beta(w) \rangle = \frac{\delta_{\alpha\beta}}{|z - w|^\Delta}$$

$$\langle \phi_\alpha(z)\phi_\beta(w)\phi_\gamma(\xi) \rangle = \frac{C_{\alpha\beta\gamma}}{|z - w|^\Delta |w - \xi|^\Delta |z - \xi|^\Delta}$$

$$\langle \phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0) \rangle = \sum_p C^p_{12} C^p_{34} \sum_{\{k\}} z^{-\sum \beta^{p,\{k\}}_{12}} \beta^{p,\{k\}}_{34} \langle \phi_p|L_{-n_1} \cdots L_{-n_k}|\phi_p\rangle.$$

The only unknowns here are the structure constants $C^p_{12}$ and $C^p_{34}$. In the BPZ paper, the four-point correlators are constructed in terms of objects called conformal blocks $F^4_{12}(p|z)$, as

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_p C^p_{12} C^p_{34} |F^4_{12}(p|z)|.$$
These are universal quantities which are fully determined by representation-theoretic considerations. In all rational theories, they satisfy differential equations called BPZ equations coming from the Ward identities. (In WZW models they are called KZ equations, coming from some rep theory of $U(g)$.) However, in Liouville theory we cannot do this, and there are other things to work with.

### 2 Davis (Oct 01): The local structure of CFT

*(Notes by Davis)*

In this talk, I’ll try to explain how vertex operator algebras are related to the local structure of conformal field theory. I’ll motivate VOAs from this point of view, explain what the Virasoro algebra is from a mathematically motivated perspective, then explain how the Heisenberg Lie algebra is related to the free boson CFT. Time permitting, I hope to say something about the OPE, or about conformal blocks.

In 2D CFT, a general correlator on a manifold $\Sigma$, 

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_{\Sigma},$$

can be viewed as fields $\phi_1, \ldots, \phi_n$ inserted at points $z_1, \ldots, z_n$ on the manifold.

Part of the content of Kostya’s talk was justifying that we could understand this picture by breaking apart $\Sigma$ into smaller pieces and cobording them together. Hence, we may break apart $\Sigma$ into pieces and assume all the fields live together on a genus zero piece. Studying this genus zero piece is what I will call the local structure of CFT: understanding higher genus pieces, and how they glue together, will be beyond this talk.

#### 2.1 The standout objects of CFT on the circle

Kostya’s talk yesterday introduced a lot of important objects in the study of CFT on a genus zero surface with boundary,

1. The space of states (boundary conditions)
2. The OPE, $C^\alpha(z)C^\beta(w) = \sum C^\gamma_{\alpha\beta}(z,w)C^\gamma(z)$
3. The state operator correspondence
4. The stress energy tensor, $T_{\mu\nu} \rightarrow T(z), \overline{T}(\bar{z})$.

**Claim.** On the circle, for a large class of CFTs called ‘chiral CFTs’, the “holomorphic and antiholomorphic sectors decouple” and we may study the holomorphic part of the CFT independently. I will justify this statement, and explain it in more detail, shortly, but for now take it on faith.

#### 2.2 VOAs

Formalising these ideas, observing that we may derive the OPE from the state-operator correspondence, we arrive at the long definition of a VOA.

**Definition 2.1. A vertex operator algebra**

1. A space of states $V$ (a vector space)
2. A vacuum vector $|0\rangle \in V$
3. A translation operator $T : V \rightarrow V,$
4. Vertex operators, $Y(\bullet, z) : V \rightarrow \text{End}_V[[z, z^{-1}]],$ sending $A \rightarrow Y(A, Z),$ a field.
such that
1. $Y(|0\rangle, z) = id$
2. $Y(A, z) |0\rangle \in V[[z]], Y(A, z) |0\rangle_{z=0} = A$;
3. $[T, Y(A, z)] = \partial_z Y(A, z), T |0\rangle = 0$;
4. The $Y(A, z)$ are pairwise local.

where
1. $\sum A_j z^{-j} = A(z) \in EndV[[z, z^{-1}]]$ is a field if for any $v \in V$, $A_j v = 0$ for $j$ large enough.
2. Two fields $A(z), B(w)$ are local pairwise if $\exists N \geq 0, (z - w)^N [A(z), B(w)] = 0$.

Where did the stress energy tensor go in this definition? We’ll see after we’ve discussed Virasoro.

I don’t want to get into the weeds of formal power series, and BenZvi/Frenkel’s book covers this background well. But here’s one example:

Example 2.2. The formal delta function, $\delta(z - w)$, is the power series

$$\delta(z - w) = \sum z^m w^{-m - 1}.$$  

It satisfies the property

$$A(z) \delta(z - w) = \sum A_k w^k z^m w^{-m - 1} = \sum A_{m+n+1} z^m w^n = A(w) \delta(z - w).$$

In particular, $z \delta(z - w) = w \delta(z - w) \implies (z - w) \delta(z - w) = 0$.

The easiest VOAs are commutative ones.

Example 2.3. Suppose $[Y(A, z), Y(B, w)] = 0$ for all $A, B$. Then

$$Y(A, z) B = Y(A, z) Y(B, w) |0\rangle_{w=0} = Y(B, w) Y(A, z) |0\rangle_{w=0}.$$  

Because this is true for all $B$ and $Y(A, z) |0\rangle \in EndV[[z]]$ by the axioms, this means $Y(A, z) \in EndV[[z]]$ for all $A$.

So we can equip the VOA with the structure of a commutative algebra,

$$A \ast B := Y(A, 0) B.$$  

$T$ is a derivation for this algebra.

In the other direction, with mild assumptions, $Y(A, z) := e^{zT} A$ makes an algebra with a derivation into a VOA.

2.3 Virasoro

Our first examples of interesting vertex operators will come from the Virasoro algebra.

Recall that we have identified states with boundary conditions. So $Diff(S^1)$ acts on states on the disc. We really want the ‘complexified’ $Diff(S^1)$ action. However, $Diff(S^1)$ admits no complexification as a Lie group. (There is a semigroup which serves as a partial answer, as exposited in Andre Henriques cobordism-centred CFT notes).

So, we look at its Lie alg and complexify. Complexified $LieDiff(S^1)$ has a basis

$$\ell_n = -z^{n+1} \partial_z$$
$$\ell_n = -z^{n+1} \partial_z$$

$$[\ell_n, \ell_m] = (n - m) \ell_{n+m}.$$  

$$[\ell_n, \ell_m] = (n - m) \ell_{n+m}$$

So, $(LieDiff(S^1))_c = Witt \oplus Witt$, where $Witt$ has basis $\{\ell_n\}$.  

7
Remark. This is what I mean by decoupling of holomorphic/antiholomorphic sectors. The assumption here is that the \( \text{Diff}(S^1) \) global action, whatever I really mean by this, upgrades to a local action: this the definition of a chiral CFT.

Primary fields, for instance, are ones where this action so upgrades. For general CFTs, you can only use the VOA to study the **chiral sector**, which basically means fields coming from primary fields.

**Goal.** We want to study projective representations of the Witt algebra, because scaling by a constant doesn’t change the physical state. The standard yoga is that projective representations of Witt are ordinary reps of centrally extended Witt.

**Definition 2.4.** The **Virasoro algebra** is \([L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}\).

It is the universal central extension of the Witt algebra.

**Remark.** We can now write a generating functional

\[
T(z) = \sum L_n z^{-n-2}
\]

\[
[T(z), T(w)] = \frac{c}{12} \partial^n \delta(z - w) + 2T(w)\partial_n \delta(z - w) + \partial_n T(w) \delta(z - w)
\]

and this relation is equivalent to the \([L_n, L_m]\) relation. So, the stress-energy tensor pops up!

Finally, we can make a VOA

**Example 2.5.** Let \( \text{Vir}_c = U(Vir) \otimes_U (L_{\text{Diff}}(S^1)) \subset \mathbb{C}_c \), i.e. the universal enveloping algebra of Virasoro with the element \( c \) sent to be constant number \( c \).

This algebra has a basis

\[
L_{j_1} \ldots L_{j_m} |0\rangle
\]

where \( j_1 \leq \cdots \leq j_m \leq -2 \)

we can declare \( T = L_{-1}, Y(L_{-2} |0\rangle, z) := T(z) \), and other vertex operators ‘generated from this’.

What we mean by ‘generated by this’ calls for another messy theorem which will let us practically define a lot of VOAs.

**Theorem 2.6.** Let \( V \) a vector space, \(|0\rangle \neq 0 \in V, T \in \text{End}(V)\). Let \( \{a^\alpha\}_{\alpha \in \mathbb{Z}} \) vectors, and \( \{a^\alpha(z)\} \) fields so that

1. \( a^\alpha(z) |0\rangle = a^\alpha + O(z^{\geq 1}) \)
2. \( T |0\rangle = 0, [T, a^\alpha(z)] = \partial_z a^\alpha(z) \)
3. \( a^\alpha(z) \) are pairwise local;
4. \( V \) has a basis of vectors \( a^{\alpha_1}_{(j_1)} \ldots a^{\alpha_m}_{(j_m)} |0\rangle \), with \( j_1 \leq \cdots \leq j_m < 0 \) and \( j_i = j_{i+1} \Rightarrow \alpha_i = \alpha_{i+1} \)

then, we can equip \( V \) with a VOA structure by

\[
Y(a_{j_1}^{\alpha_1} \ldots a_{j_m}^{\alpha_m} |0\rangle, z) = \frac{1}{(-j_1 - 1)! \ldots (-j_m - 1)!} : \partial_z^{-j_1 - 1} a_{\alpha_1} \ldots a_{\alpha_m} (z) :
\]

where \( AB : \) denotes **normal ordering**: if \( A(z) = \sum A_m z^{-m-1}, B(w) = \sum B_n w^{-n-1}, \) we define \( A(z)B(w) : \) to be \( \sum_n (\sum_{m<0} A_m B_n z^{-m-1} + \sum_{m \geq 0} B_n A_m z^{-m-1}) w^{-n-1} \). We then extend \( : AB : = A : BC : \).

Normal ordering has a weird definition, basically it means removing the singular point as \( z \to w \). In physics speak, we demand that vacuum expectation values of normal ordered correlators vanish.

I hope to explain this theorem’s formula via the OPE, time permitting.
Example 2.7. So in the Virasoro case, we may write $Y(L_{j_1} \ldots L_{j_m} |0 \rangle, z)$ in this way as

$$\text{const} \cdot \partial_z^{-j_1-2}T(z) \ldots \partial_z^{-j_m-2}T(z):$$

Most VOAs we care about are related to the Virasoro VOAs, in the following sense.

**Definition 2.8.** A VOA is **conformal with central charge** $c$ if

1. The space of states $V$ is $\mathbb{Z}$-graded;

2. $\exists \omega \in V_2$, a conformal vector, so $Y(\omega, z) = \sum L^V_n z^{-n-2}$ and the $L^V_n$ have the Virasoro $L_n$ commutation relations;

3. Further, $T = L^V_{-1}, L^V_0 |_{V_n} = n \cdot \text{id}$.

Of course, $Vir_c$ is conformal, with central charge $c$ and $\omega = L_{-2} |0 \rangle$. (From the physics perspective, all CFTs have stress energy tensors, so the state-operator map should define a conformal vector, so this sort of structure should be universally expected.)

### 2.4 Free boson

One CFT we care about is the free boson. I will say some physics words to try to make contact with the physics. If these words mean nothing to you, don’t worry: it should be brief. It has Lagrangian

$$\frac{g}{2} \int dx (\partial_t \phi)^2 - (\partial_x \phi)^2.$$

On a cylinder, $\phi(x + L, t) = \phi(x, t)$, we may Fourier expand and go to a Hamiltonian, we find

$$H = \frac{1}{2gL} \sum_n \pi_n \pi_{-n} + (2\pi n g)^2 \phi_n \phi_{-n}.$$

Which is an infinite sum of harmonic oscillators, plus one zero mode, $\pi_0^2$.

The commutation relations of the $\pi_n, \phi_n$ are $[\pi_n, \phi_m] = i\delta_{nm}$. These are like ‘energy and momentum pairs’. The algebra generated by this is called the Heisenberg algebra.

**Definition 2.9.** Heisenberg is the central extension

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{H} \rightarrow \mathbb{C}[t, t^{-1}] \rightarrow 0.$$

with basis $\{t^n\}_{n \in \mathbb{Z}}$ and $\bar{1}$, and rule $[t^n, t^m] = -m \delta_{n,-m} \bar{1}$.

(This is a basis of creation and annihilation operators, related to our $\phi, \pi$ by

$$a_n = \frac{1}{\sqrt{4\pi g |n|}} (2\pi g |n| \phi_n + i \pi_{-n})$$

$$n > 0 \implies t^n = -i \sqrt{n} a_n$$

$$n \leq 0 \implies t^n = i \sqrt{-n} a^\dagger_{-n}$$

The **Weyl algebra** is Heisenberg letting $\bar{1} = 1$.

The simplest VOA we can make out of this is the **Fock space** of the CFT, the subalgebra of the Weyl algebra generated by $\{t^i = b_i\}_{i < 0}$ acting on $|0 \rangle = \bar{1}$.

We define

- $T |0 \rangle = 0, [T, b_i] = -ib_{i-1}$
• $Y(b_{-1}, z) = b(z) = \sum b_n z^{-n-1}$
• $y(b_{-n}, z) = \frac{1}{(n-1)!} \partial_z^{n-1} b(z)$

**Example 2.10.** In fact the Fock space VOA admits a one-parameter family of conformal structures. Let

$$\omega_\lambda = \left( \frac{1}{2} b_{-1}^2 + \lambda b_{-2} \right) |0\rangle.$$

then $Y(\omega_\lambda, z) = \frac{1}{2} : b(z)^2 : + \lambda \partial_z b(z)$ satisfy the conformal relations, and equip the Fock space VOA with the structure of a conformal VOA with central charge $c_\lambda = 1 - 12\lambda^2$.

What’s going on here? Because the Hamiltonian is independent of $\phi_0$, $\pi_0$ ‘commutes with everything’, so we can simultaneously diagonalise eigenstates of $H$ to also be eigenstates of $\pi_0$. In the physical picture, we can view the Fock space as being built on a one-parameter family of vacua $|\lambda\rangle$, where $\lambda$ is related by normalisation to the eigenvalue of $\pi_0$ by normalisation. No operators relate the vacua, so the theory decouples into conformal VOAs with different central charges.

By the way, why did we want to consider these vertex operator fields/states rather than our original free boson, $\phi$? For one, $\phi$ itself doesn’t factor into holomorphic and anti-holomorphic components: it is not primary.

### 2.5 OPE

OK, the final thing I want to do is give a general derivation of the OPE from the state-operator correspondence, at the level of physics rigor.

State-operator says that $Y(A, z) Y(B, w)$ is determined by the state $Y(A, z) Y(B, w) |0\rangle$ as $w \to 0$. (This is a theorem in Ben-Zvi/Frenkel, called Goddard’s uniqueness theorem). Translate to $Y(A, z-w) Y(B, 0) |0\rangle$ and expand $Y(A, z-w) = \sum A_n(z-w)^{-n-1}$. We find

$$Y(A, z) Y(B, w) = \sum_{n \leq 0} \frac{Y(A_n B, w)}{(z-w)^{n+1}} + : Y(A, z) Y(B, w) :,$$

which is the OPE for vertex operators. (The non-singular term, sort of by definition, is $: Y(A, z) Y(B, w) :$, but I haven’t justified this adequately.)

OPE can be used to construct the weird formula we had for the VOA of a product of endomorphisms.

We need one more formula:

**Claim.** $Y(B, z) |0\rangle = e^{z T} B$

**Proof.** It suffices to prove $B_{(-n-1)} |0\rangle = \frac{T^n}{n!} B$.

By the axioms, $\partial_z Y(B, z) |0\rangle = [T, Y(B, z)] |0\rangle = TY(B, z) |0\rangle$.

Equate coefficients to find $nB_{-1-n} |0\rangle = TB_{-n} |0\rangle$. Induct. \qed

**Claim.** $Y(TA, z) = \partial_z Y(A, z)$ for all $A$.

**Proof.** $Y(TA, z) |0\rangle = \partial_z Y(A, z) |0\rangle$, use state-operator to go back. \qed

Which implies $Y(B_{(-n-1)} |0\rangle, z) = \frac{\partial^n}{n!} Y(B, z)$.

Contour integrating, the OPE we find

$$Y(A_n B, z) = \frac{1}{(-n-1)!} : \partial_z^{n-1} Y(A, z) Y(B, z) :.$$

Now using the above claim to expand replace $B$ with $B_{-n-1}$, if desired, and inducting to include more fields, we get the desired ‘reconstruction formula’.