Notes for Learning Seminar on Conformal Field Theory (Fall 2019)

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November 7, 2019


Abstract

These are my live-texed notes for the Fall 2019 student learning seminar on conformal field theory. Let me know when you find errors or typos. I’m sure there are plenty.

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1 Kostya (Sep 24): Intro to CFT

This will be an informal talk from a physics perspective. We’ll try to make some mathematical connections, but there will be very few mathematical statements. The main purpose is to give a feeling of how CFT formalism appeared and why.

We’ll follow the paper by Belavin, Polyakov and Zamolodchikov from ’84. This paper actually arise as an attempt to understand an earlier paper in ’81 by Polyakov about string theory. String theory is a theory on Riemann surfaces (Σ, g), with or without boundary, genus, and marked points. It feels as few of the features of the surfaces as possible, because we want it to embed into spacetime R^{1,3}. The theory should be:

1. covariant, i.e. independent of all coordinate transformations;

2. conformal, i.e. the Lagrangian should only depend on the conformal class of the metric g.

For example, for a quantum particle, there is a worldline and a map from the worldline to R^{1,3}. In principle when we write a Lagrangian for the particle, it depends on the metric of the worldline. But in string theory we don’t want this dependence.
The partition function $Z$ of string theory looks like

$$Z = \int_{\text{all } (\Sigma,g)} DX Dg \exp \left( \frac{1}{\hbar} \left[ S_{\text{matter}}[X] + S_{\text{gravity}}[g] \right] \right).$$

In general, we think of $DX$ as a measure on something like $\Gamma(E)$ where $E$ is some bundle over the moduli of some $(\Sigma,g)$. When we consider conformal matter, gauge fixing for diffeomorphisms turns this action into

$$Z = \int_{\text{conformal classes}} DX Dg \exp \left( \frac{1}{\hbar} \left[ S_{\text{matter}}[X] + S_{\text{gravity}}[g] + S_{\text{gauge fixing}} \right] \right).$$

In a particular case, called critical strings, the gravity term $S_{\text{gravity}}[g]$ is trivial and therefore we get a theory that only depends on the topology of $\Sigma$. In the critical case, the central charge of matter is $c_{\text{matter}} = 26$; in the supersymmetric version $c_{\text{matter}} = 15$. But in general we can consider arbitrary central charge, giving different gravity theories. One such example is Liouville gravity. Classically it coincides with the Liouville equation, which is also conformal. The most useful result in this direction appeared in the ’84 paper, which showed that CFTs could actually be solved. However the technique did not work for Liouville theory, which was worked out about a decade later.

What is a CFT? The first thing to discuss we should discuss, which is very particular to CFTs, is the state-operator correspondence. First forget about conformal structure and think only about topological theories. For such 2d theories there are Atiyah–Segal axioms, which basically say that such a theory is a functor

$$\mathcal{F} : \text{Cob} \to \text{Vect}$$

from the category of cobordisms to the category of vector spaces. Boundaries are always disjoint unions of circles $S^1$, and

$$\mathcal{F}(S^1) = \mathcal{H} = (\text{state space of the theory}).$$

For a cobordism represented by a 2d manifold $\Sigma$ with boundary, we get

$$\mathcal{F}(\Sigma) = v \in \mathcal{F}(\partial \Sigma).$$

For example, if $\Sigma$ is a cylinder, its boundary is two circles with different orientation, i.e.

$$\partial \Sigma = S^1 + (S^1)_{\text{op}}.$$

Part of the axioms say that orientation is encoded by dualizing, so

$$\mathcal{F}(\text{cylinder}) \in \mathcal{H} \otimes \mathcal{H}^\vee$$

which we can view as an operator. This is actually the identity operator of the theory.

A state is something living on the boundary. So it is a vector in the Hilbert space $\mathcal{H}$. For example, taking the cylinder, we can plug in two states to get a correlation number of the theory. Another nice way to think of a state is, again, in the path integral formalism. For a non-topological theory, think of $\mathcal{H}$ as $L^2(S^1)$ in some sense. Say $\Sigma$ is a disk $D^2$. Then we can take expectation values

$$Z = \int DX \exp \frac{1}{\hbar} S \in \mathbb{C}$$

where the action is an integral over a disk

$$S = \int_{\Sigma} d^2 \xi L[X, \partial X].$$

The point is that this integral is not well-defined unless we specify boundary conditions on the disk. A boundary condition is e.g.

$$X(e^{i\varphi}) = X_0(e^{i\varphi}).$$
A **local operator** are differential polynomials (or slight generalization) of the basic fields of the theory. (Usually operators in the CFT context are always local.) An example of a local operator would be something like $\partial X \cdot X^3$. We can also think about *non-local* operators, e.g. $\langle X(z_0)X(z_1) \rangle$. They depend not on one point, but on multiple.

At the classical level, an example is as follows. Let $X(z) \in \mathbb{C}$ be a section of the trivial line bundle. After quantization, in the Hamiltonian formalism, it will be an operator. Then we can compute correlation functions such as

$$\langle X(z_0) \rangle = \int DX \cdot X(z_0) \exp \frac{1}{\hbar} S.$$ 

This defines an element of $\mathcal{H}^\vee$ associated to the same boundary of the disk, because the action still requires a boundary condition on the disk. Hence local operators define states. In fact *any* state in CFT can be obtained in this way.

In general, think of the operator-state correspondence as follows. Consider a cylinder, infinitely long in the negative direction.

At the negative end, plug in a state $|\phi\rangle$. It produces a state $e^{i\hbar H} |\phi\rangle$ on the other $S^1$. But since the theory is conformal, we can convert this picture into a disk

The boundary on the lhs of the cylinder is mapped to a puncture at the origin, and the boundary on the rhs becomes the boundary $S^1$. Then the object at the puncture is a local object $A(0)$, which is a local operator at 0. This is in general called **radial quantization**.

An important consequence is the following. Take a sphere with three punctures. Put two states $|\phi\rangle$ and $|\psi\rangle$ in two of the punctures. Then we get an element $|v\rangle \in \mathcal{H}$ associated to the third puncture. Apply the state-operator correspondence. Pick a basis $\{\phi_\alpha\}_{\alpha \in I}$ for $\mathcal{H}$ and decompose

$$|v\rangle = \sum C^\alpha_\gamma(z,w) \phi_\alpha(w)$$

for some coefficients $C$. Hence we get

$$\phi_\alpha(z) \phi_\beta(z) = \sum C^\alpha_\gamma(z,w) \phi_\gamma(w).$$

This gives an **operator algebra** structure to $\mathcal{H}$. In general, this expression can be put into any correlation function, as

$$\langle A(z_1, \ldots, z_N) \phi_\alpha(z) \phi_\beta(w) \rangle = \langle \sum C^{\gamma}_{\alpha\beta} \phi_\gamma(w) A(z_1, \ldots, z_N) \rangle.$$

Associativity of such a product structure imposes constraints on $C^{\gamma}_{\alpha\beta}$, and allows us to solve for them. This is the *bootstrap* approach to solving CFTs.

A very important object to work with in CFT is the **stress-energy tensor**. It is the generator of time transformations. If we think about a cylinder, pick some slice $t = 0$ in it. The Hamiltonian is

$$H(t) = \int_{t=t_0} dx T_{0,0}(\xi, \bar{\xi}).$$
In classical field theory, the Hamiltonian generates time translations, i.e.
\[ \{ H(t), X \} = \partial_t X. \]

In CFT we can get much more. The theory being conformal is equivalent to the stress-energy tensor \( T_{ab} \) being traceless, i.e. \( T_{aa} = 0 \). Noether’s theorem gives \( \partial_a T^a_b = 0 \). In complex coordinates, these two properties become:
\[ T_{zz} = \text{tr} T = 0, \quad \overline{T}_{\overline{z}z} = 0, \quad \partial \overline{T}_{\overline{z}z} = 0. \]

Due to this, we write the holomorphic and anti-holomorphic functions
\[ T(z) = T_{zz}(z), \quad \overline{T}(\overline{z}) = T_{\overline{z}\overline{z}}(\overline{z}). \]

They are rank-2 tensors, so they transform like
\[ T(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^2 T(\xi). \]

After quantization, \( [H(t), X] = \partial_t X \), so
\[ \left[ \int_{t=t_0} dx \, T(z), X \right] = \partial_z X. \]

But more generally we can use any complex transformation \( \epsilon(z) = \sum_n \epsilon_n z^n \), to get
\[ \left[ \int dx \, \epsilon(z) T(z), X \right] = \delta_{\epsilon} X. \]

In radial quantization, this commutator becomes just
\[ \partial_{\epsilon} X = \int dt \, \epsilon(z) T(z) X(z_0) |0\rangle \]

where \(|0\rangle\) is the state inserted at 0. This is called a Ward identity.

A field \( \phi \) is primary if it transforms as
\[ \phi_\alpha(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^{\Delta_\alpha} \left( \frac{\partial \overline{\xi}}{\partial \overline{z}} \right)^{\overline{\Delta}_\alpha} \phi_\alpha(\xi), \]
i.e., it is a “tensor” of rank \((\Delta_\alpha, \overline{\Delta}_\alpha)\), called the conformal dimension, which can be non-integers. For such fields, the operator product expansion is
\[ T(z)\phi_\alpha(w, \overline{w}) = \frac{\Delta_\alpha}{(z-w)^2} \phi_\alpha + \frac{\partial_w \phi_\alpha}{z-w}. \]

However it turns out that \( T \) is not a primary field. (In the classical theory it of course is, because it is a rank-2 tensor. But in the quantum theory it is not.) It transforms as
\[ T(z) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^2 T(\xi) + \frac{c}{12} \{ \xi, z \} \]

where the second (correction) term is the Schwartz derivative and \( c \) is a constant called the central charge. One can try to understand this term physically in many ways, but it is believed to be the most possible such expression. Using this transformation law with the Ward identities, we get
\[ T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \]
Out of this, we can get the Virasoro algebra as follows. Formally decompose
\[ T(w) = \sum \frac{L_n}{z^{n+2}}. \]
(The +2 here is because of the conformal dimension 2.) Then
\[ \oint dz z^{n+1} T(z) = L_n. \]
Applying this to the operator product expansion, we get Virasoro relations
\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}. \]
Note that if we define vector fields \( \ell_n := z^{n+1}\partial_z \), we get just the first term in this commutator:
\[ [\ell_n, \ell_m] = (n - m)\ell_{n+m}. \]
This comes from the relation between the classical stress-energy tensor and the Hamiltonian. Note also that this means \( \{L_{-1}, L_0, L_1\} \) span an \( \text{SL}_2 \) subalgebra.

Using the OPE \( T(z)\phi_\alpha(0) \), one can show that the state operator correspondence \( \phi(z) \rightarrow |\phi\rangle = \phi(0)|0\rangle \) sends the operator \( \phi \) to a highest-weight vector for the Virasoro algebra. In other words,
\[ L_n |\phi\rangle = 0, \quad n > 0 \]
and the highest weight is exactly the conformal weight \( \Delta \). Generically,
\[ L_{-n_1} \cdots L_{-n_k} |\phi\rangle, \quad n_1 < \cdots < n_k \]
are independent. By the state-operator correspondence, they can be represented by some contour integrals. These are descendants of the primary field \( \phi(0) \).

To sum up in representation-theoretic language, the entire Hilbert space \( \mathcal{H} \) decomposes into a sum of irreducible highest weight representations of the Virasoro. Primary fields are highest weight vectors, and descendants are generated by primary fields. Correlation functions for all descendants are computed through the ones for primary fields, precisely because we know the OPE of the stress-energy tensor with any field. So to study CFTs, we only care about primary fields.

If in the decomposition \( \mathcal{H} = \bigoplus_{\alpha \in \mathcal{A}} V_\alpha \) the indexing set \( \mathcal{A} \) is finite, then we have a finite number of primaries and the CFT is called rational. Also, when
\[ \text{Vir} \subset U(\mathfrak{g}) \]
for some \( \mathfrak{g} \), e.g. Heisenberg, loop/affine algebra (like WZW theory), then we can ask about reps of this bigger symmetry algebra \( U(\mathfrak{g}) \). Examples include minimal models, WZW theories, KS models.

Another consequence of the Ward identities, which leads to differential equations for conformal blocks, comes from considering
\[ \langle T(z)\phi_{\alpha_1}(z_1) \cdots \phi_{\alpha_n}(z_n) \rangle = \sum_{i=1}^n \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{\partial / \partial z_i}{z - z_i} \right) \times \langle \phi_{\alpha_1}(z_1) \cdots \phi_{\alpha_n}(z_n) \rangle. \]
What is a conformal block? Recall the product \( \phi_1(z)\phi_2(w) = \sum C_{\alpha_2}^{\alpha_1}(z)\phi_\alpha(0) \). Decompose \( \phi_\alpha \) into primary(descendants) to get
\[ \phi_1(z)\phi_2(z) = \sum_{\alpha} C_{\alpha_2}^{\alpha_1} \sum_{\{k\}} \beta_{\{\{k\}\}}^{\alpha_2} L_{-n_1} \cdots L_{-n_k} \phi_\alpha(0). \]
Apart from the structure constants $\beta^{\alpha k}_{12}$, everything else is determined by conformal symmetry, e.g.

$$\langle \phi_\alpha(z) \rangle = 0 \text{ if } \phi_\alpha \neq \text{id}$$

$$\langle \phi_\alpha(z) \phi_\beta(w) \rangle = \frac{\delta_{\alpha\beta}}{|z - w|^{\Delta_\alpha}}$$

$$\langle \phi_\alpha(z) \phi_\beta(w) \phi_\gamma(\xi) \rangle = \frac{C_{\alpha\beta\gamma}}{|z - w|^{|\Delta_\alpha| + |\Delta_\beta| + |\Delta_\gamma|}}$$

$$\langle \phi_1(\infty) \phi_2(1) \phi_3(x) \phi_4(0) \rangle = \sum_p C_{p12}^p C_{34}^p \sum_{\{k\}} z^{\beta_{12}^p(k)} \beta_{34}^p(k) \langle \phi_p|L_{n_1'} \cdots L_{n_p'}|L_{n_1} \cdots L_{n_k}^p|\phi_p\rangle.$$

The only unknowns here are the structure constants $C_{12}^p$ and $C_{34}^p$. In the BPZ paper, the four-point correlators are constructed in terms of objects called conformal blocks $F_{12}^{12}(p|z)$, as

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \sum_p C_{12}^p C_{34}^p |F_{12}^{12}(p|z)|.$$

These are universal quantities which are fully determined by representation-theoretic considerations. In all rational theories, they satisfy differential equations called BPZ equations coming from the Ward identities. (In WZW models they are called KZ equations, coming from some rep theory of $U(n)$. However, in Liouville theory we cannot do this, and there are other things to work with.

## 2 Davis (Oct 01): The local structure of CFT

### (Notes by Davis)

In this talk, I’ll try to explain how vertex operator algebras are related to the local structure of conformal field theory. I’ll motivate VOAs from this point of view, explain what the Virasoro algebra is from a mathematically motivated perspective, then explain how the Heisenberg Lie algebra is related to the free boson CFT. Time permitting, I hope to say something about the OPE, or about conformal blocks.

In 2D CFT, a general correlator on a manifold $\Sigma$,

$$\langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle_\Sigma,$$

can be viewed as fields $\phi_1, \ldots, \phi_n$ inserted at points $z_1, \ldots, z_n$ on the manifold.

Part of the content of Kostya’s talk was justifying that we could understand this picture by breaking apart $\Sigma$ into smaller pieces and cobording them together. Hence, we may break apart $\Sigma$ into pieces and assume all the fields live together on a genus zero piece. Studying this genus zero piece is what I will call the local structure of CFT: understanding higher genus pieces, and how they glue together, will be beyond this talk.

### 2.1 The standout objects of CFT on the circle

Kostya’s talk yesterday introduced a lot of important objects in the study of CFT on a genus zero surface with boundary,

1. The space of states (boundary conditions)
2. The OPE, $C^\alpha(z)C^\beta(w) = \sum C^\gamma_{\alpha\beta}(z, w)C^\gamma(z)$
3. The state operator correspondence
4. The stress energy tensor, $T_{\mu\nu} \rightarrow T(z), \bar{T}(\bar{z})$.

**Claim.** On the circle, for a large class of CFTs called ’chiral CFTs’, the “holomorphic and antiholomorphic sectors decouple” and we may study the holomorphic part of the CFT independently. I will justify this statement, and explain it in more detail, shortly, but for now take it on faith.
2.2 VOAs

Formalising these ideas, observing that we may derive the OPE from the state-operator correspondence, we
arrive at the long definition of a VOA.

Definition 2.1. A vertex operator algebra

1. A space of states $V$ (a vector space)
2. A vacuum vector $|0\rangle \in V$
3. A translation operator $T : V \to V$,
4. Vertex operators, $Y(\bullet, z) : V \to \text{End} V[[z, z^{-1}]]$, sending $A \to Y(A, z)$, a field.

such that
1. $Y(|0\rangle, z) = id$
2. $Y(A, z) |0\rangle \in V[[z]], Y(A, z) |0\rangle_{z=0} = A$;
3. $[T, Y(A, z)] = \partial_z Y(A, z), T |0\rangle = 0$;
4. The $Y(A, z)$ are pairwise local.

where
1. $\sum A_j z^{-j} = A(z) \in \text{End} V[[z, z^{-1}]]$ is a field if for any $v \in V$, $A_j v = 0$ for $j$ large enough.
2. Two fields $A(z), B(w)$ are local pairwise if $\exists N \geq 0, (z - w)^N [A(z), B(w)] = 0$.

Where did the stress energy tensor go in this definition? We’ll see after we’ve discussed Virasoro.
I don’t want to get into the weeds of formal power series, and BenZvi/Frenkel’s book covers this background well. But here’s one example:

Example 2.2. The formal delta function, $\delta(z - w)$, is the power series

$$\delta(z - w) = \sum z^m w^{-m-1}.$$ 

It satisfies the property

$$A(z) \delta(z - w) = \sum A_k w^k z^m w^{-m-1} = \sum A_{m+n+1} z^m w^n = A(w) \delta(z - w).$$

In particular, $z \delta(z - w) = w \delta(z - w) \implies (z - w) \delta(z - w) = 0$.

The easiest VOAs are commutative ones.

Example 2.3. Suppose $[Y(A, z), Y(B, w)] = 0$ for all $A, B$. Then

$$Y(A, z) B = Y(A, z) Y(B, w) |0\rangle_{w=0} = Y(B, w) Y(A, z) |0\rangle_{w=0}. $$

Because this is true for all $B$ and $Y(A, z) |0\rangle \in \text{End} V[[z]]$ by the axioms, this means $Y(A, z) \in \text{End} V[[z]]$ for all $A$.

So we can equip the VOA with the structure of a commutative algebra,

$$A \ast B := Y(A, 0) B.$$ 

$T$ is a derivation for this algebra.

In the other direction, with mild assumptions, $Y(A, z) := e^{zT} A$ makes an algebra with a derivation into a VOA.
2.3 Virasoro

Our first examples of interesting vertex operators will come from the Virasoro algebra.

Recall that we have identified states with boundary conditions. So $Diff(S^1)$ acts on states on the disc. We really want the 'complexified' $Diff(S^1)$ action. However, $Diff(S^1)$ admits no complexification as a Lie group. (There is a semigroup which serves as a partial answer, as exposited in Andre Henriques cobordism-centred CFT notes).

So, we look at its Lie alg and complexify. Complexified $\text{LieDiff}(S^1)$ has a basis

\[ \ell_n = -z^{n+1} \partial_z \]
\[ \ell_n = -z^{n+1} \partial_z \]
\[ [\ell_n, \ell_m] = (n-m)\ell_{n+m} \]
\[ [\ell_n, \ell_m] = (n-m)\ell_{n+m} \]

So, $(\text{LieDiff}(S^1))_C = \text{Witt} \oplus \text{Witt}$, where $\text{Witt}$ has basis $\{\ell_n\}$.

Remark. This is what I mean by decoupling of holomorphic/antiholomorphic sectors. The assumption here is that the $Diff(S^1)$ global action, whatever I really mean by this, upgrades to a local action: this the definition of a chiral CFT.

Primary fields, for instance, are ones where this action so upgrades. For general CFTs, you can only use the VOA to study the chiral sector, which basically means fields coming from primary fields.

Goal. We want to study projective representations of the Witt algebra, because scaling by a constant doesn’t change the physical state. The standard yoga is that projective representations of Witt are ordinary reps of centrally extended Witt.

Definition 2.4. The Virasoro algebra is $[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}.$

It is the universal central extension of the Witt algebra.

Remark. We can now write a generating functional

\[ T(z) = \sum L_n z^{-n-2} \]
\[ [T(z), T(w)] = \frac{c}{12} \partial_z^3 \delta(z-w) + 2T(w)\partial_w \delta(z-w) + \partial_w T(w) \delta(z-w) \]

and this relation is equivalent to the $[L_n, L_m]$ relation. So, the stress-energy tensor pops up!

Finally, we can make a VOA

Example 2.5. Let $Vir_c = U(Vir) \otimes_U (\text{LieDiff}(S^1))_C \otimes \mathbb{C}_c$, i.e. 'the universal enveloping algebra of Virasoro with the element $c$ sent to be constant number $c$.

This algebra has a basis

\[ L_{j_1} \ldots L_{j_m} |0\rangle \]

where $j_1 \leq \cdots \leq j_m \leq -2$

we can declare $T = L_{-1}, Y(L_{-2} |0\rangle, z) := T(z)$, and other vertex operators ‘generated from this’.

What we mean by ‘generated by this’ calls for another messy theorem which will let us practically define a lot of VOAs.

Theorem 2.6. Let $V$ a vector space, $|0\rangle \neq 0 \in V, T \in \text{End}(V)$. Let $\{a^\alpha\}_{\alpha \in \mathbb{Z}}$ vectors, and $\{a^\alpha(z)\}$ fields so that

1. $a^\alpha(z) |0\rangle = a^\alpha + O(z^{\geq 1})$
2. $T |0\rangle = 0, [T, a^\alpha(z)] = \partial_z a^\alpha(z)$
3. $a^a(z)$ are pairwise local;

4. $V$ has a basis of vectors $a_{(j_1)}^{\alpha_1} \cdots a_{(j_m)}^{\alpha_m} |0\rangle$, with $j_1 \leq \cdots \leq j_m < 0$ and $j_i = j_{i+1} \implies \alpha_i = \alpha_{i+1}$
then, we can equip $V$ with a VOA structure by

$$Y(a_{j_1}^{\alpha_1} \cdots a_{j_m}^{\alpha_m} |0\rangle, z) = \frac{1}{(-j_1 - 1)! \cdots (-j_m - 1)!} : \partial_{z}^{-j_1 - 1} a^{\alpha_1} \cdots \partial_{z}^{-j_m - 1} a^{\alpha_m} (z) :$$

where $AB$ denotes normal ordering: if $A(z) = \sum A_m z^{-m-1}, B(w) = \sum B_n w^{-n-1},$ we define $A(z)B(w)$ to be $\sum_n (\sum_{m<0} A_m B_n z^{-m-1} + \sum_{m \geq 0} B_n A_m z^{-m-1}) w^{-n-1}$. We then extend $ABC := A : BC :.$

Normal ordering has a weird definition, basically it means removing the singular point as $z \to \infty$. In physics speak, we demand that vacuum expectation values of normal ordered correlators vanish.

I hope to explain this theorem’s formula via the OPE, time permitting.

Example 2.7. So in the Virasoro case, we may write $Y(L_{j_1} \cdots L_{j_m} |0\rangle, z)$ in this way as

$$\text{const} \times : \partial_{z}^{-j_1 - 2} T(z) \cdots \partial_{z}^{-j_m - 2} T(z) :$$

Most VOAs we care about are related to the Virasoro VOAs, in the following sense.

Definition 2.8. A VOA is conformal with central charge $c$ if

1. The space of states $V$ is $Z$-graded;

2. $\exists \omega \in V_2$, a conformal vector, so $Y(\omega, z) = \sum L_n^V z^{-n-2}$ and the $L_n^V$ have the Virasoro $L_n$ commutation relations;

3. Further, $T = L_{-1}^V, L_0^V|_{V_n} = n \cdot \text{id}.$

Of course, $Vir_c$ is conformal, with central charge $c$ and $\omega = L_{-2} |0\rangle$. (From the physics perspective, all CFTs have stress energy tensors, so the state-operator map should define a conformal vector, so this sort of structure should be universally expected.)

2.4 Free boson

One CFT we care about is the free boson. I will say some physics words to try to make contact with the physics. If these words mean nothing to you, don’t worry: it should be brief. It has Lagrangian

$$\frac{g}{2} \int dx (\partial_t \phi)^2 - (\partial_x \phi)^2.$$

On a cylinder, $\phi(x + L, t) = \phi(x, t)$, we may Fourier expand and go to a Hamiltonian, we find

$$H = \frac{1}{2gL} \sum_n \pi_n \pi_{-n} + (2\pi ng)^2 \phi_n \phi_{-n}.$$

Which is an infinite sum of harmonic oscillators, plus one zero mode, $\pi_0^2$.

The commutation relations of the $\pi_n, \phi_n$ are $[\pi_n, \phi_m] = i\delta_{nm}$. These are like ‘energy and momentum pairs’. The algebra generated by this is called the Heisenberg algebra.

Definition 2.9. Heisenberg is the central extension

$$0 \to \mathbb{C} \to \mathcal{H} \to \mathbb{C}[t, t^{-1}] \to 0.$$
with basis \( \{ t^n \}_{n \in \mathbb{Z}} \) and \( \tilde{1} \), and rule \([ t^n, t^m ] = -m \delta_{n,-m} \tilde{1}\). (This is a basis of creation and annihilation operators, related to our \( \phi, \pi \) by

\[
a_n = \frac{1}{\sqrt{4\pi g|n|}} (2\pi g|n|\phi_n + i\pi_{-n})
\]

\( n > 0 \implies t^n = -i\sqrt{n}a_n \cdot \)

\( n \leq 0 \implies t^n = i\sqrt{-na}_{-n}^\dagger \).

The **Weyl algebra** is Heisenberg letting \( \tilde{1} = 1 \).

The simplest VOA we can make out of this is the **Fock space** of the CFT, the subalgebra of the Weyl algebra generated by \( \{ t^i = b_i \}_{i < 0} \) acting on \(|0\rangle = \tilde{1}\).

We define

- \( T |0\rangle = 0, [T, b_i] = -ib_{i-1} \)
- \( Y(b_{-1}, z) = b(z) = \sum b_n z^{-n-1} \)
- \( y(b_{-n}, z) = \frac{1}{(n-1)!} \partial_z^{n-1} b(z) \)

**Example 2.10.** In fact the Fock space VOA admits a one-parameter family of conformal structures.

Let

\[
\omega_\lambda = (\frac{1}{2} b_{-1}^2 + \lambda b_{-2}) |0\rangle.
\]

then \( Y(\omega_\lambda, z) = \frac{1}{2} : b(z)^2 : + \lambda \partial_z b(z) \) satisfy the conformal relations, and equip the Fock space VOA with the structure of a conformal VOA with central charge \( c_\lambda = 1 - 12\lambda^2 \).

What’s going on here? Because the Hamiltonian is independent of \( \phi_0, \pi_0 \) ‘commutes with everything’, so we can simultaneously diagonalise eigenstates of \( H \) to also be eigenstates of \( \pi_0 \). In the physical picture, we can view the Fock space as being built on a one-parameter family of vacua \(|\lambda\rangle\), where \( \lambda \) is related by normalisation to the eigenvalue of \( \pi_0 \) by normalisation. No operators relate the vacua, so the theory decouples into conformal VOAs with different central charges.

By the way, why did we want to consider these vertex operator fields/states rather than our original free boson, \( \phi \)? For one, \( \phi \) itself doesn’t factor into holomorphic and antiholomorphic components: it is not primary.

### 2.5 OPE

OK, the final thing I want to do is give a general derivaton of the OPE from the state-operator correspondence, at the level of physics rigor.

State-operator says that \( Y(A, z) Y(B, w) \) is determined by the state \( Y(A, z) Y(B, w) |0\rangle \) as \( w \to 0 \). (This is a theorem in Ben-Zvi/Frenkel, called Goddard’s uniqueness theorem). Translate to \( Y(A, z-w) Y(B, 0) |0\rangle \) and expand \( Y(A, z-w) = \sum A_n (z-w)^{-n-1} \). We find

\[
Y(A, z) Y(B, w) = \sum_{n \leq 0} \frac{Y(A_n B, w)}{(z-w)^{n+1}} + : Y(A, z) Y(B, w) :,
\]

which is the OPE for vertex operators. (The non-singular term, sort of by definition, is \( : Y(A, z) Y(B, w) : \), but I haven’t justified this adequately.)

OPE can be used to construct the weird formula we had for the VOA of a product of endomorphisms.

We need one more formula:
Claim. \( Y(B, z) \, |0\rangle = e^{zt} B \)

Proof. It suffices to prove \( B((-n-1)) \, |0\rangle = \frac{T^n}{n!} B \).

By the axioms, \( \partial_z Y(B, z) \, |0\rangle = [T, Y(B, z)] \, |0\rangle = TY(B, z) \, |0\rangle \).

Equate coefficients to find \( nB_{-n-1} \, |0\rangle = T B_{-n} \, |0\rangle \). Induct.

Claim. \( Y(Ta, z) = \partial_z Y(A, z) \) for all \( A \).

Proof. \( Y(Ta, z) \, |0\rangle = \partial_z Y(A, z) \, |0\rangle \), use state-operator to go back.

Which implies \( Y(B((-n-1))) \, z \) = \( \frac{\partial^n}{n!} Y(B, z) \).

Contour integrating, the OPE we find

\[
Y(A_n B, z) = \frac{1}{(-n-1)!} \partial_z^{-n-1} Y(A, z) Y(B, z) :.
\]

Now using the above claim to expand replace \( B \) with \( B_{-n-1} \), if desired, and inducting to include more fields, we get the desired ‘reconstruction formula’.

3 Cailan (Oct 08): VOA: Examples and Representations

Definition 3.1. A VOA is:

1. a vector space \( V \);
2. a vacuum vector \( |0\rangle \in V \);
3. a translation operator \( T : V \to V \);
4. vertex operators, which are linear maps \( Y(-, z) : V \to \text{End} V[[z, z^{-1}]] \)

such that \( Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \) is a field, i.e. for all \( v \in V \) we have \( A_n v = 0 \) for \( n \gg 0 \). We call the \( A_n \) the Fourier coefficients of \( A \).

This data is subject to the following axioms:

1. \( Y(|0\rangle, z) = \text{id} \), and \( Y(A, z) \, |0\rangle \in V[[z]], \ Y(A, z) \, |0\rangle \big|_{z=0} = A \);
2. \( [T, Y(A, z)] = \partial_z Y(A, z) \);
3. (locality) there exists \( N \geq 0 \) such that

\[
(z-w)^N [Y(A, z), Y(B, w)] = 0.
\]

Example 3.2. The Heisenberg algebra \( \mathcal{H} \) is the central extension

\[
0 \to \mathbb{C} \cdot 1 \to \mathcal{H} \to \mathbb{C}[t, t^{-1}] \to 0
\]

given by the 2-cocycle \( c(f, g) = -\text{Res}_{t=0} f dg \).

Let \( b_n = t^n \). Then the commutation relations are

\[
[b_n, b_m] = n\delta_{n,-m} 1, \quad [1, b_n] = 0.
\]
What we will do in most of our examples is turn representations of Lie algebras into VOAs. Let
\[ \tilde{\mathcal{H}} := U\mathcal{H}/(1 - 1). \]

Let \( \tilde{\mathcal{H}}_+ \subset \tilde{\mathcal{H}} \) be the positive subalgebra. Since it is commutative, it has a trivial rep \( \mathbb{C} \). The **Fock representation** is
\[ \pi := \tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{H}}_+} \mathbb{C}. \]

By PBW, this has a basis
\[ \{ \cdots b_{-2}^1 b_{-1}^1 \otimes 1 \}. \]

The VOA structure on \( \pi \) is defined as follows.

1. The vacuum is \( |0\rangle := 1 \otimes 1 \).
2. Translation is defined inductively using
   \[ [T, b_i] = -i b_{i-1}. \]
   This means \( T \) behaves as a formal derivative, with
   \[ T(b_{j_1} \cdots b_{j_k} \otimes 1) = \sum_i j_i b_{j_1} \cdots b_{i-1} \cdots b_{j_k} \otimes 1. \]
3. Vertex operators are defined as follows:
   \[ Y(b_{-1} \otimes 1, z) := b(z) := \sum_{n \in \mathbb{Z}} b_n z^{-n-1} \]
   \[ Y(b_{-k} \otimes 1, z) = \left( \frac{i}{(k - 1)!} \right) \partial_z^{k-1} b(z) \]
   \[ Y(b_{-j_1} \cdots b_{-j_k} \otimes 1, z) = \frac{1}{(j_1 - 1)! \cdots (j_k - 1)!} : \partial_z^{j_1-1} b(z) \cdots \partial_z^{j_k-1} b(z): \]
   where \( : A(z) B(w) : = A(z)_+ B(w) + B(w) A(z)_- \).

**Claim.** This gives \( \pi \) the structure of a VOA.

**Proof.** By construction, \( [T, b_i] = -i b_{i-1} \) implies
\[ [T, b(z)] = \partial_z b(z). \]
So the translation axiom is satisfied by \( Y(b_{-1} \otimes 1, z) \). For the general case, use that normal ordering satisfies the Leibniz rule
\[ \partial_z : A(z) B(z) : = : \partial_z A(z) B(z) : + : A(z) \partial_z B(z) :. \]
The real content of this claim is in checking the locality axiom.

1. Check that \( b(z) \) is local with itself. Recall \( \delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^m \). Compute that
   \[ [b(z), b(w)] = \sum_{n, m} [b_n, b_m] z^{-n-1} w^{-m-1} = \sum_n [b_n, b_{-n}] z^{-n-1} w^{n-1} = \partial_w \delta(z - w). \]
   Locality means this commutator is a sum of delta functions and their derivatives, which is true here.
2. Check that \( \partial_z^n b(z) \) is local with \( \partial_z^m b(z) \). Start with
   \[ (z - w)^N [b(z), b(w)] = 0, \]
   which is the definition of locality for \( b(z) \) with itself, and differentiate with respect to \( z \). This yields
   \[ (z - w)^N [\partial_z b(z), b(w)] + N(z - w)^{N-1} [b(z), b(w)] = 0. \]
   Multiply by \( z - w \) to get rid of the last term and get locality of \( \partial_z b(z) \) with \( b(z) \). Then induct.
3. Apply induction using Dong’s lemma, which says if \( A(z), B(z), C(z) \) are mutually local fields then the fields \( :A(z)B(z): \) and \( C(z) \) are local.

Remark. We can’t avoid normal ordering. One of the consequences of OPE is the following identity:

\[
Y(A_n \cdot B, z) = \frac{1}{(-n-1)!} \partial_z^{-n-1} Y(A, z) Y(B, z); \quad n < 0.
\]

In the case of the Heisenberg, note that \( b_{-1} \) generates all coefficients of \( Y(b_{-1} \otimes 1, z) = b(z) \). Hence the moment we specify \( Y(b_{-1} \otimes 1, z) \) we have specified everything. This is why only \( b(z) \) and its derivatives occur in \( Y(A, z) \).

Example 3.3 (Affine Kac–Moody algebras). Define \( \hat{\mathfrak{g}} \) as

\[
0 \to CK \to \hat{\mathfrak{g}} \to L\mathfrak{g} \to 0.
\]

Let \( C_K \) be the 1-dimensional rep of the subalgebra \( g[t] \otimes CK \). Set \( K \cdot v = kv \). The vacuum rep of level \( k \) is

\[
V_k(\mathfrak{g}) := \text{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[t] \otimes CK} C_K = U(\hat{\mathfrak{g}}) \otimes U(\mathfrak{g}[t] \otimes CK) C_K.
\]

Let \( \{J^a\}_{a=1}^{\dim \mathfrak{g}} \) be a basis of \( \mathfrak{g} \), and set \( J^a_n := J^a \otimes t^n \). Then by PBW, a basis of \( V_k(\mathfrak{g}) \) is

\[
\{J_{n_1}^a \cdots J_{n_m}^a \otimes 1\}.
\]

The VOA structure is defined as follows.

1. The vacuum is \( |0\rangle := 1 \otimes 1 \).
2. Translation is defined inductively by \( T|0\rangle = 0 \) and \( [T, J^a_n] := -nJ^a_{n-1} \).
3. Vertex operators are defined as follows:

\[
Y(J^a_{-1} \otimes 1, z) := J^a(z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n+1} \partial_z^{-n},
\]

\[
Y(J_{n_1}^a \cdots J_{n_m}^a \otimes 1, z) = \frac{1}{(n_1 - 1)! \cdots (n_m - 1)!} \partial_z^{n_1 - 1} J^a_1(z) \cdots \partial_z^{n_m - 1} J^a_m(z). \]

Checking locality is analogous to the previous case.

Definition 3.4. Let \( (V, |0\rangle, T < Y) \) be a vertex algebra. A vector space \( M \) is a \( V \)-module if it is equipped with an action

\[
Y_M : V \to \text{End} M[[z, z^{-1}]], \quad A \mapsto \sum_{n \in \mathbb{Z}} A_m^{(n)} z^{-n+1}.
\]

This action must satisfy the following axioms:

1. \( Y_M(|0\rangle, z) = \text{id}_M; \)
2. for \( A, B \in V \) and \( m \in M \), the elements

\[
Y_M(A, z) Y_M(B, w) m \in M((z))(w) \]

\[
Y_M(Y(A, z - w)B, w) m \in M((w))(z - w)
\]

represent the same element in \( M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \).
Remark. What does this mean? By Taylor expansion, we get a map
\[ M((z))((w)) \to M((w))((z - w)), \quad z \mapsto w + (z - w). \]

On one hand,
\[ Y_M(A, z)Y_M(B, w) = Y_M(A, w + z - w)Y_M(B, w) = \sum_{n=0}^{\infty} \frac{\partial^n Y_M(A, w)}{n!} Y_M(B, w)(z - w)^n. \]

On the other hand,
\[ Y_M(Y(A, z - w)B, w) = Y_M(\sum_n A_n \cdot B(z - w)^{-n-1}, w) = \sum_n Y_M(A_n B, w)(z - w)^{-n-1}. \]

If \( M = V \), this is an OPE.

Remark. There is a Lie algebra associated to every VOA, such that \( V \)-modules are the same as \( g \)-modules. This is as follows. A consequence of OPEs is that for \( Y(A, z) \) and \( Y(B, z) \),
\[ [A_m, B_k] = \sum_{n \geq 0} \binom{m}{n} (A_n B)_{m+k-n}. \]

Hence the span of all Fourier coefficients of all vertex operators form a Lie subalgebra \( U'(V) \subset \text{End}(V) \).

(That this satisfies the Jacobi identity is automatic from it being a commutator.)

**Theorem 3.5.** There is an equivalence of categories
\[ \text{Mod}(V) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{smooth and coherent modules} \\ \text{of Lie algebra generated by } U'(V) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{smooth modules of } \tilde{U}(V) \end{array} \right\}. \]

**Example 3.6** (VOAs associated to 1-dimensional lattices). Let \( C_{\lambda} \) be the 1-dimensional rep of \( \tilde{H} \) where \( b_0 |\lambda\rangle = \lambda |\lambda\rangle \). Let \( \pi_{\lambda} := \tilde{H} \otimes_{\tilde{H}^+} C_{\lambda} \), with basis \( \{ b_0^z b_1^z \otimes |\lambda\rangle \} \). This is a locally finite \( \tilde{H} \)-module, and therefore a \( \pi \)-module. Actually one can show it is irreducible. Let \( N \in \mathbb{Z}^+ \) and set
\[ V_{\sqrt{N}} := \bigoplus_{m \in \mathbb{Z}} \pi_{m \sqrt{N}}. \]

This turns out to be a VOA.

**Theorem 3.7.** For any even \( N \) (resp. odd \( N \)), the module \( V_{\sqrt{N}} \) carries the structure of a VOA (resp. super VOA) such that \( \pi_0 \) is a vertex subalgebra.

**Proof.** Once we define \( V_{\lambda}(z) := Y(1 \otimes |\lambda\rangle, z) \) for \( \lambda \in \sqrt{N} \mathbb{Z} \), we are done. It looks like
\[ V_{\lambda}(z) := S_{\lambda} z^{\lambda b_0} \exp \left( -\lambda \sum_{n<0} \frac{b_n}{n} z^{-n} \right) \exp \left( -\lambda \sum_{n>0} \frac{b_n}{n} z^{-n} \right). \]

**Remark.** When \( N = 1 \), actually there is an identification
\[ V_{\mathbb{Z}} \cong \Lambda := \text{Cl} \otimes_{\text{Cl}^+} \mathbb{C} \]
where \( \text{Cl} \) is the Clifford algebra on \( \{ \psi_n, \psi_n^* \} \). This is the boson-fermion correspondence, and e.g. the Jacobi triple product comes from looking at graded dimension on both sides.
4 Yasha (Oct 15): Minimal models

Recall that $\varphi(z)$ is a primary field if it is highest weight in a representation of the Virasoro algebra. In terms of OPEs,

$$T(z)\varphi(w) = \frac{\Delta\varphi(w)}{(z-w)^2} + \frac{\partial\varphi(w)}{z-w} + \text{reg.}$$

In terms of the Virasoro algebra, this is equivalent to

$$L_n\varphi(z) = \begin{cases} 0 & n > 0 \\ \Delta\varphi(z) & n = 0. \end{cases}$$

**Theorem 4.1. Correlators of descendant fields are determined by correlators of primary fields.**

**Proof.** Consider a correlator

$$\langle L_{-n}A(z)A_1(z_1)\cdots A_N(z_N) \rangle \quad n \geq 0.$$ 

We can write

$$L_{-n} = \oint T(\zeta)\zeta^{-n+1} \frac{d\zeta}{2\pi i}.$$ 

Substituting this into the correlator gives

$$\oint \frac{d\zeta}{2\pi i} (\zeta - z)^{-n+1} \langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N) \rangle$$

where the contour is around poles of $A(z)$ and not poles of the other $A_i$. The stress-energy tensor $T(\zeta)$ decays very quickly as $\zeta \to \infty$, so we can expand the contour to include all other poles, giving

$$-\sum_{i=1}^N \oint \frac{d\zeta}{2\pi i} (z_i - z + \zeta - z_i)^{-n+1} \langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N) \rangle.$$ 

The series expansion of this will have $L_n$ acting on the other fields $A_i(z_i)$:

$$-\sum_{i=1}^N \sum_{k \geq 0} \oint_{C_i} \frac{d\zeta}{2\pi i} \left( \frac{1 - n}{k} \right) (z_i - z)^{1-n-k} (\zeta - z_i)^k \langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N) \rangle$$

$$= -\sum_{i=1}^N \sum_{k \geq 0} \left( \frac{1 - n}{k} \right) (z_i - z)^{1-n-k} \langle A(z)A_1(z_1)\cdots L_{k-1}A_1(z_i)\cdots A_N(z_N) \rangle.$$ 

If all fields $A_1, \ldots, A_N$ are primary, then only $k = 0$ and $k = 1$ act non-trivially. Hence there are two terms

$$L_{-n}A(z)\phi_1(z_1)\cdots \phi_N(z_N) = \sum_{i=1}^N \left( \frac{(n-1)\Delta_i}{(z_i - z_i)^n} - \frac{1}{(z_i - z)^{n-1}} \frac{\partial}{\partial z_i} \right) \langle A(z)A_1(z_1)\cdots A_N(z_N) \rangle.$$ 

In fact, the contribution of descendants to OPEs (three-point correlators) is also fully determined by the contribution of primary fields. Suppose we have two primary fields $\varphi_n(z)$ and $\varphi_m(w)$. Then we can always expand

$$\varphi_n(z)\varphi_m(0) = \sum_{p,\lambda} C^{p\lambda}_{nm} z^{\Delta_p - \Delta_n - \Delta_m + |\lambda|} \varphi_p(0)$$

where $\lambda$ encodes the descendant $\varphi_p^\lambda := L_{-\lambda_1} \cdots L_{-\lambda_n} \varphi_p$. The claim is that it suffices to know $C^{p\lambda}_{nm}$ only for $\lambda = \emptyset$, and all the other coefficients are determined by conformal symmetry. Rewrite the expansion as

$$\varphi_n(z)\varphi_m(0) = \sum_{p,\lambda} C^{q\lambda}_{nm} z^{\Delta_p - \Delta_n - \Delta_m} \Psi_p(z) \Psi_m(0) = \sum_{p} \beta^{p\lambda}_{nm} z^{\Delta_p} \varphi^\lambda_p(0).$$
How do we determine the $\beta$ coefficients? Assume for simplicity that $\Delta_p = \Delta_n = \Delta_m = \Delta$. Then

$$
\varphi_\Delta(z) |\Delta\rangle = \sum C_{\Delta\Delta}' \varphi_{\Delta'}(z) |\Delta_p\rangle, \quad \varphi_\Delta(z) = \sum z^{[\lambda]} \beta^\lambda L_{-\lambda_1} \ldots L_{-\lambda_k}.
$$

Apply $L_n$ to both sides, for $n > 0$, to get

$$
L_n \varphi_1(z) \varphi_2(0) = \oint d\zeta T(\zeta) \zeta^{-n+1} \varphi_1(z) \varphi_2(0)
= (z^{n+1} \frac{\partial}{\partial z} + (n + 1) z^n \Delta_1) \varphi_1(z) \varphi_2(0) + \varphi_1(z) L_n \varphi_2(0).
$$

(The two terms come from $L_1$ and $L_0$ respectively.) The second term is zero, because $\varphi_2$ is primary.

Now suppose we want to compute four-point correlators

$$
\langle \varphi_k(\zeta_1) \varphi_\ell(\zeta_2) \varphi_n(\zeta_3) \varphi_m(\zeta_4) \rangle = \langle k| \varphi_\ell(1) \varphi_n(x) |m\rangle = G_{nm}^{lk}(x).
$$

Here we used an automorphism to set $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$. By gluing,

$$
G_{nm}^{lk}(x) = \sum_p C_{nm}^p A_{nm}^{lk}(p, x).
$$

This function

$$
A_{nm}^{lk}(p, x) = (C_{kl}^p)^{-1} \langle k| \varphi_\ell(1) \varphi_p(x) |0\rangle
$$

is called a conformal block.

In minimal models, we consider degenerate representations of the Virasoro algebra $\text{Vir} = \bigoplus_i \mathbb{C} L_i \oplus \mathbb{C} c$.

**Theorem 4.2** (Kac, Feigin–Fuchs). *The Verma module $|\Delta, c\rangle$ is irreducible for generic $\Delta$ and $c$. Set*

$$
\alpha_\pm := \frac{\sqrt{1-c} \pm \sqrt{25-4c}}{2}, \quad \Delta_0 := (1/4)(\alpha_+ + \alpha_-)^2.
$$

*For fixed $c$, there is a singular vector at the following levels:*

$$
\Delta_{mn} := \Delta_0 + \frac{1}{4}(m\alpha_+ + n\alpha_-), \quad m, n \in \mathbb{Z}_{> 0}.
$$

**Example 4.3** (Level 0). The simplest example is if $L_{-1} |\Delta\rangle$ is a singular vector. Then we have

$$
L_1 L_{-1} |\Delta\rangle = 0.
$$

But this is equal to

$$
[L_1, L_{-1}] |\Delta\rangle = 2L_0 |\Delta\rangle = 2\Delta |\Delta\rangle.
$$

Hence this is only possible when $\Delta = 0$. This corresponds to $m = n = 0$.

**Example 4.4** (Level 2). The next simplest example is that

$$
v = (L_{-2} + aL_{-1}^2) |\Delta\rangle
$$

is a singular vector. Then

$$
0 = L_1 v = [L_1, L_{-2}] |\Delta\rangle + (a[L_1, L_{-1}] L_{-1} + a L_{-1} [L_1, L_{-1}]) |\Delta\rangle
= 3L_{-1} |\Delta\rangle + 2a L_0 L_{-1} |\Delta\rangle + 2a L_{-1} L_0 |\Delta\rangle
= (3 + 2a(2\Delta + 1)) |\Delta\rangle.
$$
Hence

\[ a = -\frac{3}{2(2\Delta + 1)}. \]

By similar calculations,

\[ 0 = L_2 v = \left(4\Delta + \frac{c}{2} - \frac{9\Delta}{2\Delta + 1}\right) |\Delta\rangle. \]

This constrains \( c \).

Let

\[ \varphi^{(2)}_{12} = \left(L_{-2} - \frac{3}{2(2\Delta + 1)} L_{-1}\right) \phi_{12} = 0. \]

This vector has norm zero and therefore we require it to really be zero. Transforming it into an operator gives

\[ -\frac{3}{2(2\Delta + 1)} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{N} \left(\frac{\Delta_i}{(z_i - z)^2} - \frac{1}{z_i - z} \frac{\partial}{\partial z_i}\right) \langle \phi_{12}(z) \phi_1(z_1) \cdots \phi_N(z_N) \rangle = 0. \]

The general OPE looks like

\[ \phi_{12}(z) \phi_\Delta(z_1) = \sum_{\Delta'} C_{12,\Delta}^{\Delta'} (z - z_1)^{\Delta' - \Delta} + \cdots. \]

Applying the operator to this, we get a constraint

\[ \frac{3\mathcal{H}(\mathcal{H} - 1)}{2(2\Delta_{12} + 1)} + \Delta - \mathcal{H} = 0 \]

where \( \mathcal{H} := \Delta' - \Delta_{12} - \Delta \). The two solutions are

\[ \Delta' = \Delta(\alpha - \alpha_-), \Delta(\alpha + \alpha_+). \]

Hence we have shown

\[ \phi_{12} \phi_\alpha = [\phi_{\alpha - \alpha_-}] + [\phi_{\alpha + \alpha_-}]. \]

Here we are using fusion ring notation, where we forget about coefficients and just remember whether or not a field appears in the OPE. Similarly,

\[ \phi_{21} \phi_\alpha = [\phi_{\alpha - \alpha_+}] + [\phi_{\alpha + \alpha_+}]. \]

In particular, applying both rules,

\[ [\phi_{02}] + [\phi_{22}] = \phi_{12} \phi_{21} = [\phi_{20}] + [\phi_{22}]. \]

Hence it must actually just be \([\phi_{22}]\). As another example,

\[ \phi_{12} \phi_{12} = [\phi_{11}] + [\phi_{13}]. \]

But \([\phi_{11}]\) is the identity operator. So we now know how to apply \( \phi_{13} \):

\[ \phi_{13} \phi_\alpha = [\phi_{\alpha + 2\alpha_-}] + [\phi_\alpha] + [\phi_{\alpha + 2\alpha_-}]. \]

In general,

\[ \phi_{m_1 n_1} \phi_{m_2 n_2} = \sum_{\ell=0}^{k_0} \sum_{k=0}^{k_0} \phi_{m_0 + 2\ell, n_0 + 2k}. \]
where
\[ m_0 = |m_1 - m_2| + 1 \]
\[ n_0 = |n_1 - n_2| + 1 \]
\[ \ell_0 = \min(m_1, m_2) - 1 \]
\[ k_0 = \min(n_1, n_2) - 1. \]

This corresponds to multiplication in SL\(_q(2)\).

Now let’s consider some minimal models. For minimal models, pick two integers \( p \neq q \). Suppose we have
\[ \alpha_+ := -\sqrt{\frac{q}{p}}, \quad \alpha_- := \sqrt{\frac{p}{q}}. \]

Let \( m < p \) and \( n < q \) and consider \( |\Delta_{mn}\rangle \). Note that
\[ \Delta_{p+m,q+n} = \Delta_{mn}, \quad \Delta_{p-m,q-n} = \Delta_{mn}. \]

The central charge is
\[ c = 1 - \frac{6(p - q)}{4pq}. \]

We have
\[ \mathcal{M}(p/q) = \frac{1}{2} \bigoplus [\phi]. \]

For minimal models it is possible to determine all the structure constants, by crossing symmetry. The four-point function turns out to be a hypergeometric function in \( x \). We know how to analytically continue it, to get it in \( 1/x \). This gives explicit constraints on structure constants. For example,
\[ \mathcal{M}\left(\frac{2}{5}\right) = \mathbb{C}[\phi_{11}] + \mathbb{C}[\phi_{12}], \]

since \( \phi_{12} = \phi_{13} \) and \( \phi_{11} = \phi_{14} \). Hence
\[ \phi_{12}\phi_{12} = C\phi_{12} + \cdots \]

for some constant \( C \). From the explicit constraints, we can compute
\[ C = i \frac{\Gamma(1/5)\Gamma(2/5)\Gamma(3/5)}{\Gamma(4/5)} \sqrt{\frac{\sqrt{5} - 1}{2}}. \]

**Example 4.5.** Consider \( \mathcal{M}(p/p + 1) \). These are called *unitary* minimal models. It is conjectured that these are the only minimal models with real structure constants. In \( \mathcal{M}(3/4) \), we have
\[
\begin{array}{ccc}
\epsilon & 22 & 23 \\
11 & 12 & 13 \\
I & \sigma & \epsilon \\
\end{array}
\]

Then we have
\[ \epsilon\epsilon = \phi_{21}\phi_{21} = [I] + [\phi_{31}], \]

But we can also write it as
\[ \epsilon\epsilon = \phi_{13}\phi_{13} = [I] + [\phi_{13}] + [\phi_{15}]. \]

The dimension of \( \phi_{31} \) is \( \Delta = 5/3 \), whereas for \( \phi_{15} \) it is \( \Delta = 5/2 \). Hence they are different fields, and so
\[ \epsilon\epsilon = [I]. \]

Similarly one can compute (in the fusion ring)
\[ \epsilon\sigma = \sigma, \quad \sigma\sigma = I + \epsilon. \]
When we pair with the antiholomorphic part, this example is actually the Ising model (at criticality). For \( \phi_{21} = \phi_{13} = \epsilon \), we have \( \Delta = 1/2 \). Take
\[
\psi \in V_{12} \otimes V_{11}, \quad \overline{\psi} \in V_{11} \otimes V_{12}.
\]
Then
\[
\psi(z)\psi(0) = \frac{I}{z} + \text{reg.}
\]
\[
\overline{\psi}(z)\overline{\psi}(0) = \frac{I}{z} + \text{reg.}
\]
\[
\psi(z)\overline{\psi}(0) = i\epsilon + \text{reg.}
\]
We see a free fermion subalgebra \([I] \oplus [\psi] \oplus [\overline{\psi}] \oplus [\epsilon]\), with Hamiltonian
\[
H = \frac{1}{2} \int (\overline{\psi} \partial \psi + \overline{\psi} \partial \psi) \, d^2x
\]
and \( T = (-1/2) :\psi \partial \psi: \). Analogously, \( \phi_{21} = \phi_{22} \) has \( \Delta = 1/16 \). Take \( V_{21} \otimes V_{21} \) and the subalgebra
\[
\mathcal{A}_{IM} := [I] \oplus [\sigma] \oplus [\mu] \oplus [\epsilon] \oplus [\psi] \oplus [\overline{\psi}]
\]
where
\[
\psi(z)\sigma(0) = z^{-1/2}\mu(0) + \cdots.
\]
This space corresponds to the Ising model, where \( \sigma \) is the parameter of order and \( \mu \) is the parameter of disorder. There is some duality between them at high temperatures.

5 Guillaume (Oct 22): A probabilistic approach to Liouville CFT

We’ll start with some motivation. Liouville field theory first appeared in Polyakov’s 1981 paper “Quantum geometry of bosonic strings”. The idea is that when we have a quantum theory, to model a particle going from point A to point B, we sum over all paths connecting the two points. In string theory, points are replaced by loops, and therefore paths by surfaces connecting the loops. In this talk, the surface will always be the Riemann sphere \( S^2 \). Let \( \mathcal{M} \) be the space of all Riemannian metrics on \( S^2 \). Polyakov tried to understand what the canonical uniform measure on \( \mathcal{M} \). This is highly non-trivial because it is an infinite-dimensional, highly non-linear space. In particular we are interested in quantities
\[
\int_{\mathcal{M}} Dg \, F[g]
\]
for some formal measure \( Dg \) and functional \( F[g] \). Recall the uniformization theorem from Riemannian geometry, which says that
\[
\mathcal{M} = \{ e^{\phi}g : \phi : S^2 \to \mathbb{R} \}
\]
for some fixed metric \( g \) on \( S^2 \). Physicists understood that choosing a uniform measure on \( \mathcal{M} \) is essentially choosing a measure on \( \phi \) given by Liouville field theory.

What is Liouville theory? In the path integral formalism, let \( \Sigma \) be the space of all \( X : S^2 \to \mathbb{R} \). Define
\[
\langle F(\phi) \rangle := \frac{1}{Z} \int_{\Sigma} DX \, F(X) e^{-S_L(X)}
\]
for some uniform measure \( DX \), and “weight” \( e^{S_L(X)} \). The Liouville action is
\[
S_L(X) = \frac{1}{4\pi} \int_{S^2} (|\partial^g X|^2 + QR_g X + \mu e^\gamma X) \, d\lambda_g.
\]
• The $|\partial^g X|^2$ term is kinetic energy. It is the most basic term we can put into any energy functional.

• The $e^{\gamma X}$ is a non-linear term which makes the whole theory non-trivial. It is the total volume of $S^2$ using the metric given by $X$.

• The coupling constant $\mu$ is called the **cosmological constant**. The whole theory in the end depends trivially on $\mu$ so it doesn’t really matter.

• $\gamma \in (0, 2)$. In physics, the notation is $b = \gamma/2$.

• $Q = 2/\gamma + \gamma/2$, and $c_L := 1 + 6Q^2$.

Mark some points $z_i \in S^2$, with associated weights $\alpha_i \in \mathbb{R}$. The geometric interpretation is some conical singularities at those points. Then set

$$F(X) := \prod_{i=1}^N e^{\alpha_i X(z_i)}.$$

For the correlator $\langle F(X) \rangle$ to exist, the **Seiberg bounds** must hold:

$$\sum_i \alpha_i > 2Q, \quad \alpha_i < Q \forall i.$$

Hence the minimum number of points for it to be well-defined is three. Note that these $e^{\alpha_i X(z_i)}$ are vertex operators, and primary fields.

In Liouville theory, we can make the path integral formalism rigorous using probability (following David, Kupiainen, Rhodes, Vargas, 2014). Start with

$$\frac{1}{Z} \int_{\Sigma} DX e^{-\frac{1}{4\pi} \int_{S^2} |\partial^g X|^2 d\lambda_g \hat{F}(X)}.$$

Integrating by parts,

$$\int_{S^2} |\partial^g X| d\lambda_g = -\int_{S^2} X \Delta_g X d\lambda_g.$$

So the kinetic term will give a Gaussian free field, with covariance given by the Green’s function. If we diagonalize $-\Delta_g$ as

$$-\Delta_g \varphi_j(x) = \lambda_j \varphi_j(x) \quad \lambda_j > 0.$$

Then $X = c + \sum_{j \geq 1} c_j \varphi_j(x)$. We think of $c = \int_{S^2} X d\lambda_g$, and the $\varphi_j$ are chosen with $\int_{S^2} \varphi_j(x) d\lambda_g = 0$. Then

$$-\frac{1}{4\pi} \int_{S^2} X \Delta_g X d\lambda_g = -\frac{1}{4\pi} \sum_{j \geq 1} \lambda_j c_j^2.$$

Hence the correlator becomes

$$\frac{1}{Z} \int_{\mathbb{R}} dc \int_{\mathbb{R}^N} \prod_{j \geq 1} e^{-u_j^2/2} \frac{du_j}{\sqrt{2\pi}} \mathbb{E} \left[ \hat{F} \left( c + \sqrt{2\pi} \sum_{j \geq 1} u_j \frac{\varphi_j}{\sqrt{\lambda_j}} \right) \right].$$

Here $X_{GFF}$ is a Gaussian free field. So we define the correlator as

$$\int_{\mathbb{R}} dc \mathbb{E}[\hat{F}(c + X_{GFF})].$$

The $\hat{F}$ will include the rest of the terms in the Liouville action. In summary,

$$\langle \prod e^{\alpha_i \delta(z_i)} \rangle := \int_{\mathbb{R}} dc \mathbb{E} \left[ \prod_{i=1}^N e^{\alpha_i X_{GFF}(z_i) + c} \exp \left( -\frac{1}{4\pi} \int_{S^2} QR_g(X_{GFF} + c) d\lambda_g - \frac{\mu}{4\pi} \int_{S^2} e^{\gamma(X_{GFF} + c)} d\lambda_g \right) \right].$$
Strictly speaking we need to introduce a regularization \( \epsilon \) and send \( \epsilon \to 0 \). This integral over the zero mode \( c \) can be computed, giving

\[
\frac{2\gamma^{-s}}{\mu} \Gamma(s) \prod_{i<j} \frac{1}{|z_i - z_j|^\alpha_{ij}} \mathbb{E} \left[ \left( \int_{S^2} e^{\gamma X_G F(x)} \prod_{i=1}^N \frac{1}{|x - z_i|^N} d\lambda_g(x) \right)^{-s} \right]
\]

with \( s = (\sum \alpha_i - 2Q)/\gamma \).

Now that this is well-defined, we can ask for the usual structures of CFT (e.g. OPE, BPZ equations) in this language. Given a Mobius map \( \psi : S^2 \to S^2 \), correlators behave as conformal tensors

\[
\langle \prod_{i=1}^N e^{\alpha_i \phi(\psi(z_i))} \rangle = \prod_{i=1}^N |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle
\]

with conformal weight \( \Delta_{\alpha_i} = (\alpha_i/2)(Q - \alpha_i/2) \). This is like seeing global conformal invariance. We can also ask for the BPZ equations, which is like seeing local conformal invariance. Let \( \chi := -\gamma/2 \) or \( -2/\gamma \). Then

\[
\left( \frac{1}{\chi^2} \frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z_i - z_j)^2} + \sum_{i=1}^N \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \langle e^{\chi \phi(z)} \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle = 0.
\]

By solving the BPZ equations, we get an explicit formula for 3-point functions called the **DOZZ formula**. Move the points \( z_1, z_2, z_3 \) to 0, 1, \( \infty \) to get

\[
\langle \prod_{i=1}^3 e^{\alpha_i \phi(z_i)} \rangle = \frac{1}{|z_1 - z_2|^{|\Delta_{\alpha_2}|z_1 - z_3|^{|\Delta_{\alpha_3}|z_2 - z_3|^{|\Delta_{\alpha_3}|}}} C_\gamma(\alpha_1, \alpha_2, \alpha_3).
\]

The DOZZ formula is for \( C_\gamma \), and is in terms of double gamma functions. It is an analytic expression in the \( \alpha_i \).

**Theorem 5.1** (Kupiainen, Rhodes, Vargas, 2017). *The three-point correlator defined via probability is equal to the DOZZ formula.*

6 Ivan (Oct 29): WZW

No notes, sorry!

7 Gus (Nov 05): Free field realizations

Recall last time Ivan discussed WZW theory. The input is a simple Lie algebra \( g \) with \( g \)-invariant bilinear form \((\cdot, \cdot)\). Then we consider the affine Lie algebra \( \hat{g} = g[t, t^{-1}] \oplus \mathbb{C} c \), which has commutators

\[
[x_n, y_m] = [x, y]_{n+m} + \delta_{n+m,0} c(x, y).
\]

We considered various representations of \( \hat{g} \). The first kind is Verma modules. If \( V_\lambda \) is a Verma for \( g \), then there is a corresponding Verma \( V_{\lambda,k} := \text{Ind}_{\hat{g}}^\hat{g} (V_\lambda) \) for \( \hat{g} \). We always suppose \( \lambda \) is generic, so that \( V_{\lambda,k} \) is irreducible. The other kind is evaluation modules. If \( V \) is any rep of \( g \), then \( V(z) \) is the rep of \( \hat{g} \) where \( x_n \) acts as \( z^n x \).

Using these representations, we considered intertwiners

\[
\Phi(z) : V_{\lambda_1,k} \to V_{\lambda_2,k} \otimes V_{\mu}(z).
\]
Take such $\Phi_{i}(z_{i})$ for $\lambda_{0}, \ldots, \lambda_{n+1}$ and consider matrix elements

$$
\Psi(z_{1}, \ldots, z_{n}) := \langle u_{0}, \Phi_{1}(z_{1}) \cdots \Phi_{n}(z_{n}) u_{n+1} \rangle
$$

for fixed vectors $u_{0} \in V_{\lambda_{0}, k}$ and $u_{n+1} \in V_{\lambda_{n+1}, k}$. These are correlation functions, and satisfy KZ equations

$$(k + h^{\vee}) \partial_{z_{i}} \Psi = \sum_{j \neq i} \Omega_{ij} \frac{\Psi}{z_{i} - z_{j}}.$$

Today we will focus on how to actually compute these intertwiners $\Phi$ explicitly, in a form suitable for calculating matrix elements.

**Reminder on OPEs.** Given two fields $a(z) = \sum a_{n} z^{-n-1}$ and $b(z) = \sum b_{n} z^{-n-1}$, they are *local* if

$$
[a(z), b(w)] = N^{-1} \sum_{j=0}^{N-1} c_{j}(w) \partial_{z}^{j} \delta(z - w)
$$

is a sum of delta functions and their derivatives. The $c_{j}(w)$ are OPE *coefficients*. We set

$$
a_{\pm}(z) := \sum_{n<0} a_{n} z^{-n-1}, \quad a_{\pm}(z) := \sum_{n \geq 0} a_{n} z^{-n-1}.
$$

This splitting is done so that $(\partial a)_{\pm} = \partial (a_{\pm})$. The normally ordered product of fields is

$$
:a(z)b(w): = a_{+}(z)b(w) + b(w)a_{-}(z) = a_{+}b_{+} + a_{+}b_{-} + b_{+}a_{-} + b_{-}a_{-},
$$

i.e. we put all minus terms before plus terms. Then

$$
\partial :a_{\pm}b: = :\partial a_{\pm}b: + :a_{\pm}\partial b: .
$$

The only distinction between $:a(z)b(w):$ and $a(z)b(w)$ is the term $b_{+}a_{-}$, so

$$
a(z)b(w) = :a(z)b(w): + [a_{-}(z), b(w)]
$$

$$
\sim :a(z)b(w): + \iota_{z,w} \sum_{j=0}^{N-1} \frac{c_{j}(w)}{(z - w)^{j+1}}
$$

where $\iota_{z,w}$ means expansion in $|z| > |w|$. Hence to compute OPEs, we just need the difference between $a(z)b(w)$ and the normal ordered $:a(z)b(w):$. The notation is

$$
a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c_{j}(w)}{(z - w)^{j+1}}
$$

in this case.

**Example 7.1 (Free boson).** For the free boson, we have $a(z) = \sum a_{n} z^{-n-1}$ with $[a_{n}, a_{m}] = n \delta_{n+m, 0}$. This corresponds to the OPE

$$
a(z)a(w) \sim \frac{1}{(z - w)^{2}}.
$$

Equivalently, this says

$$
[a(z), a(w)] = \partial \delta(z - w).
$$
Example 7.2 (Current algebras). Given a simple $\mathfrak{g}$, for each $x \in \mathfrak{g}$ we produce a field

$$J_x(z) := \sum_n x_n z^{-n-1}.$$  

The commutator $[x_n, y_m] = [x, y]_{n+m} + n c(x, y) \delta_{n+m,0}$ yields the OPE

$$J_x(z)J_y(w) \sim \frac{J_{[x,y]}(w)}{z-w} + \frac{c(x,y)}{(z-w)^2}.$$  

Now we can talk about free field realizations of $\hat{\mathfrak{g}}$. Some motivation comes from the finite dimensional setting. The easiest Lie algebras to handle are the abelian ones; the next easiest, with non-trivial commutators, is the Heisenberg algebra $\text{Heis}$ with commutator

$$[\partial, x] = c.$$  

In the Heisenberg, $[a, b] = \text{scalar}$ for any elements, so it is very easy to compute nested commutators. On the other end of the spectrum, for simple Lie algebras we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, giving rise to highly non-trivial terms in the BCH formula.

The difference between simple $\mathfrak{g}$ and the Heisenberg algebra “goes away” if we pass to universal enveloping algebras. Namely, there is an algebra homomorphism $U\mathfrak{g} \to U \text{Heis}$. It comes from the action of $G$ on $G/B$, which gives a homomorphism $\mathfrak{g} \to \text{Vect}(G/B)$. This extends to

$$U\mathfrak{g} \to \text{Diff}(G/B).$$  

More generally, we can pick a line bundle $\lambda \in \text{Pic}(G/B) \otimes \mathbb{C}$ and look at differential operators twisted by $\lambda$.

Example 7.3. For $\mathfrak{sl}_2$, the flag variety is $G/B = \mathbb{P}^1$. We have

$$\text{Heis}_1 = \text{Diff}(\mathbb{C}) \subset \text{Diff}(\mathbb{P}^1).$$  

The homomorphism is

$$e \mapsto \partial, \quad f \mapsto -x^2 \partial, \quad h \mapsto -2x \partial,$$

given by restriction to a specific chart $\mathbb{C} \subset \mathbb{P}^1$. In general we can restrict to any chart, which is equivalent to changing a Borel $B$. More generally, picking a line bundle $\lambda \in \mathbb{C}$ gives

$$e \mapsto \partial, \quad f \mapsto -x^2 \partial + \lambda, \quad h \mapsto -2x \partial + \lambda x.$$  

We want to affinize this construction. A free theory is a collection of pairwise local fields $\{a^j(z)\}$ such that

$$[a^j_+ (z), a^j_+ (w)] = 0$$

and all OPE coefficients $c^j(w)$ are just scalars. Roughly this is the affine analogue of $[A, B] = \text{scalar}$. The idea is to build fields with more interesting OPEs in terms of normally ordered products of free fields, in the same way that the example built $\mathfrak{sl}_2$ from $\text{Heis}_1$.

For this purpose, we need some tools for computing products of normally ordered products (of free fields).

1. (Taylor’s theorem) Given a field $a(z, w)$ and some cutoff $N \in \mathbb{N}$, there is a Taylor expansion

$$a(z, w) = \sum_{j=0}^{n-1} \frac{c^j(w)(z-w)^j + (z-w)^N}{a^j_+ (z, w)|_{z=w}}.$$  

where $c^j(w) = \partial^j a(z, w)|_{z=w}$.
2. (Wick’s theorem) If \( \{a_1, \ldots, a_M\} \) and \( \{b_1, \ldots, b_N\} \) are two collections of free fields,

\[
: a_1(z) \cdots a_M(z) : b_1(w) \cdots b_N(w): \]

is the sum over all possible subset of pairs of “contractions”. Formally, it is

\[
\sum_{s=0}^{\min(N,M)} \sum_{i_1 < \cdots < i_s, j_1 \neq \cdots \neq j_s} \prod_{k=1}^{s} [a_{i_k}, -](z), b_{j_k}(w) : A(i)B(j) : \]

where \( A(i) \) denotes the product of all \( a_i(z) \) where \( i \) is \textit{not} one of the chosen indices \( i_k \), and similarly for \( B(j) \). The proof is roughly that commuting + across − produces terms \( [a_-, b](w) \), but because all OPE coefficients are scalar these can be collected in front. Notation:

\[
[a_-, b](w) =: \langle a(z)b(w) \rangle.
\]

\textbf{Example 7.4 (Virasoro).} Let \( a(z) \) be a free boson. Then

\[
L(z) := \frac{1}{2} : a(z)^2 : \]

satisfies the OPE

\[
L(z)L(w) \sim \frac{1}{(z-w)^4} + \frac{L(z)}{(z-w)^2} + \frac{2L'(z)}{z-w},
\]

which is exactly the OPE for the Virasoro algebra.

\textbf{Example 7.5 (\( \hat{\mathfrak{sl}}_2 \)).} Take three free fields: a free boson \( \alpha(z) = \sum \alpha_n z^{-n-1} \), and a \( \beta \gamma \) system \( \beta(z) = \sum \beta_n z^{-n-1} \) and \( \gamma(z) = \sum \gamma_n z^{-n} \). This indexing of \( \gamma_n \) is done so that

\[
[\beta_{\pm}, \gamma_{\pm}] = 0.
\]

Normalize \( \alpha \) so that

\[
\alpha(z)\alpha(w) \sim \frac{2}{(z-w)^2}
\]

because we want to think of it as the root \( \alpha \) in \( \mathfrak{sl}_2 \). We also set

\[
\beta(z)\gamma(w) \sim \frac{1}{z-w}
\]

and all other OPEs are trivial. Unpacking,

\[
[\beta_n, \gamma_m] = \delta_{n+m,0}.
\]

For \( \lambda \in \mathbb{C} \), consider the rep \( \mathcal{H}_\lambda \) of \( \{\alpha_n, \beta_n, \gamma_n\} \) generated by a vacuum vector \( v \) satisfying

\[
\alpha_0 v = \lambda v, \quad \beta_0 v = 0, \quad \alpha_n v = \beta_n v = \gamma_n v = 0 \ \forall n > 0.
\]

Now define

\[
J_e(z) = \beta(z) \quad J_h(z) = -2 : \gamma(z)\beta(z) : +\kappa^{1/2} \alpha(z) \quad J_f(z) = - : \gamma^2(z)\beta(z) : +\kappa^{1/2} \alpha(z)\gamma(z) + k \gamma'(z)
\]

where \( \kappa := k + \hbar' \) with \( k \) the level. For \( \mathfrak{sl}_2 \) we have \( \hbar' = 2 \).
Theorem 7.6. This prescription defines a rep of $\hat{sl}_2$ on $\mathcal{H}_\lambda(k)$. If $\lambda, k$ are generic, i.e. $V_{\lambda,k}$ is irreducible, then
\[ \mathcal{H}_{\lambda/\sqrt{\kappa}}(k) \cong V_{\lambda,k} \]
as $\hat{sl}_2$-modules.

For special $\lambda, k$, it is not always an isomorphism. For example, if $\lambda = 0$ then it is not hard to see $f_0 v = 0$. Note that there is a strong analogy with the finite case.

Why is this important? It lets us write down a formula for the intertwining operator. For $\lambda \in \mathfrak{h}$, consider free fields
\[ h^\lambda(z) h^\mu(w) \sim \frac{(\lambda, \mu)}{(z - w)^2}. \]

Construct a vertex operator
\[ X(\mu, z) := \exp \left( \sum_{n < 0} \frac{h_n^\mu}{-n} z^n \right) \exp \left( \sum_{n > 0} \frac{h_n^\mu}{-n} z^n \right) e^\mu z^0 \]
where $e^\mu : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\lambda + \mu}$ induced from $v_\lambda \mapsto v_{\lambda + \mu}$. So $X(\mu, z)$ has components in $\text{Hom}(\mathcal{H}_\lambda, \mathcal{H}_{\lambda + \mu})$. They satisfy
\[ X(\mu, z) X(\nu, w) = (z - w)^{(\mu, \nu)} : X(\mu, z) X(\nu, w) : \]
where $(z - w)^{(\mu, \nu)}$ means
\[ \exp \left( - (\mu, \nu) \sum_{n \geq 0} \frac{w^n}{n z^n} \right). \]

This is like doing the calculation $e^A B e^{-A} = B + [A, B]$. Hence we can view $X(\mu, z) X(\nu, w)$ \textit{analytically}, since we know how to analytically continue $(z - w)^{(\mu, \nu)}$ and the normally ordered product is holomorphic.

To construct $\Phi_m(z) : V_{\lambda,k} \rightarrow V_{\lambda + \mu - 2m,k} \otimes V_\mu(z)$, we think of $V_{\lambda,k} \cong \mathcal{H}_{\lambda/\sqrt{\kappa}}(k)$ and similarly for the other term.

1. $(m = 0)$ In the easy case of level zero, we take
\[ \Phi^0(z) u := X \left( \frac{\mu}{\sqrt{\kappa}}, z \right) \exp \left( - \gamma(z) \otimes e \right) u \otimes v_\mu. \]

One can check via Wick’s theorem that this is an intertwiner for $\hat{sl}_2$.

2. $(m > 0)$ In this case we need a screening current
\[ U(t) := X \left( - \frac{\alpha}{\sqrt{\kappa}}, t \right) \beta(t). \]

It satisfies
\[ [U(t), J_e(w)] = 0, \quad [J_f(z), U(w)] = \kappa \partial_w \left( \delta(z - w) X \left( - \frac{\alpha}{\sqrt{\kappa}}, w \right) \right). \]

Then form
\[ \Theta(z, t_1, \ldots, t_m) u = X \left( \frac{\mu}{\sqrt{\kappa}}, z \right) \exp \left( - \gamma(z) \otimes e \right) U(t_1) \cdots U(t_m) u \otimes v_\mu. \]

This can be analytically continued to cycles $C \subset (\mathbb{C}^*)^m$, in the $t$ variables. Then
\[ \Phi^m(z) := \int_C \Theta(z, t) dt \]
is an intertwiner, because its commutator with $J_f$ is a total $t$-derivative.
8 Sam (Nov 07): Representations of quantum affine algebras and qKZ equations

Our goal will be to understand the second column of the following table:

<table>
<thead>
<tr>
<th>$U_{\hat{g}}$</th>
<th>$U_q(\hat{g})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlators $\Phi$: $V_{\lambda,k} \to V_{\mu,k} \otimes V(z)$</td>
<td>correlators quantum currents $L_a(z)$</td>
</tr>
<tr>
<td>currents $J_a(z)$</td>
<td>qKZ equation</td>
</tr>
<tr>
<td>$\Omega_{\alpha_{i,j}}$ satisfies CYB</td>
<td>$R$ satisfies YB.</td>
</tr>
</tbody>
</table>

Then we’ll hopefully have enough time to see this for the simplest case $\mathfrak{sl}_2$.

**Definition 8.1.** Let $\mathfrak{g}$ be finite dimensional with Cartan matrix $A = (a_{ij})$. The quantum affine algebra $U_q(\hat{g})$ is a Hopf algebra over $\mathbb{C}[[q-1]]$, with generators $e_i, f_i, q^h$ where $h \in \mathfrak{h}$ and relations:

1. $[q^{h_1}, q^{h_2}] = 0$;
2. $q^h e_i = q^{\varepsilon_i(h)} e_i q^h$ and similarly $q^h f_j = q^{-\varepsilon_j(h)} f_j q^h$;
3. $[e_i, f_j] = 0$ unless $i = j$ in which case $[e_i, f_i] = q^{d_i h_i} - q^{-d_i h_i} / (q - q^{-1})$;
4. $q$-Serre relations.

Since this is a Hopf algebra we have the coproduct $\Delta$, counit $\epsilon$, and antipode $\gamma$. For example

$$\Delta(e_i) = e_i \otimes q^{h_i} + 1 \otimes e_i.$$  

**Definition 8.2.** Let $U_q(\hat{g})$ be the algebra generated by $U_q(\hat{g})$ and symbols $q^{ad}$ where $a \in \mathbb{C}$. The additional relations arise from writing

$$q^{ad} = \sum_{n>0} \frac{1}{n!} (\tau ad)^n$$

and set

$$[d, e_i] = \delta_{i0} e_i, \quad [d, f_j] = \delta_{j0} f_j, \quad [d, q^h] = 0.$$

We want an analogue of evaluation reps for $U_q(\hat{g})$, i.e. some composition

$$U_q(\hat{g}) \to U_q(\hat{g}) \xrightarrow{\rho} V.$$  

Unfortunately, unless $\mathfrak{g} = \mathfrak{sl}_n$, this first map does not exist. Instead, define

$$D_z: U_q(\hat{g}) \to U_q(\hat{g})$$

$$e_0 \mapsto z e_0$$

$$f_0 \mapsto z^{-1} f_0$$

and all other generators map to themselves. Given some representation $\rho: U_q(\hat{g}) \to \text{End}(V)$, we introduce the $z$ by the twist

$$\rho \circ D_z: U_q(\hat{g}) \to \text{End}(V(z)).$$

**Definition 8.3.** A rep $V$ is an evaluation rep if:

1. it is finite length over $U_q(\mathfrak{g}) \hookrightarrow U_q(\hat{g})$;
2. $V \cong \bigoplus \lambda V_\lambda^q$ where $V_\lambda^q$ are highest weight reps.
Construction 1. Given a highest weight rep $V^q_\lambda$ of $U_q(\mathfrak{g})$, we can induce it up to $U_q(\hat{\mathfrak{g}})$ to get $V^q_{\lambda,k}$.

Construction 2. If $V$ is a $U_q(\mathfrak{g})$-module and $g: L^q_{\lambda_1} \otimes L^q_{\lambda_2} \otimes V$ is an $U_q(\tilde{\mathfrak{g}})$-module, we can extend it uniquely to an intertwiner

$$\Phi^g(z): V^q_{\lambda_1,k} \to V^q_{\lambda_2,k} \otimes z^\Delta V[z^\pm]$$

where $\Delta = \Delta(\lambda_1) - \Delta(\lambda_2)$ with

$$\Delta(\lambda) := \frac{\langle \lambda, \lambda + \rho \rangle}{2(k + h^\vee)}.$$

Definition 8.4. A universal R-matrix is an invertible $R \in U_q(\tilde{\mathfrak{g}}) \hat{\otimes} U_q(\mathfrak{g})$ such that:

1. $R\Delta(x) = \Delta^\text{op}(x)R$;
2. $(\Delta \otimes \text{id})R = R_{13}R_{23}$;
3. $(\text{id} \otimes \Delta)R = R_{13}R_{12}$.

One can construct such a thing via the Drinfeld double construction. If $H$ is any Hopf algebra, its double is $D(H) := H \otimes H^\vee$. Then

$$R = 1 \otimes \text{id} \otimes 1 \in H \otimes H^\vee \otimes H \otimes H^\vee$$

is an R-matrix in $D(H)$. We can apply this to $U_q(\tilde{\mathfrak{g}}) = D(U_q(\tilde{\mathfrak{b}}_+))/U_q(\mathfrak{b})$ to get the universal R-matrix. The resulting formula is

$$\hat{R} = q^{c \otimes d + d \otimes c + \sum e_i \otimes f_i} \sum a_j \otimes a^j$$

where $a_j$ is a basis for $U_q(\tilde{\mathfrak{g}})$ as a module over the base ring. We'll actually use the R-matrix for $U_q(\hat{\mathfrak{g}})$

$$R := q^{-c \otimes d - d \otimes c} \hat{R}.$$

We also need to add the spectral parameter $z$, by

$$R(z) := (D_z \otimes \text{id})R = (\text{id} \otimes D_{z^{-1}})R.$$

These R-matrices satisfy the Yang–Baxter (YB) equation. (Aside: this means that given any rep $W$, the braid group $B_n$ acts on $W^\otimes n$.)

Definition 8.5. Let $R^{op} := \text{flip} \circ R$. The q-currents are

$$L^{+,V}(z) := (\text{id} \otimes \rho_V)R^{op}(z) \in U_q(\tilde{\mathfrak{g}}) \hat{\otimes} \text{End}(V)[[z]]$$

$$L^{-,V}(z) := (\text{id} \otimes \rho_V)R^{-1}(z^{-1}) \in U_q(\tilde{\mathfrak{g}}) \hat{\otimes} \text{End}(V)[[z^{-1}]]$$

These satisfy

$$L^\pm(z) = 1 \otimes 1 + (q - q^{-1}) \sum_{a \in B} J^\pm_{a}(z) \otimes \rho_V(a) + O((q - q^{-1})^2).$$
**Definition 8.6** ($q$-Sugawara). We need an action of $q^d$ in $V^q_{\lambda,k}$ when $k \neq -h^\vee$. Use that there is an adjoint action

$$\text{Ad}_{q^d} : U_q(\hat{g}) \rightarrow U_q(\hat{g})$$

Let $\tilde{\rho} := \rho + h^\vee d$ and $\tilde{\lambda} := \lambda + dk - \Delta c$ such that

$$(\tilde{\lambda}, \tilde{\lambda} + \tilde{\rho}) = 0.$$ 

Recall there is a quantum Casimir $C$ in $U_q(\hat{g})$ which acts by $q^{(\tilde{\lambda},\tilde{\lambda})}$ in $V_{\tilde{\lambda}}$. If we pick $\tilde{\lambda}$ as above, then

$$1 = C^{-1} = q^{-2\tilde{\rho}}q^{-2kd} \text{mult} ((\gamma \circ \text{Ad}_{q^{2kd}}) \otimes 1 R^{\text{op}}).$$

Hence we can define an action of

$$q^{2(k+h^\vee)d} = \text{mult} ((\gamma \circ \text{Ad}_{q^{2kd}}) \otimes 1 R^{\text{op}})$$

on $V^q_{\lambda,k}$.

Finally we can talk about correlators and qKZ equations. As usual we define correlators

$$\Psi := \langle u_0 \tilde{\Phi}^\rho(z_1) \cdots \tilde{\Phi}^\rho(z_N) u_{N+1} \rangle$$

where $u_0$ is lowest weight and $u_{N+1}$ is highest weight. If we consider

$$\Psi(z_1, \ldots, q^{2(k+h^\vee)} z_j, \ldots, z_N)$$

this is like acting by $d$ in the $j$-th insertion. By our construction of what this means, it is the same as inserting $L^+$ on one side and $L^-$ on the other. Because the $\tilde{\Phi}$ are intertwining operators, these $L^\pm$ pass through to either side, moduli picking up an R-matrix every time. Finally, the action of $L^\pm$ on lowest and highest weight vectors yields $q^{\lambda_0}$ and $q^{\lambda_N}$. Putting this all together, we get

$$\Psi(z_1, \ldots, p z_j, \ldots, z_N) = q^{\lambda_0 + \lambda_N + 2\rho} |V_j R^{j-1} \left( \frac{p z_j}{z_1} \right) R^{j-1} \left( \frac{p z_j}{z_{j-1}} \right) (R^{-1})^{j-1+1} \left( \frac{z_j+1}{z_j} \right) \cdots \Psi(z_1, \ldots, z_N),$$

called the **qKZ equation**. They form a consistent system, i.e. if we write

$$\Psi(z_1, \ldots, p z_j, \ldots, z_N) = A_j \Psi(z_1, \ldots, z_N),$$

then we have

$$A_i(z_1, \ldots, p z_j, \ldots, z_N) A_j = A_j (\cdots) A_i.$$