1 Sep 17 (Xujia): Lifting cobordisms and Kontsevich-type recursions for counts of real curves

Let $(X, \omega)$ be a symplectic manifold of dimension $2n$, and let $J$ be an almost-complex structure. Fix $B \in H_2(X)$ and take closed submanifolds $H_1, \ldots, H_\ell \subset X$ such that $2(c_1(X)B + n - 3 + \ell) = \sum_i \text{codim}_R H_i$. Then we can ask about the number of $J$-holomorphic curves of degree $B$ and genus 0 passing through the $H_i$. These are usual genus-0 Gromov–Witten invariants $\langle [H_1], \ldots, [H_\ell] \rangle_B$.

This quantity is independent of the choice of (generic) $J$.

More precisely, define the moduli space

$$\mathcal{M}_\ell(B) := \{ u: \mathbb{P}^1 \to X: J\text{-holomorphic, } [u] = B, z_1, \ldots, z_\ell \in \mathbb{P}^1 \}/\text{Aut}(\mathbb{P}^1).$$

For today we can think of this as a smooth manifold. It has a natural Gromov compactification

$$\overline{\mathcal{M}}_\ell(B) := \{ u: C \to X: C \text{ nodal curve, } J\text{-holomorphic, } [u] = B, z_1, \ldots, z_\ell \in C \}/\text{Aut}(C).$$

For today we think of this as a compact smooth manifold. There are evaluation maps

$$\overline{\mathcal{M}}_\ell(B) \overset{ev}{\to} X^\ell$$

$$[(u, z_1, \ldots, z_\ell)] \mapsto (u(z_1), \ldots, u(z_\ell)).$$

To compute these Gromov–Witten invariants, we use the WDVV relations (Kontsevich ’92, Ruan–Tian ’93). These say that

$$\# D(H_1, H_2|H_3, H_4) = \# D(H_1, H_3|H_2, H_4)$$

where $D(H_1, H_2|H_3, H_4)$ is the divisor consisting of curves with a single node, with one component passing through $H_1, H_2$ and another passing through $H_3, H_4$. To see this, recall that

$$\mathcal{M}_{0,4} = \{(z_1, z_2, z_3, z_4) \in \mathbb{P}^1\}/\text{Aut}(\mathbb{P}^1) \cong \mathbb{P}^1$$

given by the cross-ratio. Consider the points $\sigma_0, \sigma_\infty \in \mathcal{M}_{0,4}$ corresponding to $0, \infty \in \mathbb{P}^1$. They are linearly equivalent divisors, i.e.

$$[\sigma_0] = [\sigma_\infty] \in H_0(\mathcal{M}_{0,4}).$$

Any moduli $\mathcal{M}_\ell(B)$ has a forgetful map

$$f: \mathcal{M}_\ell(B) \to \mathcal{M}_{0,4}$$

given by forgetting the map and all but the first 4 marked points. Hence this relation on $\mathcal{M}_{0,4}$ pulls back via $f$:

$$[f^{-1}(\sigma_0)] = [f^{-1}(\sigma_\infty)] \in H_*(\mathcal{M}_\ell(B)).$$
This is exactly the WDVV relation.

Today we will discuss a similar story for the real case. Let \((X, \omega, \phi)\) be a real symplectic manifold of dimension \(2n\). This means \((X, \omega)\) is a symplectic manifold and \(\phi: X \to X\) is an anti-symplectic involution, i.e.

\[
\phi^2 = \text{id}, \quad \phi^* \omega = -\omega.
\]

We usually refer to \(\phi\) as the conjugation, because typical examples include \(\mathbb{P}^n\) with \(\phi\) as the actual conjugation map. Note that the almost holomorphic structure \(J\) behaves as

\[
\phi^* J = -J.
\]

Let \(X^\phi\) denote the fixed locus of \(\phi\). We take \(B \in H_2(X)\) and \(p_1, \ldots, p_k \in X^\phi\) and \(H_1, \ldots, H_\ell \subset X\). Now we can count real rational curves \(C \subset X\), namely curves with \(\phi(C) = C\). (In general, “real” means \(\phi\)-invariant.) Now if we impose the usual dimension restriction \(c_1(X)B + n - 3 = k(n - 1) + \sum_i (\text{codim} H_i - 2)\), then we can count curves like we did in the complex case. However this is not in general an invariant, because in the real setting we don’t have Bézout’s theorem and so on.

**Theorem 1.1** (Welschinger ’03, ’05, Solomon ’06). If \(n = 2\), or \(n = 3\) and \(X^\phi\) is oriented, then the number of degree \(B\) real rational curves passing through \(H_1, \ldots, H_\ell, p_1, \ldots, p_k\), counted with appropriate signs, is an invariant of \(J\), \(p_i\), and \(H_i \in [H_i] \in H_1(X - X^\phi)\).

Now we can ask: are there WDVV relations in this real case?

**Theorem 1.2** (Xujia Chen ’18). In \(n = 2\), the relations for Welschinger’s invariants proposed by Solomon ’07 hold.

**Theorem 1.3** (Chen–Zinger ’19). In \(n = 3\), similar relations hold if \((X, \omega, \phi)\) has some symmetry, e.g. \(\mathbb{P}^3\) with real hyperplane reflection.

In the case of \(\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1\) these WDVV relations reduce to formulas in Alcalado’s ’17 thesis. For e.g. \(\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1, (\mathbb{P}^1)^3\), and real blow-ups of \(\mathbb{P}^2\), we get complete recursion formulas.

**Idea of proof.** The usual conjugation map \(\text{conj}: \mathbb{P}^1 \to \mathbb{P}^1\) sends \(z \to \overline{z}\), with fixed locus \(\mathbb{R}\mathbb{P}^1\). The real moduli is

\[
\mathcal{R}\mathcal{M}_{k,\ell}(B) := \left\{ u: \mathbb{P}^1 \to X : \begin{array}{l}
\text{J-hol, } [u] = B, \ u \circ \text{conj} = \phi \circ u, \\
x_1, \ldots, x_k \in \mathbb{R}\mathbb{P}^1, \ z_{i}^{\pm}, \ldots, z_{\ell}^{\pm} \in \mathbb{P}^1, \ z_{i}^{-} = \text{conj}(z_{i}^{+}) \end{array} \right\}/ \text{Aut}(\mathbb{P}^1).
\]

This is, again, not compact, but it has a Gromov compactification \(\mathcal{R}\overline{\mathcal{M}}_{k,\ell}(B)\) by replacing \(\mathbb{P}^1\) with nodal curves.

We can lift relations from \(\mathcal{R}\overline{\mathcal{M}}_{1,2}\) and \(\mathcal{R}\overline{\mathcal{M}}_{0,3}\). However the problem is that the spaces

\[
\mathcal{R}\overline{\mathcal{M}}_{k,\ell}(B), \quad X^\phi, \quad \mathcal{R}\overline{\mathcal{M}}_{1,2}
\]

may not be orientable. In Solomon’s thesis, he showed that \(\text{ev}|_{\mathcal{R}\overline{\mathcal{M}}_{k,\ell}(B)}\) is relatively orientable, i.e. pullback of the first Stiefel–Whitney class of the target equals that of the domain. The relative orientation extends through some codimension-1 strata, but not all of them.

Let \(\Gamma \subset \mathcal{R}\overline{\mathcal{M}}_{1,2}\) (or \((0, 3)\)) consist of curves such that \(z_{2}^{\pm}\) or \(z_{3}^{\pm}\) coincide with \(z_{1}^{\mp}\). Georgieva–Zinger ’13 shows that \(\Gamma\) bounds in \(\mathcal{R}\overline{\mathcal{M}}_{0,3}\). Take \(Y \subset \mathcal{R}\overline{\mathcal{M}}_{1,2}\) such that \(\partial Y = \Gamma\) and

\[
\mathcal{R}\overline{\mathcal{M}}_{k,\ell}(B) \xrightarrow{\text{ev} \times f} (X^\phi)^k \times X^\ell \times \mathcal{R}\overline{\mathcal{M}}_{1,2} \leftrightarrow (p_1 \times \cdots \times H_\ell) \times Y.
\]

Let \(C\) denote the constraints \(p_1 \times \cdots \times H_\ell\). Then

\[
\mathcal{R}\overline{\mathcal{M}}_{k,\ell}(B) \cdot (C \times \Gamma) = \pm 2(\text{bad strata}) \cdot (C \times Y).
\]
This comes from cutting $\mathbb{R}M$ open along the bad strata. Call the resulting space $\mathbb{R}\hat{M}$. It is relatively orientable now. Then

$$\partial(\mathbb{R}\hat{M} \cdot (C \times Y)) = (\partial\mathbb{R}\hat{M}) \cdot (C \times Y) \pm \mathbb{R}\hat{M} \cdot \partial(C \times Y).$$

The lhs is 0, and the rhs gives the desired formula \[1\].

Finally, the lifted relations \[1\], with splitting relations, give the desired relations between Welschinger’s invariants. Splitting works as follows. A dimension count together with a good choice of $Y$ shows that, for all bad strata contributing to the rhs of \[1\],

1. the first bubble is rigid,
2. the condition “cut out by $Y$” is the same as specifying the position of the node on the first bubble.

Hence the count of such nodal curves is exactly the count of first bubbles, and the count of second bubbles with one additional real point specifying the position of the node.

As for the lhs, when $n = 2$ the splitting is immediate. When $n = 3$, a dimension count gives two cases.

1. The real bubble is rigid. Then

$$\#(\text{nodal}) = \#(\text{first bubble}) \cdot \#(\text{second bubble with curve insertion}).$$

2. The complex bubble is rigid, and the real bubble passes through it. Then

$$\#(\text{nodal}) = \#(\text{complex bubble}) \cdot \#(\text{real bubble passing through } C_1 \sqcup \cdots \sqcup C_N).$$

This case is why we need to assume the symmetry property in the theorem. Assume there is a $G \subset \text{Aut}(X,\omega,\phi)$ such that

$$H_2(X - X^\phi)^G \sim H_2(X).$$

Then if we take $H_1, \ldots, H_\ell$ to be $G$-invariant, $C_1 \sqcup \cdots \sqcup C_N$ is also $G$-invariant. This way we can express the second term above in terms of usual GW invariants.

\[\square\]

2 Nathan (Sep 24): BHK mirror symmetry and beyond

The outline for today will be:

1. LG models and BHK mirror symmetry;
2. GW theory and LG/CY correspondence;
3. other forms of mirror symmetry.

What is a Landau–Ginzburg model? Mathematically this is called FJRW theory. The input is a pair $(W, G)$.

- $W$ is a quasi-homogeneous polynomial, meaning that

$$W(\lambda^{w_1}x_1, \ldots, \lambda^{w_N}x_N) = \lambda^d W(x_1, \ldots, x_N), \quad \gcd(w_1, \ldots, w_N, d) = 1.$$

- $W$ is non-degenerate, meaning that there is an isolated critical point at 0 and there are no terms like $x_i x_j$. This implies there is exactly one choice of weights $w_i$ to make homogeneity work.

- (Calabi–Yau condition) $\sum w_i = d$. This is not strictly necessary for FJRW theory, but we’ll assume it for today.
• $G \subset \text{Aut}(W)$, where
\[
\text{Aut}(W) := \{(g_1, \ldots, g_N) \in (\mathbb{Q}/\mathbb{Z})^N : W(e^{2\pi i g_1}x_1, \ldots, e^{2\pi i g_N}x_N) = W(x_1, \ldots, x_N)\}.
\]

• (A-admissible) $G$ must contain the exponential grading operator $j_W := (w_1/d, \ldots, w_N/d)$. This is a condition we always need for FJRW theory, since it is the LG A-model.

• (B-admissible) $\sum g_i \equiv 0 \mod \mathbb{Z}$. This is required for a LG B-model. It is necessary for us today.

The output of FJRW theory is the following.

• A state space
\[
\mathcal{H}_{W,G} = \bigoplus_{g \in G} \mathcal{H}_{\text{middle}}(\mathbb{C}^{N_g, W_g^+\infty}, \mathbb{C})^G
\]
where $N_g$ is the dimension of the fixed locus of $g$, and $W_g^+\infty$ is the Milnor fiber.

• A moduli space of $W$-curves
\[
\mathcal{W}_{g,k} := \{(C, p_1, \ldots, p_k, \mathcal{L}_1, \ldots, \mathcal{L}_N, \varphi_1, \ldots, \varphi_s) : \text{genus}(C) = g, \ varphi_i : W_i(\mathcal{L}_1, \ldots, \mathcal{L}_N) \overset{\sim}{\rightarrow} \omega_{\log}(C)\}
\]
where we write $W = \sum_{i=1}^s W_i$ as a sum of monomials. This moduli has a “virtual class”
\[
[W_{g,k}]^{\text{ir}} \in H_*(\mathcal{W}_{g,k}(W, g_1, \ldots, g_n)) \otimes \prod H_{\text{middle}}(\mathbb{C}^{N_g, W_g^+\infty}, \mathbb{Q})
\]
constructed analytically, not via an obstruction theory.

• FJRW invariants
\[
\langle \tau_{\ell_1}(\alpha_1), \ldots, \tau_{\ell_k}(\alpha_k) \rangle_{W,G} := \int_{[\mathcal{W}_{g,k}]} \Lambda_{g,k}^{W,G}(\alpha_1, \ldots, \alpha_k) \prod_{i=1}^k \psi_i^\ell_i
\]
where $\Lambda_{g,k}^{W,G}$ is the factor arising from pushing down from $\mathcal{W}_{g,k}$ to the moduli of curves $\mathcal{M}_{g,k}$.

BHK mirror symmetry constructs a “dual” $(W^T, G^T)$ to a pair $(W, G)$, in the situation where $W$ is invertible, meaning that there is the same number of variables as monomials.

• $W^T$ comes from taking the exponent matrix $A_W$ of $W$, taking its transpose, and getting the resulting polynomial. For example,
\[
W = x_1^4 + x_2^2x_3 + x_3^4 + x_4^8
\]
is quasi-homogeneous with respect to $(2, 3, 2, 1; 8)$. Then
\[
A_W = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}.
\]
The dual polynomial is therefore
\[
W^T = x_1^4 + x_2^2 + x_2x_3^4 + x_4^8,
\]
which is quasi-homogeneous with respect to $(2, 4, 1, 1; 8)$.

• $G^T$ comes from
\[
G^T := \{h \in \text{Aut}(W^T) : hA_Wg^T \in \mathbb{Z} \forall g \in G\}.
\]
Equivalently, this is $\text{Hom}(\text{Aut}(W)/G, \mathbb{C}^\times)$. 

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Given a pair \((W, G)\), we can get an associated stack \([X_W/\widetilde{G}]\), where
\[ X_W := \{ W = 0 \} \subset \mathbb{P}(w_1, \ldots, w_N), \quad \widetilde{G} := G/\langle j_W \rangle. \]
Then we can look at the GW theory with target \([X_W/\widetilde{G}]\). The output is a state space \(H^*_\text{CR}([X_W/\widetilde{G}], \mathbb{C})\), a moduli space of stable curves, and invariants.

**Theorem 2.1** (Chiodo–Ruan, LG/CY correspondence),
\[ \mathcal{H}_{W,G} \cong H^*_\text{CR}([X_W/\widetilde{G}]) \]
as bigraded vector spaces.

**Conjecture 2.2.**
\[ \text{FJRW}(W, G) \cong \text{GW}([X_W/\widetilde{G}]). \]

This has been proved in a handful of cases and different levels. For example, if \(W\) is a sum of Fermat polynomials then this correspondence is proved in genus 0. We can compare it to other forms of mirror symmetry.

1. Quintic/mirror quintic:
\[
W = \sum_{i=1}^{5} x_i^5, \quad G = \langle j_W \rangle \leftrightarrow Q = \{ W = 0 \} \subset \mathbb{P}^4 \\
W^T = \sum_{i=1}^{5} x_i^5, \quad G^T = \langle (1/5, 0, 0, 0), (1/5, 0, 4/5, 0), (1/5, 0, 0, 4/5) \rangle \leftrightarrow M = \{ W = 0 \}/\widetilde{G}, \quad \widetilde{G} \cong \mathbb{Z}_5^3
\]
where
\[ \widetilde{G} = \langle (\xi_5, \xi_5^4, 1, 1, 1), (\xi_5, 1, \xi_5^4, 1, 1), (\xi_5, 1, 1, \xi_5^4, 1) \rangle. \]
In this setting, there is a diagram
\[
\begin{array}{ccc}
\text{FJRW}(W, G) & \longrightarrow & \text{GW}(Q) \\
\uparrow \text{mirror theorem} & & \downarrow \text{mirror theorem} \\
\text{B-model}(W^T, G^T) & \longrightarrow & \text{B-model}(M)
\end{array}
\]
On the rhs, the invariants involved are
\[ J^{\text{FJRW}} = \sum (\tau_{\ell_1}(\alpha_1), \ldots, \tau_{\ell_N}(\alpha_N))_{W,G} t_{\ell_1}^{(\alpha_1)} \cdots t_{\ell_N}^{(\alpha_N)} \]
and one can show
\[ (J^{\text{GW}} = \frac{I^{\text{GW}}}{I_0^{\text{GW}}}) \text{ after change of vars.} \]
A similar thing holds on the lhs.

2. Mirror symmetry for K3 surfaces. Take a K3 surface \(X\). Then \(H^2(X, \mathbb{Z}) = U^3 \oplus E_8^2 =: L_{K3}\) where \(U\) is the hyperbolic lattice. Take a polarization, i.e. a lattice \(M\) with a primitive embedding \(M \hookrightarrow \text{Pic}(X)\); there is a moduli of \(M\)-polarized K3 surfaces. If there is a decomposition
\[ M_{L_{K3}}^\perp = U \oplus M^\perp, \]
we say \(X^\perp\) is mirror to \(X\) if there exists \(M^\perp \hookrightarrow \text{Pic}(X^\perp)\).
If we let 

\[ W := x_0^n + f(x_1, x_2, x_3) \]

and we choose \( W \) right, then 

\[ X_{W,G} = \overline{X_W / \tilde{G}} \]

is a K3 surface. This means for such K3 surfaces we have two candidates for how to do mirror symmetry. It turns out the dual \( X_{W^T,G^T} \) is also a K3 surface.

**Theorem 2.3.** \( X_{W,G} \) and \( X_{W^T,G^T} \) are K3 mirror.

Why? If we take the automorphism 

\[ \sigma_n : (x_0, x_1, x_2, x_3) \mapsto (\xi_n x_0, x_1, x_2, x_3) \]

then we can look at the invariant lattice 

\[ S(\sigma_n) = \{ x \in H^2(X, \mathbb{Z}) : \sigma_n^* x = x \} \]

and one can show \( S(\sigma)^\vee = S(\sigma_n^T) \).

Finally we can look at non-abelian LG models. Take \( W \) as before, but now 

\[ G = H \cdot K, \quad H \subset \text{Aut}(W) \subset \text{GL}_n(\mathbb{C}), \quad K \in A_n(\text{permuting vars}). \]

Then there is a mirror \((W^\vee, G^\vee)\), where 

\[ W^\vee := W^T \]

as before, but 

\[ G^\vee := H^T \cdot K. \]

This should be mirror symmetry. The first indication this should work is an isomorphism of state spaces.

**Theorem 2.4.**

\[ (\mathcal{H}_{W,G})_{\text{untwisted}} \cong (B_{W^\vee,G^\vee})_{\text{narrow diagonal}} \]
\[ (\mathcal{H}_{W,G})_{\text{narrow diagonal}} \cong (B_{W^\vee,G^\vee})_{\text{untwisted}}. \]

This allows us to think about e.g. GW theory of quotients of the quintic, or symmetric products of elliptic curves.