1 Xujia (Sep 17): Lifting cobordisms and Kontsevich-type recursions for counts of real curves

Let \((X, \omega)\) be a symplectic manifold of dimension \(2n\), and let \(J\) be an almost-complex structure. Fix \(B \in H_2(X)\) and take closed submanifolds \(H_1, \ldots, H_\ell \subset X\) such that \(2(c_1(X)B + n-3+\ell) = \sum_i \text{codim}_R H_i\). Then we can ask about the number of \(J\)-holomorphic curves of degree \(B\) and genus 0 passing through the \(H_i\). These are usual genus-0 Gromov–Witten invariants

\[\langle [H_1], \ldots, [H_\ell]\rangle_B.\]

This quantity is independent of the choice of (generic) \(J\).

More precisely, define the moduli space

\[\mathcal{M}_\ell(B) := \{u: \mathbb{P}^1 \to X : J\text{-holomorphic}, [u] = B, z_1, \ldots, z_\ell \in \mathbb{P}^1\}/\text{Aut}(\mathbb{P}^1).\]

For today we can think of this as a smooth manifold. It has a natural Gromov compactification

\[\overline{\mathcal{M}}_\ell(B) := \{u: C \to X : C \text{ nodal curve, } J\text{-holomorphic}, [u] = B, z_1, \ldots, z_\ell \in C\}/\text{Aut}(C).\]

For today we think of this as a compact smooth manifold. There are evaluation maps

\[\overline{\mathcal{M}}_\ell(B) \xrightarrow{\text{ev}} X^\ell\]

\[[(u, z_1, \ldots, z_\ell)] \mapsto (u(z_1), \ldots, u(z_\ell)).\]

To compute these Gromov–Witten invariants, we use the WDVV relations (Kontsevich ’92, Ruan–Tian ’93). These say that

\[#D(H_1, H_2|H_3, H_4) = #D(H_1, H_3|H_2, H_4)\]

where \(D(H_1, H_2|H_3, H_4)\) is the divisor consisting of curves with a single node, with one component passing through \(H_1, H_2\) and another passing through \(H_3, H_4\). To see this, recall that

\[\overline{\mathcal{M}}_{0,4} = \{(z_1, z_2, z_3, z_4) \in \mathbb{P}^1\}/\text{Aut}(\mathbb{P}^1) \cong \mathbb{P}^1\]

given by the cross-ratio. Consider the points \(\sigma_0, \sigma_\infty \in \overline{\mathcal{M}}_{0,4}\) corresponding to \(0, \infty \in \mathbb{P}^1\). They are linearly equivalent divisors, i.e.

\[\sigma_0 = \sigma_\infty \in H_0(\overline{\mathcal{M}}_{0,4}).\]

Any moduli \(\overline{\mathcal{M}}_\ell(B)\) has a forgetful map

\[f: \overline{\mathcal{M}}_\ell(B) \to \overline{\mathcal{M}}_{0,4}\]

given by forgetting the map and all but the first 4 marked points. Hence this relation on \(\overline{\mathcal{M}}_{0,4}\) pulls back via \(f\):

\[[f^{-1}(\sigma_0)] = [f^{-1}(\sigma_\infty)] \in H_*(\overline{\mathcal{M}}_\ell(B)).\]
This is exactly the WDVV relation.

Today we will discuss a similar story for the real case. Let $(X,\omega,\phi)$ be a real symplectic manifold of dimension $2n$. This means $(X,\omega)$ is a symplectic manifold and $\phi: X \to X$ is an anti-symplectic involution, i.e.
\[ \phi^2 = \text{id}, \quad \phi^*\omega = -\omega. \]
We usually refer to $\phi$ as the conjugation, because typical examples include $\mathbb{P}^n$ with $\phi$ as the actual conjugation map. Note that the almost holomorphic structure $J$ behaves as
\[ \phi^*J = -J. \]

Let $X^\phi$ denote the fixed locus of $\phi$. We take $B \in H_2(X)$ and $p_1,\ldots,p_k \in X^\phi$ and $H_1,\ldots,H_\ell \subset X$. Now we can count real rational curves $C \subset X$, namely curves with $\phi(C) = C$. (In general, “real” means $\phi$-invariant.) Now if we impose the usual dimension restriction $c_1(X)B + n - 3 = k(n - 1) + \sum_i (\text{codim} H_i - 2)$, then we can count curves like we did in the complex case. However this is not in general an invariant, because in the real setting we don’t have Bézout’s theorem and so on.

**Theorem 1.1** (Welschinger ’03, ’05, Solomon ’06). If $n = 2$, or $n = 3$ and $X^\phi$ is oriented, then the number of degree $B$ real rational curves passing through $H_1,\ldots,H_\ell,p_1,\ldots,p_k$, counted with appropriate signs, is an invariant of $J$, $p_i$, and $H_i \in [H_i] \in H_2(X - X^\phi)$.

Now we can ask: are there WDVV relations in this real case?

**Theorem 1.2** (Xujia Chen ’18). In $n = 2$, the relations for Welschinger’s invariants proposed by Solomon ’07 hold.

**Theorem 1.3** (Chen–Zinger ’19). In $n = 3$, similar relations hold if $(X,\omega,\phi)$ has some symmetry, e.g. $\mathbb{P}^3$ with real hyperplane reflection.

In the case of $\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1$ these WDVV relations reduce to formulas in Alcalado’s ’17 thesis. For e.g. $\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1, (\mathbb{P}^1)^3$, and real blow-ups of $\mathbb{P}^2$, we get complete recursion formulas.

**Idea of proof.** The usual conjugation map conj: $\mathbb{P}^1 \to \mathbb{P}^1$ sends $z \to \overline{z}$, with fixed locus $\mathbb{R}\mathbb{P}^1$. The real moduli is
\[ \mathbb{RM}_{k,\ell}(B) := \left\{ u: \mathbb{P}^1 \to X : \begin{array}{l} J\text{-hol}, \ [u] = B, \ u \circ \text{conj} = \phi \circ u, \ x_1,\ldots,x_k \in \mathbb{R}\mathbb{P}^1, \ z_1^\pm,\ldots,z_\ell^\pm \in \mathbb{P}^1, \ z_i^- = \text{conj}(z_i^+) \end{array} \right\} / \text{Aut}(\mathbb{P}^1). \]
This is, again, not compact, but it has a Gromov compactification $\overline{\mathbb{RM}}_{k,\ell}(B)$ by replacing $\mathbb{P}^1$ with nodal curves.

We can lift relations from $\overline{\mathbb{RM}}_{1,2}$ and $\overline{\mathbb{RM}}_{0,3}$. However the problem is that the spaces
\[ \overline{\mathbb{RM}}_{k,\ell}(B), \ X^\phi, \ \overline{\mathbb{RM}}_{1,2} \]
may not be orientable. In Solomon’s thesis, he showed that $\text{ev}|_{\overline{\mathbb{RM}}_{k,\ell}(B)}$ is relatively orientable, i.e. pullback of the first Stiefel–Whitney class of the target equals that of the domain. The relative orientation extends through some codimension-1 strata, but not all of them.

Let $\Gamma \subset \overline{\mathbb{RM}}_{1,2}$ (or $(0,3)$) consist of curves such that $z_2^\pm$ or $z_3^\pm$ coincide with $z_1^\pm$. Georgieva–Zinger ’13 shows that $\Gamma$ bounds in $\overline{\mathbb{RM}}_{0,3}$. Take $Y \subset \overline{\mathbb{RM}}_{1,2}$ such that $\partial Y = \Gamma$ and
\[ \overline{\mathbb{RM}}_{k,\ell}(B) \xrightarrow{\text{ev} \times f} (X^\phi)^k \times X^\ell \times \overline{\mathbb{RM}}_{1,2} \leftrightarrow (p_1 \times \cdots \times H_\ell) \times Y. \]
Let $C$ denote the constraints $p_1 \times \cdots \times H_\ell$. Then
\[ \overline{\mathbb{RM}}_{k,\ell}(B) \cdot (C \times \Gamma) = \pm 2(\text{bad strata}) \cdot (C \times Y). \] (1)
This comes from cutting $\mathbb{R}\hat{M}$ open along the bad strata. Call the resulting space $\mathbb{R}\hat{M}$. It is relatively orientable now. Then

$$\partial(\mathbb{R}\hat{M} \cdot (C \times Y)) = (\partial\mathbb{R}\hat{M}) \cdot (C \times Y) \pm \mathbb{R}\hat{M} \cdot \partial(C \times Y).$$

The lhs is 0, and the rhs gives the desired formula [1].

Finally, the lifted relations [1], with splitting relations, give the desired relations between Welschinger’s invariants. Splitting works as follows. A dimension count together with a good choice of $Y$ shows that, for all bad strata contributing to the rhs of [1],

1. the first bubble is rigid,
2. the condition “cut out by $Y$” is the same as specifying the position of the node on the first bubble.

Hence the count of such nodal curves is exactly the count of first bubbles, and the count of second bubbles with one additional real point specifying the position of the node.

As for the lhs, when $n = 2$ the splitting is immediate. When $n = 3$, a dimension count gives two cases.

1. The real bubble is rigid. Then
   $$\#(\text{nodal}) = \#(\text{first bubble}) \cdot \#(\text{second bubble with curve insertion}).$$

2. The complex bubble is rigid, and the real bubble passes through it. Then
   $$\#(\text{nodal}) = \#(\text{complex bubble}) \cdot \#(\text{real bubble passing through } C_1 \sqcup \cdots \sqcup C_N).$$

This case is why we need to assume the symmetry property in the theorem. Assume there is a $G \subset \text{Aut}(X,\omega,\phi)$ such that

$$H_2(X - X^G) \sim H_2(X).$$

Then if we take $H_1, \ldots, H_t$ to be $G$-invariant, $C_1 \sqcup \cdots \sqcup C_N$ is also $G$-invariant. This way we can express the second term above in terms of usual GW invariants. □

2 Nathan (Sep 24): BHK mirror symmetry and beyond

The outline for today will be:

1. LG models and BHK mirror symmetry;
2. GW theory and LG/CY correspondence;
3. other forms of mirror symmetry.

What is a Landau–Ginzburg model? Mathematically this is called FJRW theory. The input is a pair $(W,G)$.

- $W$ is a quasi-homogeneous polynomial, meaning that
  $$W(\lambda^{w_1}x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \ldots, x_N), \quad \text{gcd}(w_1, \ldots, w_N, d) = 1.$$

- $W$ is non-degenerate, meaning that there is an isolated critical point at 0 and there are no terms like $x_i x_j$. This implies there is exactly one choice of weights $w_i$ to make homogeneity work.

- (Calabi–Yau condition) $\sum w_i = d$. This is not strictly necessary for FJRW theory, but we’ll assume it for today.
• $G \subset \text{Aut}(W)$, where
\[
\text{Aut}(W) := \{(g_1, \ldots, g_N) \in (\mathbb{Q}/\mathbb{Z})^N : W(e^{2\pi i g_1 x_1}, \ldots, e^{2\pi i g_N x_N}) = W(x_1, \ldots, x_N)\}.
\]

• (A-admissible) $G$ must contain the exponential grading operator $j_W := (w_1/d, \ldots, w_N/d)$. This is a condition we always need for FJRW theory, since it is the LG A-model.

• (B-admissible) $\sum g_i \equiv 0 \mod \mathbb{Z}$. This is required for a LG B-model. It is necessary for us today.

The output of FJRW theory is the following.

• A state space
\[
\mathcal{H}_{W,G} = \bigoplus_{g \in G} H^{\text{middle}}(\mathbb{C}^N, W_g^{+\infty}, \mathbb{C})^G
\]
where $N_g$ is the dimension of the fixed locus of $g$, and $W_g^{+\infty}$ is the Milnor fiber.

• A moduli space of $W$-curves
\[
W_{g,k} := \{(C, p_1, \ldots, p_k, L_1, \ldots, L_N, \varphi_1, \ldots, \varphi_s) : \text{genus}(C) = g, \varphi_i : W_i(L_1, \ldots, L_N) \sim \omega_{\log}(C)\}
\]
where $W = \sum_{i=1}^s W_i$ as a sum of monomials. This moduli has a “virtual class”
\[
[W_{g,k}]^{\text{vir}} \in H_*(W_{g,k}(W, g_1, \ldots, g_n)) \otimes \prod H^{\text{middle}}(\mathbb{C}^N, W_g^{+\infty}, \mathbb{Q})
\]
constructed analytically, not via an obstruction theory.

• FJRW invariants
\[
\langle \tau_{x_1}(\alpha_1), \ldots, \tau_{x_k}(\alpha_k) \rangle_{W,G} := \int_{[\overline{M}_{g,k}]} \Lambda_{g,k}^{W,G}(\alpha_1, \ldots, \alpha_k) \prod_{i=1}^k \psi_i^{\ell_i}
\]
where $\Lambda_{g,k}^{W,G}$ is the factor arising from pushing down from $W_{g,k}$ to the moduli of curves $\overline{M}_{g,k}$.

BHK mirror symmetry constructs a “dual” $(W^T, G^T)$ to a pair $(W, G)$, in the situation where $W$ is invertible, meaning that there is the same number of variables as monomials.

• $W^T$ comes from taking the exponent matrix $A_W$ of $W$, taking its transpose, and getting the resulting polynomial. For example,
\[
W = x_1^4 + x_2^2 x_3 + x_3^4 + x_4^8
\]
is quasi-homogeneous with respect to $(2, 3, 2, 1; 8)$. Then
\[
A_W = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}.
\]
The dual polynomial is therefore
\[
W^T = x_1^4 + x_2^2 + x_2 x_3^4 + x_4^8,
\]
which is quasi-homogeneous with respect to $(2, 4, 1, 1; 8)$.

• $G^T$ comes from
\[
G^T := \{h \in \text{Aut}(W^T) : hA_W g^T \in \mathbb{Z} \forall g \in G\}.
\]
Equivalently, this is $\text{Hom}(\text{Aut}(W)/G, \mathbb{C}^\times)$. 

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Given a pair \((W, G)\), we can get an associated stack \([X_W/\tilde{G}]\), where
\[ X_W := \{ W = 0 \} \subseteq \mathbb{P}(w_1, \ldots, w_N), \quad \tilde{G} := G/\langle j_W \rangle. \]
Then we can look at the GW theory with target \([X_W/\tilde{G}]\). The output is a state space \(H^*_CR([X_W/\tilde{G}], \mathbb{C})\), a moduli space of stable curves, and invariants.

**Theorem 2.1** (Chiodo–Ruan, LG/CY correspondence).
\[ \mathcal{H}_{W,G} \cong H^*_CR([X_W/\tilde{G}]) \]
as bigraded vector spaces.

**Conjecture 2.2.**
\[ FJRW(W, G) \cong GW([X_W/\tilde{G}]). \]

This has been proved in a handful of cases and different levels. For example, if \(W\) is a sum of Fermat polynomials then this correspondence is proved in genus 0. We can compare it to other forms of mirror symmetry.

1. Quintic/mirror quintic:
   \[ W = \sum_{i=1}^{5} x_i^5, \quad G = \langle j_W \rangle \leftrightarrow Q = \{ W = 0 \} \subseteq \mathbb{P}^4 \]
   \[ W^T = \sum_{i=1}^{5} x_i^5, \quad G^T = \langle 0, 0, 0, 4/5, 0, 4/5 \rangle \leftrightarrow M = \{ W = 0 \}/\tilde{G}, \quad \tilde{G} \cong \mathbb{Z}_5^3 \]
   where
   \[ \tilde{G} = \langle (\xi_5, \xi_5^4, 1, 1, 1), (\xi_5, 1, \xi_5^4, 1, 1), (\xi_5, 1, 1, \xi_5^4, 1) \rangle. \]
   In this setting, there is a diagram
   \[
   \begin{array}{cccc}
   \text{FJRW}(W, G) & \xrightarrow{\text{mirror theorem}} & \text{GW}(Q) \\
   \text{B-model}(W^T, G^T) & \xrightarrow{\text{analytic continuation}} & \text{B-model}(M) \\
   \end{array}
   \]
   where \(U : \tilde{I}^GW = I^{FJRW}\) after change of vars.

2. Mirror symmetry for K3 surfaces. Take a K3 surface \(X\). Then \(H^2(X, \mathbb{Z}) = U^3 \oplus E_8^2 =: L_{K3}\) where \(U\) is the hyperbolic lattice. Take a polarization, i.e. a lattice \(M\) with a primitive embedding \(M \hookrightarrow \text{Pic}(X)\); there is a moduli of \(M\)-polarized K3 surfaces. If there is a decomposition
   \[ M_{L_{K3}} = U \oplus M' \]
   we say \(X'\) is mirror to \(X\) if there exists \(M' \hookrightarrow \text{Pic}(X')\).
If we let
\[ W := x_0^n + f(x_1, x_2, x_3) \]
and we choose \( W \) right, then
\[ X_{W,G} = X_W / \tilde{G} \]
is a K3 surface. This means for such K3 surfaces we have two candidates for how to do mirror symmetry. It turns out the dual \( X_{W^T,G^T} \) is also a K3 surface.

**Theorem 2.3.** \( X_{W,G} \) and \( X_{W^T,G^T} \) are K3 mirror.

Why? If we take the automorphism
\[ \sigma_n : (x_0, x_1, x_2, x_3) \mapsto (\xi_n x_0, x_1, x_2, x_3) \]
then we can look at the invariant lattice
\[ S(\sigma_n) = \{ x \in H^2(X, \mathbb{Z}) : \sigma_n^* x = x \} \]
and one can show \( S(\sigma^\vee) = S(\sigma_n^{T^*}) \).

Finally we can look at non-abelian LG models. Take \( W \) as before, but now \( G = H \cdot K, \quad H \subset \text{Aut}(W) \subset \text{GL}_n(\mathbb{C}), \quad K \in A_n(\text{permuting vars}) \).

Then there is a mirror \( (W^\vee, G^\vee) \), where
\[ W^\vee := W^T \]
as before, but
\[ G^\vee := H^T \cdot K. \]
This should be mirror symmetry. The first indication this should work is an isomorphism of state spaces.

**Theorem 2.4.**

\[ (\mathcal{H}_{W,G})_{\text{untwisted}} \cong (B_{W^\vee,G^\vee})_{\text{narrow diagonal}} \]
\[ (\mathcal{H}_{W,G})_{\text{narrow diagonal}} \cong (B_{W^\vee,G^\vee})_{\text{untwisted}}. \]

This allows us to think about e.g. GW theory of quotients of the quintic, or symmetric products of elliptic curves.

### 3 Fenglong (Oct 01): Structures of relative Gromov–Witten theory

Let \( X \) be a smooth projective variety. In GW theory we are interested in counting curves in \( X \). To define invariants, we consider the moduli space of stable maps
\[ \overline{M}_{g,n,d}(X) := \{ f : (C, p_i) \to X : \text{genus } g \text{ degree } d \text{ and } n \text{ markings} \}. \]
Let \( ev_i : \overline{M}_{g,n}(X) \to X \) be the \( i \)-th evaluation map \( p_i \mapsto f(p_i) \). To define invariants we use \( ev_i \) to pull back some cohomology classes \( \gamma_i \in H^*(X) \) and cap with the virtual fundamental class of \( \overline{M}_{g,n}(X) \). There are also classes \( \psi := c_1(L) \) where \( L_i \) is the line bundle over \( \overline{M}_{g,n} \) whose fibers are the cotangent spaces of the \( i \)-th marking. We define GW invariants
\[ \langle \prod_{i=1}^n \tau_{a_i} (\gamma_i) \rangle_{g,n,d}^X : = \int_{\overline{M}_{g,n,d}} \prod_i \psi_i^{a_i} ev_i^* (\gamma_i). \]
Such invariants have many structural properties. For example, we can define quantum cohomology, WDVV relations, topological recursion relations, Givental’s formalism, Virasoro constraints, and cohomological field theory.

To define relative GW invariants we need a smooth divisor $D \subset X$, and we count curves with tangency conditions along the divisor. The relevant moduli space is

$$\overline{M}_{g,k,n,d}(X,D) := \{ f : (C,p_i) \to X : \text{genus } g \text{ degree } d \text{ and } n \text{ interior markings} \}$$

with relative condition $k = (k_1, \ldots, k_m)$ where $k_i \in \mathbb{Z}_{\geq 0}$ and $\sum k_i = \int_D |D|$. There are now additional evaluation maps $ev_j : \overline{M} \to D$ for $1 \leq j \leq m$, using which we can pull back $\delta_i \in H^*(D)$. Then invariants are

$$\langle \prod \tau_n(\delta_i) \prod \tau_n(\gamma_i) \rangle_{g,k,n,d}^{(X,D)} = \int_{\overline{M}_{g,n,d}} \prod \psi_i^{\delta_i} ev_i^*(\delta_i) \prod \psi_i^{\gamma_i} ev_i^*(\gamma_i).$$

The virtual dimension constraint of this relative moduli space is

$$\text{vdim} = (1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d |D| + m + n$$

$$= \frac{1}{2} \sum \deg(\gamma_i) + \frac{1}{2} \sum \deg(\delta_i) + \sum a_i.$$

Question: how to obtain structural properties of relative GW theory? (E.g. relative quantum cohomology, WDVV, topological recursion, Givental’s formalism, mirror theorem.) Answer: using stacks to impose tangency conditions. This was originally part of C. Cadman’s dissertation.

The specific stack we will use is the $r$-th root stack $X_{D,r}$ of $X$ along $D$, where $r \in \mathbb{Z}_{>0}$. Geometrically, $X_{D,r}$ is smooth away from $D$ and has $\mu_r$ stabilizer along $D$. Then consider the evaluation map

$$\overline{M}_{g,k,n,d}(X_{D,r}) \xrightarrow{ev} IX_{D,r}.$$

Here $IX_{D,r}$ is the inertia stack. In general inertia stacks are complicated, but in this case

$$IX_{D,r} = X \cup \bigcup_{\text{twisted sectors}} D^{r-1}.$$

The twisted sectors are labeled by ages $1/r, 2/r, \ldots, (r-1)/r$. The ages of the divisor evaluations $ev_j : \overline{M} \to D$ now have ages $k_i/r$. We can define GW invariants for $X_{D,r}$ as well. The virtual dimension constraint is

$$(1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d |D| + \int_d |D| \frac{m+1}{r} + m + n - \sum \text{ages} = \frac{1}{2} \sum \deg(\gamma_i) + \frac{1}{2} \sum \deg(\delta_i) + \sum a_i.$$

Remark. Not all orbifold GW invariants of $X_{D,r}$ are defined using relative data; there are extra orbifold invariants which will be very important later.

Question: what is the relation between the orbifold invariant $\langle \cdots \rangle_{X_D,r}$ and the relative invariant $\langle \cdots \rangle^{(X,D)}$? Before we talk about this, we state some facts.

1. (Cadman) $\overline{M}_{g,k,n,d}(X_{D,r})$ provides an alternative compactification of the space of relative stable maps.

2. (Maulik–Pandharipande) Relative invariants can be determined from absolute invariants of $X$ and $D$. (Tseng–You) The orbifold invariants can also be determined from this data, with the extra data of $r$.

3. $-K_{X_{D,r}} = -K_X - D + D/r$. As $r \to \infty$ note that this becomes $-K_X - D$. 

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What is the precise relation? In genus-0, they are equal (Abramovich–Cadman–Wise 2017, arXiv 2010):

\[ \langle \cdots \rangle^{(X,D)} = \langle \cdots \rangle^{X_{D,r}}, \quad r \gg 1. \]

This ACW result includes all relative GW invariants of \((X, D)\) but not all orbifold GW invariants of \(X_{D,r}\). Orbifold GW invariants may involve large ages \(1 - k_i/r\), which are not included in this relation. It would be nice if the ACW relation holds in higher genus as well, but in \(g = 1\) there is a counterexample given by Maulik: take

\[ X = E \times \mathbb{P}^1, \quad D = E_0 \cup E_\infty \]

and compute for \([f] \in H_2(E)\) the invariants

\[ \langle \rangle_1^{(X,D)} = 0, \quad \langle \rangle_1^{X_{D_0,(a),(D_{D \infty}, r\infty)}} = r_0 + r_\infty. \]

So what is the relation in higher genus? The invariant \(\langle \rangle^{X_{D,r}}\) is a function of \(r\). As \(r \to \infty\) this is constant in \(g = 0\). For general \(g\), it is a polynomial in \(r\).

**Theorem 3.1** (Tseng–You). The relation is that

\[ \langle \rangle_1^{X_{D,r}} = \langle \rangle_1^{(X,D)}, \]

i.e. the constant term of the polynomial.

**Proof.** The ACW proof doesn’t generalize to higher genus. Instead, the idea is as follows.

1. The comparison is local over \(D\), so we can degenerate to the normal cone \(Y := \mathbb{P}(N_D \oplus O_D)\), by

\[ X \sim X \cup_D Y, \quad X_{D,r} \sim X \cup_D Y_{(D_\infty,r)} \]

Here \(Y_{D_\infty,r}\) is the \(r\)-th root stack of \(Y\) along \(\infty\). By the degeneration formula,

\[ \langle \rangle^{X_{D,r}} = \sum \langle \rangle^{(X,D)} \langle \rangle^{Y_{D_\infty,r}(D_0)} \]

\[ \langle \rangle^{(X,D)} = \sum \langle \rangle^{(X,D)} \langle \rangle^{Y(D_\infty \cup D_\infty)}. \]

The sums are exactly the same, so it remains to compare the orbifold and relative invariants for \(Y\).

2. Localize with respect to the \(\mathbb{C}^\times\) scaling on the \(\mathbb{P}^1\) fibers of \(Y\). There are two fixed loci: one orbifold, one relative. The key idea is from double ramification cycles (Janda–Pandharipande–Pixton–Zvonkine, 2018), and is a polynomiality in the pushforward of the resulting Hurwitz–Hodge classes to \(\overline{M}_{g,n}\). What remains is a rubber integral.

In genus 0, we can actually get this without needing polynomiality. For higher genus we need to take the constant term, which removes some extra contributions from the orbifold side.

Question: do orbifold invariants with large ages stabilize? What is the relation with relative invariants? The answer to the first question is no! Actually \(\langle \rangle^{X_{D,r}}\) depends on \(r\) and tends to 0 when \(r \to \infty\). We should instead set

\[ m_- := \#(\text{large ages}), \]

and look at \(r^{m_-} \langle \rangle^{X_{D,r}}\) (Fan–Wu–You). In genus 0,

\[ r^{m_-} \langle \rangle^{X_{D,r}} = \langle \rangle^{(X,D)} \text{ with negative contact orders.} \]

For positive contact orders, curves in the rubber satisfy kissing conditions with the original curve ramifying at the divisor \(D\). If instead the curve in the rubber ramifies at \(D\) with no corresponding ramification in the original curve, it is a negative contact order. The \(m_-\) is the number of such negative contact orders. (This
is expected to be related to the punctured GW invariants of the Gross–Seibert program, but the precise relation is unknown.)

Using this, we get topological recursion relations, WDVV, relative quantum cohomology, Givental’s formalism, and Virasoro constraints directly from the orbifold GW theory. The underlying state space $\mathcal{H} = \bigoplus_{a \in \mathbb{Z}} \mathcal{H}_a$ has $\mathcal{H}_0 = H^*(X)$, for the untwisted sector, and $\mathcal{H}_a = H^*(D)$ when $a \neq 0$, for the twisted sectors. Write $[\gamma]_i \in \mathcal{H}_i$. There is a pairing $\langle [\alpha]_i, [\beta]_j \rangle = \begin{cases} 0 & i + j \neq 0 \\ \alpha \cup \beta & i + j = 0, \ i, j \neq 0 \\ \int_X \alpha \cup \beta & i = j = 0. \end{cases}$

In higher genus, we do have $[r^m - \langle X_D, r \rangle]_0 = \langle \Omega^{(X,D)} \rangle$ negative contact orders.

For this, we need to prove that $r^m - \langle X_D, r \rangle$ is a polynomial, which is not clear even from preceding results (Fan–Wu–You). This implies a partial CohFT structure, which is the usual CohFT structure without the loop axiom.

4 Kostya (Oct 08): Special geometry for invertible singularities and localization in GLSM

The first paper in this direction was by Candelas, de la Ossa, Green, Parker (’91) where they computed periods for the quintic threefold. Another later paper was by Cecotti and Vafa. We’ll generalizes to the cases of arbitrary number of deformations, for invertible singularities.

Mathematically, the genus zero $B$-model corresponds to variation of polarized Hodge structure. Special geometry is a case of this. Let $X \to M$ be a family of quasi-smooth CY 3-folds. The example worked out first was the family

$$X = \{ \sum_{i=1}^5 x_i^5 - \phi \prod x_i = 0 \}.$$

This is a family over $\mathbb{C}$, parameterized by $\phi$. The special fiber at $\phi = 0$ is the Fermat quintic. There is a Kähler metric on $M$; if $\phi^a$ are coordinates on $M$, then the Kähler potential is

$$\exp (-K(\phi_a, \bar{\phi}_b)) = \int_{X_\phi} \Omega_{\phi} \wedge \overline{\Omega}_{\phi},$$

where $\Omega \in \Gamma(M, H^{3,0}(X_\phi))$ is a family of 3-forms. The Kähler metric itself is

$$G_{ab} = -\partial_{\phi^a} \bar{\partial}_{\phi^b} \log \int_{X_\phi} \Omega_{\phi} \wedge \overline{\Omega}_{\phi} = \frac{\int_{X_\phi} \chi_a \wedge \overline{\chi}_b}{\int_{X_\phi} \Omega_{\phi} \wedge \overline{\Omega}_{\phi}}.$$

These $\chi$ are Beltrami $(2,1)$-forms. In the case of the quintic three-fold,

$$\Omega = \frac{x_5dx_1dx_2dx_3}{\partial W(x, \phi)} = \text{Res}_{x_5=0} \text{Res}_{w=0} \frac{d^5x}{W(x, \phi)}.$$

Pick a flat basis $\{p^a\}$ of $H^3(X_\phi, \mathbb{Z})$; this is do-able because there is a canonical Gauss–Manin connection. Let $\Omega_{\phi} = \omega_a(\phi)p^a$ by taking Poincaré duals. Then

$$\int_{X_\phi} \Omega_{\phi} \wedge \overline{\Omega}_{\phi} = \omega_a(\phi)C^{ab} \overline{\partial}_{\phi^b}(\phi)$$
where $C^{ab}$ is the intersection matrix $\int p^a \wedge p^b$. Usually we pick $\{p^a\}$ such that $C^{ab}$ is symplectic. Then we introduce

$$X^a := \int q_a \Omega_\phi$$

These are projective coordinates on the moduli space. They have dual coordinates

$$F_a = \int q_a^{-1} - h^{2,1} \Omega_\phi.$$ 

Then we can form the superpotential

$$F(X) = \frac{1}{2} \sum_a X^a F_a$$

and say $F_a = \partial X^a F(X)$.

Finally, we define Yukawa constants. These are mirror to 3-point GW invariants. In the B-model they are computed as

$$C_{abc} = \int_{X_\phi} \Omega \wedge \partial_{\phi_a} \partial_{\phi_b} \partial_{\phi_c} \Omega = \frac{1}{X_0^0} \partial_{X^a} \partial_{X^b} \partial_{X^c} F(X)$$

where $\partial_{\phi}$ is the Gauss–Manin connection. Define

$$\tilde{F}(X, X) = (\text{poly in } X, X_0) + X_0 \cdot F(X).$$

This is a superpotential defining a Frobenius manifold isomorphic to the quantum cohomology of the mirror.

What is an invertible singularity? They are natural generalizations of the quintic which do not lose many nice properties. They are defined by equations of the type

$$\{W(X, \phi) = 0\} \subset \mathbb{P}^5_{(v_1, \ldots, v_5)}$$

inside weighted projective space. Invertible singularities are distinguished by the fact that we can write

$$W(X, \phi) = W_0(X) + \sum e^5 \phi_5$$

where $W_0$ is a sum $\sum \prod x_j^{M_{ij}}$ of five monomials such that $(M_{ij})$ is an invertible matrix. Actually most formulas are written in terms of the inverse of this matrix. For the quintic,

$$\exp(-K(\phi, \bar{\phi})) = \sum_{i=0}^{3} (-1)^i \frac{\Gamma((i+1)/5)\Gamma((4-i)/5)}{5} \sum_{n \geq 0} \left( \frac{n+1}{5} \right)^5 \frac{\phi^{5n+i}}{(5n+i)!}.$$ 

We’ll see the general formula is similar to this.

To describe $H^3(X_\phi)$, there is a certain subspace $H^3_{\text{poly}}$ which stands for polynomial deformations. We have

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

and polynomial deformations live in $H^{2,1}$. In the Fermat case all deformations are polynomial. But in general weighted projective spaces have orbifold singularities. Then the hypersurface may intersect such singularities, and the blow-up may contribute non-trivially to $H^3$.

The space $H^3_{\text{poly}}(X_\phi)$ is isomorphic to the Jacobi ring $\text{Jac}(W)_{\mathbb{Z}/d}$. The Jacobi ring is the space of infinitesimal deformations of a polynomial:

$$\text{Jac}(W) = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\langle \partial_1 W, \ldots, \partial_5 W \rangle}.$$ 

Invertible singularities have a huge symmetry group:
\[ W_0(\lambda \sum_j M_{ij}^{-1} x_i) = \sum \lambda_i \prod_j x_j^{M_{ij}}. \]

If we take the \( \lambda_i \) as exponents of \( e^{2\pi i} \), then they generate symmetries. The \( \mathbb{Z}/d \) is the diagonal subgroup where \( \lambda_1 = \cdots = \lambda_5 \).

**Lemma 4.1** (Key lemma). For any \( \gamma \in H_3^{\text{pol}}(X_\phi) \),
\[ \int_{\gamma} \Omega_\phi = \int_{\Gamma} e^{-W(x,\phi)} \, d^5 x \]
where \( \Gamma \in H_5(\mathbb{C}^5, RW \ll 0, \mathbb{Q}) \) is some Lefschetz thimble.

This lemma lets us reduce computations about the CY geometry to oscillating integrals. It is a version of a well-known fact in singularity theory, that period integrals are the same as singularity integrals. Hence we just need to compute
\[ \int_{\Gamma^a} e^{W(x,\phi)} \, d^5 x \, C_{ab} \int_{\Gamma^b} e^{-W(x,\phi)} \, d^5 x. \]

What are these cycles \( \Gamma \)? The space of integrands of such integrals is just \( \Omega^5(\mathbb{C}^5)/(d + dW)\Omega^5(\mathbb{C}^5) \). This differential comes from Stokes’ formula
\[ \int d(e^{W(x,\phi)} \alpha) = \int e^{W(x,\phi)} (d\alpha + dW \wedge \alpha). \]

This space is isomorphic to \( \text{Jac}(W) \). A basis \( \{ e^a \} \) of \( \text{Jac}(W) \) gives a basis \( \{ e^a d^5 x \} \) of this space.

For example, the whole quintic has a 101-dimensional deformation
\[ \sum x_i^5 + \sum_{a=1}^{101} e^a \phi_a. \]

The corresponding oscillating integral is relatively easy to compute:
\[ \int_{\Gamma} e^{W(x,\phi)} \, d^5 x = \sum_{a=1}^{101} \phi_a^{m_1} \cdots \phi_a^{m_{101}} m_1! \cdots m_{101}! \int e^{W_0(x)} \prod_i x_i^{m_i s_i} \, d^5 x. \]

But then we can write
\[ x_1^{b_1} \cdots x_5^{b_5} \, d^5 x = \prod_{i=1}^5 \left( \frac{b_i \mod 5 + 1}{5} \right) x_1^{b_1 \mod 5} \cdots x_5^{b_5 \mod 5} \, d^5 x. \]

(Missing notes.)