Notes for Enumerative geometry seminar (Spring 2019)

Henry Liu

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Abstract

These are my live-texed notes for the Spring 2019 enumerative geometry seminar. Let me know when you find errors or typos. I’m sure there are plenty.

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1 Jan 29 (Renata): Gromov–Witten invariants of stable maps with fields

No notes, sorry!

2 Feb 05 (Renata): Gromov–Witten invariants of stable maps with fields

No notes, sorry!
Let’s first review all-genus mirror symmetry for toric CY 3-folds (or orbifolds), of finite type. This means that the fan contains only finitely many cones. In this setting, there is the remodeling conjecture, which is an all-genus statement for mirror symmetry. It was first introduced by Bouchard–Klemm–Mariño–Pasquetti (BKMP).

Before we talk about all-genus mirror symmetry, let’s first review genus-0 mirror symmetry for general toric manifolds/orbifolds.

1. In the smooth case, this was first proved by Givental and also by Lian–Liu–Yau (in 1997).
2. In the orbifold case, there are two independent proofs:
   (a) by Coates–Corti–Iritani–Tseng (in 2013) generalizing the proof in the smooth case;
   (b) by Cheung–Ciocan–Fontanine–Kim (in 2014) using quasimap theory.

The statement of mirror symmetry is the correspondence of the following data:

\[
\begin{array}{c|c}
\text{A-model} & \text{Landau–Ginzburg B-model} \\
\hline
\text{Semi-projective toric orbifold } X \text{ of dim } r & T\text{-equivariant superpotential } W^T : (\mathbb{C}^*)^r \to \mathbb{C} \\
\text{torus action } T = (\mathbb{C}^*)^r \text{ on } X & \text{the critical locus } \text{Jac}(W^T) \\
\text{equivariant quantum cohomology } QH^*_CR,T(X) & \text{a residue pairing } (f,g) \\
\text{quantum product } \star_t & \\
\text{the } T\text{-equivariant Poincaré pairing } (\cdot,\cdot)_{X,T} & \\
\end{array}
\]

(The construction of \( W^T \) comes purely from the combinatorial data of the fan of \( X \).)

**Theorem 3.1** (Genus-0 mirror symmetry). There is an isomorphism of Frobenius structures

\[
(QH^*_CR,T(X), \star_t, (-, -)_{X,T}) \cong (\text{Jac}(W^T), \cdot, (\cdot, \cdot)).
\]

Now we want to generalize this to the higher-genus case. In higher genus, we also have A-model and B-model.

\[
\begin{array}{c|c}
\text{A-model} & \text{B-model} \\
\hline
X \text{ a toric CY 3-orbifold} & \text{affine mirror curve } C \subset (\mathbb{C}^*)^2 \\
\text{a Lagrangian } L \subset X \text{ (Aganagic–Vafa brane)} & \text{Eynard–Orantin topological recursion producing} \\
\text{open GW potential } F_{g,n}^{X,L} \text{ of } (X,L) & \text{a symmetric } n\text{-form } \Omega_{g,n} \in H^0(\mathbb{C}^n, K_C \otimes \cdots) S^n \\
\end{array}
\]

**Theorem 3.2** (Remodeling conjecture). If we expand \( \Omega_{g,n} \) under suitable local coordinates on the mirror curve \( C \), we obtain the open GW potential \( F_{g,n}^{X,L} \), under the mirror map.

**Example 3.3.** Consider \( X = \text{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2) \). The fan is the cone over the polytope

\[
\begin{array}{c|c}
\hline
\text{A-model} & \text{B-model} \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
X \text{ a toric CY 3-orbifold} & \text{affine mirror curve } C \subset (\mathbb{C}^*)^2 \\
\text{a Lagrangian } L \subset X \text{ (Aganagic–Vafa brane)} & \text{Eynard–Orantin topological recursion producing} \\
\text{open GW potential } F_{g,n}^{X,L} \text{ of } (X,L) & \text{a symmetric } n\text{-form } \Omega_{g,n} \in H^0(\mathbb{C}^n, K_C \otimes \cdots) S^n \\
\end{array}
\]
From this polytope, we can draw the toric diagram by drawing edges in a dual graph perpendicular to corresponding edges in the polytope:

The Lagrangian $L$ will intersect with one of the three non-compact edges. On the mirror side, we get

$$H(X, Y) := X + Y + 1 + qX^3Y^{-1} = 0.$$ 

Here this $q$ is the parameter for complex structures. The equation comes from writing down an monomial for each point in the polytope. The topological type of the mirror curve $C$ is determined by the toric diagram, by fattening. We put a puncture at $(X, Y) = (0, -1)$. If we expand $\Omega_{g,n}$ around $(0, -1)$, we will get the A-model open GW potential.

**Proof strategy.** A priori, the A-model and B-model pictures are of very different objects. The strategy is to realize both A-model and B-model higher-genus potentials as quantizations on two isomorphic semisimple Frobenius structures. The diagram is

$\begin{array}{c}
genus 0 \text{ A-model} \xrightarrow{\text{genus 0 mirror symmetry}} \text{ genus 0 B-model} \\
ed \text{orbifold Givental quantization} \downarrow \text{dimensional reduction} \\
\text{higher genus A-model} \xrightarrow{\text{graph sum formula}} \text{higher genus B-model}
\end{array}$

On the B-model side we still have a curve

$$\hat{C} := \{H(X, Y) = 0\} \subset (\mathbb{C}^*)^2.$$ 

In the original case, the monomials in the equation $H(X, Y)$ corresponded to points in the toric polytope, but now there are infinitely many points. So $H(X, Y)$ will not be an algebraic equation, and there will be a convergence issue. It turns out convergence is not an issue. We can still run topological recursion to get $\Omega_{g,n}$.

Another more interesting case we want to study is when there exist a certain symmetry on the toric variety $Y$. Specifically, assume there exists an action of $\Gamma$ on $Y$, for a certain discrete group $\Gamma$. This induces a $\Gamma$-action on the mirror curve $\hat{C}$. We want to consider the pair

$$X := Y/\Gamma, \quad C := \hat{C}/\Gamma,$$
so $C$ is the mirror curve for $X$. The picture is summarized in the following diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{MS} & \hat{C} \subset (\mathbb{C}^*)^2 \\
\downarrow & & \downarrow \\
X = Y/\Gamma & \xrightarrow{MS} & C = \hat{C}/\Gamma.
\end{array}
\]

Of course, we want to assume the $\Gamma$ action commutes with the torus action on $Y$. So $X$ will have a non-Hamiltonian torus action in general. This is because the moment map $Y \to \mathbb{R}^2$ descends to $X \to \mathbb{R}^2/\Gamma$, which is in general not a vector space.

**Example 3.4.** The polyhedral $\Delta$ is given by the (infinite) periodic tiling which looks like

![Polyhedral Tiling Diagram]

The toric diagram arising from this polyhedral cone is a hexagonal lattice. There is a $\mathbb{Z}^2$-action on $\Delta$ given by translation $(a, b) \mapsto (a+m, b+n)$, which induces an action on the resulting toric variety $Y$. The fundamental diagram in the quotient $X = Y/\mathbb{Z}^2$ is

![Fundamental Diagram]

Here the two vertical edges are identified, and the two horizontal edges are identified, for a total of three $T$-invariant $\mathbb{P}^1$'s. This $X$ is sometimes called the local banana manifold. The generalized moment map is

\[ \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = U(1)^2, \]

i.e. it is Lie group valued. It turns out there is a generalization of Hamiltonian actions called *quasi-Hamiltonian* $T$-actions. This is when the target is Lie group valued and the moment map satisfies certain conditions.

The mirror curve lies in $A := \mathbb{C}^2/\mathbb{Z}^2$, which is an abelian surface. If we write its period matrix as

\[ P = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}, \]

then the mirror curve is given by

\[ H(X,Y) := \Delta(q)\Theta_2 \left[ \left( -\tau/2, -\rho/2 \right) \right] \left( -x/2\pi i, -y/2\pi i; \rho \right), \]

where $\Theta_2$ is the genus-2 Riemann zeta function. So the mirror curve is a genus-2 curve in the abelian surface $A$.

### 4 Feb 26 (Song): Open-closed Gromov–Witten invariants of toric CY3-orbifolds

We'll start with some basics about toric orbifolds. Recall that toric varieties can be defined via the combinatorial data of fans. Toric orbifolds arise from stacky fans, encoding data of the stabilizers. In the algebraic setting, these are called smooth toric DM stacks.
Definition 4.1. A stacky fan is the following data:

1. a finitely-generated free abelian group $N \cong \mathbb{Z}^n$ (i.e. a lattice of finite rank);
2. a simplicial fan $\Sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$;
3. vectors $b_1, \ldots, b_r$ such that the set of 1-cones in $\Sigma$ are generated by $\{\mathbb{R}_{\geq 0}b_i\}$;
4. additional vectors $b_{r+1}, \ldots, b_r$ such that $N = \langle b_1, \ldots, b_r \rangle$.

So we have a short exact sequence

$$0 \to (\mathbb{Z}^{r-n} \cong L) \to (\mathbb{Z}^r \cong \tilde{N}) \xrightarrow{\tilde{b}_i \mapsto b_i} N \to 0$$

Applying $\otimes \mathbb{C}$, we get

$$0 \to G \to \tilde{T} \to T \cong (\mathbb{C}^*)^r \to 0.$$  

This inclusion $G \to \tilde{T}$ induces an action of $G$ on $\mathbb{C}^r$. The toric orbifold $\mathcal{X}$ is the quotient by $G$ of a subvariety of $\mathbb{C}^r$ as follows.

1. Define the anti-cones

$$\mathcal{A} := \{I' \cup \{r' + 1, \ldots, r\} : I' \subset \{1, \ldots, r'\} \text{ s.t. } \sum_{i \notin I'} \mathbb{R}_{\geq 0}b_i \text{ is a cone in } \Sigma\}.$$  

2. Remove anti-cones:

$$U_{\mathcal{A}} := \mathbb{C}^r \setminus \bigcup_{I \in \mathcal{A}} \mathbb{Z}(\prod_{i \in I} x_i).$$  

3. Define the stacky quotient $\mathcal{X} := [U_{\mathcal{A}}/G]$.

Remark. If $b_1, \ldots, b_r$ are minimal generators, in the sense that on each ray we select the smallest integral point, $\mathcal{X}$ will be a simplicial toric variety, not a DM stack. We disallow torsion in $N$ because that gives rise to generic stabilizer.

Example 4.2. Let $N = \mathbb{Z}$ and $\Sigma$ be the complete fan on $\mathbb{R}$. Choose $b_1 = 3$ and $b_2 = 2$. To generate $N$, pick $b_3 = 1$. The result is

$$\mathcal{X} = \mathbb{P}[3, 2].$$

Definition 4.3. In the toric case, we have a nice description of stabilizers in terms of the stacky fan data.

1. Let $\Sigma(d)$ be the set of all $d$-dimensional cones in $\Sigma$.
2. Given $\sigma \in \Sigma(d)$, let

$$I_{\sigma'} := \{i \in \{1, \ldots, r'\} : \mathbb{R}_{\geq 0}b_i \in \sigma\}$$

$$I_{\sigma} := \{1, \ldots, r\} \setminus I_{\sigma'}.$$  

3. Define the $n - d$-dimensional closed substack

$$V(\sigma) := [(U_{\mathcal{A}} \cap \mathbb{Z}(x_i : i \in I_{\sigma'}))/G].$$

Note that this is $T$-invariant. In general, it is not an orbifold. Smaller orbits will have larger generic stabilizers.

Let $G_{\sigma} \leq G$ be the generic stabilizer of $V(\sigma)$. All of these arise combinatorially as follows.
Lemma 4.4. There is an identification of \( G_\sigma \) with

\[
\text{Box}(\sigma) := \{ \sum_{i \in I_{\sigma'}} c_i b_i \in N : 0 \leq c_i \leq 1 \}.
\]

Example 4.5. Let \( N = \mathbb{Z}^2 \) with fan \( \Sigma \) just the first quadrant. Pick

\[b_1 = (2,0), \quad b_2 = (0,3)\]

with additional vectors to generate the lattice. Then we have one 2-dimensional cone \( \sigma \) and two 1-dimensional cones \( \tau_1, \tau_2 \). The box elements are therefore

\[
\text{Box}(\tau_1) = \{(0,0), (1,0)\}, \quad \text{Box}(\tau_2) = \{(0,0), (0,1), (0,2)\}.
\]

These are identified with stabilizers \( \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}/3\mathbb{Z} \). What about \( \text{Box}(\sigma) \)? It is

\[
\text{Box}(\sigma_1) = \text{Box}(\tau_1) \cup \text{Box}(\tau_2) \cup \{(1,1)\}.
\]

This is identified with \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \).

Remark. We see that in general \( \text{Box}(\sigma) \) is always actually a box. If \( \tau \subset \sigma \) then \( \text{Box}(\tau) \) is the projection of \( \text{Box}(\sigma) \) in the appropriate direction.

Remark. The most interesting case is \( \tau \prec \sigma \) where \( \tau \in \Sigma(n-1) \) and \( \sigma \in \Sigma(n) \). This is a \( T \)-fixed point on a \( T \)-fixed curve. The quotient is always a cyclic group \( \mu_{r(\tau,\sigma)} \), where

\[
r(\tau, \sigma) := \frac{|G_{\sigma}|}{|G_{\tau}|}.
\]

Definition 4.6. For \( v := \sum_{i \in I_{\sigma'}} c_i b_i \in \text{Box}(\sigma) \), define the age

\[
\text{age}(v) := \sum c_i.
\]

Definition 4.7. We can package all this data into an object called the inertia stack \( I\mathcal{X} \). Formally, it is defined via a Cartesian square

\[
\begin{array}{ccc}
I\mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}
\end{array}
\]

Example 4.8. In our case, because we understand the stabilizers well, we can identify all the components in the inertia stack. There is a bijection

\[
\{ g \in G : g \text{ fixes a point in } U_A \} \leftrightarrow \text{Box}(\Sigma) := \bigcup_{\sigma \in \Sigma(n)} \text{Box}(\sigma).
\]

The inertia stack will be

\[
I\mathcal{X} := \coprod_{v \in \text{Box}(\Sigma)} \mathcal{X}_v
\]

where \( \mathcal{X}_v \) is the maximal closed substack fixed by the element \( g \in G \) corresponding to \( v \).

Definition 4.9. The inertia stack allows us to upgrade our usual cohomology theory to Chen–Ruan orbifold cohomology

\[
H^*_\text{CR}(\mathcal{X}) := \bigoplus_{v \in \text{Box}(\Sigma)} H^*(\mathcal{X}_v)[2\text{age}(v)].
\]
Example 4.10. Let \( \mathcal{X} = \mathbb{P}[3, 2] \). In this case, the boxes are \( \text{Box}(\sigma_1) = \{0, 1, 2\} \) and \( \text{Box}(\sigma_2) = \{0, -1\} \).

1. The generic stabilizer of the point corresponding to \( \sigma_1 \) is therefore \( Z/3Z \). Hence \( V(\sigma_1) = B\mu_3 \).

2. The generic stabilizer of the point corresponding to \( \sigma_2 \) is therefore \( Z/2Z \). Hence \( V(\sigma_2) = B\mu_2 \).

The inertia stack \( I\mathcal{X} \) therefore has four components

\[
I\mathcal{X} = \mathcal{X}_0 \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \mathcal{X}^{-1}.
\]

1. \( \mathcal{X}_0 = \mathcal{X} \), since the identity element fixes everything. The age is 0.

2. \( \mathcal{X}_1 = \mathcal{X}_2 = B\mu_3 \), corresponding to \( \sigma_1 \). The age is \( 1/3 \) in both cases.

3. \( \mathcal{X}^{-1} = B\mu_2 \), corresponding to \( \sigma_2 \). The age is \( 1/2 \).

Hence the Chen–Ruan orbifold cohomology, as a \( \mathbb{Q} \)-vector space, is

\[
H^*_{CR}(\mathcal{X}) = \mathbb{Q}1_0 \oplus \mathbb{Q}1_{1/3} \oplus \mathbb{Q}1_{1/2} \oplus \mathbb{Q}1_{2/3} \oplus \mathbb{Q}H.
\]

Remark. Note that \( I\mathcal{X} \) has an involution given by

\[
\text{inv} := (x, \varphi) \mapsto (x, \varphi^{-1}).
\]

This essentially maps \( \mathcal{X}_v \) to \( \mathcal{X}_{v^{-1}} \). Here \( \text{“}v^{-1}\text{”} \) means the inverse of the group element corresponding to \( v \).

Definition 4.11. The product structure on \( H^*_{CR}(\mathcal{X}) \) is defined via the orbifold Poincaré pairing

\[
(\alpha, \beta) := \int_{I\mathcal{X}} \alpha \cup \text{inv}^*(\beta).
\]

If \( \alpha, \beta \) are pure in that they both come from single strata \( \mathcal{X}_{v_1} \) and \( \mathcal{X}_{v_2} \), then

\[
(\alpha, \beta) = \begin{cases} 
\int_{\mathcal{X}_{v_1}} \alpha \cup \text{inv}^*(\beta) & V(\beta) = \text{inv}(V(\alpha)) \\
0 & \text{otherwise}.
\end{cases}
\]

Remark. For toric orbifolds, Borisov–Reisner–Smith writes down a presentation for the orbifold cohomology ring which looks like a Stanley–Reisner presentation.

Definition 4.12. Now let’s talk about open-closed GW invariants. Specialize to \( \mathcal{X} \) a toric CY3-orbifold. So \( N = \mathbb{Z}^3 = \langle u_1, u_2, u_3 \rangle \). The CY condition says that

\[
b_i \in \text{hyperplane } N_i := \{u_3 = 1\}.
\]

In this situation, the data of the stacky fan is realized on the \( N_i \) plane as a planar graph. This is equivalent to giving the data called the toric graph, consisting of a point for each top-dimensional cone and a ray for each edge.

Definition 4.13. Since we want to study open invariants, we need Aganagic–Vafa branes. These are Lagrangian sub-orbifolds \( \mathcal{L} \), preserved under the compact CY torus \( T' \subset T \). Such \( \mathcal{L} \) intersect unique \( T \)-fixed curves \( V(\tau) \).

1. If \( \mathcal{L} \) intersects a non-compact orbit, call it outer.

2. If \( \mathcal{L} \) intersects a compact orbit, call it inner.

The computations for the two cases will be very similar.
Remark. In general, \( \mathcal{L} = [S^1 \times \mathbb{C}/G_{\sigma}] \). This is saying that in addition to the winding numbers around \( \mathcal{L} \), we also need to know stabilizers. We get a sequence

\[
\begin{array}{cccccc}
0 & \downarrow & & & \\
& \downarrow & & & \\
& G_{\tau} & \downarrow & & \\
& & \downarrow & & \\
H_1(\mathcal{L}, \mathbb{Z}) & \longrightarrow & G_{\sigma} & \longrightarrow & 0 \\
& \downarrow & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mu_{r,\tau,\sigma} & \longrightarrow & 0 \\
& & & & & \downarrow & & \\
& & & & & 0 \\
\end{array}
\]

Definition 4.14. Now we can define stable maps to \((\mathcal{X}, \mathcal{L})\)

\[
\overline{\mathcal{M}}_{(g,h), n}(\mathcal{X}, \mathcal{L}|_{\beta'}, \vec{\mu}).
\]

1. The domain is a bordered Riemann orbi-surface

\((\Sigma, x_1, \ldots, x_n)\)

with \( n \) marked points. The compact components are genus \( g \), with \( h \) disks attached to these components. We require that orbifold points only come from marked points or the nodes.

2. The map is

\((\Sigma, \partial\Sigma) \xrightarrow{u} (\mathcal{X}, \mathcal{L})\).

The data of this map is

\[
\beta' := u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L})
\]

and \( \vec{\mu} = (\mu_1, \ldots, \mu_n) \) where

\[
\mu_j := u_*([\partial D_j]) \in H_1(\mathcal{L}, \mathbb{Z}).
\]

Remark. We can write the usual perfect obstruction theory, but evaluation maps are more subtle than in the non-orbifold case. Fixed points \([\text{pt}/\mu_r]\) can have non-trivial stabilizer, so evaluation maps must keep track of where the generator of the stabilizer group is sent. So

\[
ev_i: \overline{\mathcal{M}}_{(g,h), n}(\mathcal{X}, \mathcal{L}|_{\beta'}, \vec{\mu}) \to I\mathcal{X}.
\]

Hence we must pull back classes in \( H^*(I\mathcal{X}) = H^*_{CR}(\mathcal{X}) \). In general, \( \mathcal{X} \) is not compact, so we must use the torus action of \( T' \) to define invariants via localization:

\[
\langle \gamma_1, \ldots, \gamma_n \rangle_{\mathcal{L}, \beta', \vec{\mu}} := \prod_{[E] \in \text{vir}} \frac{\prod_{E'} \text{ev}_i^*(\gamma_{E'})}{e_{T'}(N_{E'})}.
\]

How do we compute invariants? We can decompose the data of a stable map into two parts:

1. invariants on the closed components, with \( n + h \) fixed points total;

2. (disk factors) invariants of the \( h \) open components.
In the case of disk factors, our moduli space is simple since \((h, n) = (1, 1)\). Consider the case that \(L\) is inner, on an edge \(V(\tau)\) between fixed points \(V(\sigma^+)\) and \(V(\sigma^-)\). It splits the edge into two orbi-disks \(D^\pm\). Let \(\alpha \in H_2(\mathfrak{X}, \mathcal{L})\) be the class of \(V(\tau)\), and let \(\beta\) and \(\alpha - \beta\) be the classes \([D^+]\) and \([D^-]\) respectively. Given the data 
\[(d_0, \lambda) \in \mathbb{Z} \times G_\tau,\]
define \(h^\pm(d_0, \lambda) \in G_{\sigma^\pm}\) so that \((d_0, h^+, h^-) \in H_1(\mathcal{L}, \mathbb{Z})\). Define the disk factor

\[D_{d_0, \lambda} := \left\{ \begin{array}{ll}
(1_{h^+(d_0, \lambda)}) & d_0 > 0 \\
(1_{h^-(d_0, \lambda)}) & d_0 < 0.
\end{array} \right. \]

These are still a little difficult to compute. We reinterpret them as relative invariants and compute them that way. Idea: degenerate the orbit such that the generic stabilizer \(G_{\tau}\) of the resulting node \(p_0\) is the generic stabilizer of the entire orbit. So for \(d_0 > 0\), we can compute

\[D_{d_0, \lambda} = \int_{[\mathfrak{X}_{0,1}(I_+ / p_0, (d_0, \lambda))]\text{vir}} \text{ev}^* (1_{h^+(d_0, \lambda)}) e(V^+)\]

where \(e(V^+)\) is the contribution of the obstruction bundle. Explicitly,

\[V^+ := R\pi_* f^* (\text{degeneration of } N_{V(\tau)} / \mathfrak{X} \text{ to } I^+).\]

So in the localization calculation, all we need are the additional torus weights at the degeneration point on \(V(\tau)\), and along the divisor corresponding to the degeneration.

5 Mar 05 (Song): Mirror symmetry and crepant resolution conjecture for disks

Last time we talked about open-closed GW invariants. These are roughly just virtual counts of holomorphic orbifold maps

\[f: \left( \begin{array}{c}
\text{bordered prestable} \\
\text{orbifold Riemann surface}
\end{array} \right) \to (\mathfrak{X}, \mathcal{L})\]

where \((\mathfrak{X}, \mathcal{L})\) is a toric CY3-orbifold. This means we have the following data.

1. The action of a 3d complex torus \(T\) on \(\mathfrak{X}\), with compact real sub-torus \(T'_{\mathbb{R}} := U(1)^2\), called the Calabi–Yau torus.

2. Combinatorially, the data of \(\mathfrak{X}\) is given by the stacky fan. By the CY condition, this amounts to some two-dimensional fan, or equivalently a toric graph.

3. \(\mathcal{L}\) is a Lagrangian called the Aganagic–Vafa brane. It must be invariant under the \(T'_{\mathbb{R}}\) action. It intersects a unique \(T\)-invariant curve of \(\mathfrak{X}\).

   (a) If the curve is non-compact, we call \(\mathcal{L}\) outer.

   (b) If the curve is compact, we call \(\mathcal{L}\) inner.

Because we are in the orbifold, each fixed point \(\sigma\) and curve \(\tau\) will have stabilizers. There is a short exact sequence

\[1 \to G_\tau \to G_\sigma \to \mu_{r(\tau, \sigma)} \to 1\]

where \(r(\tau, \sigma)\) is the appropriate index. We saw last time that we can get \(r(\tau, \sigma)\) from the box description. We also get a bijection

\[H_1(\mathcal{L}, \mathbb{Z}) \leftrightarrow \mathbb{Z} \times \mu_m.\]
**Example 5.1.** Let $\mathcal{X} := [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}$. The first factor is an $A_2$-singularity. In terms of the stacky fan, we have

![Stacky Fan Diagram](image)

The stacky fan data is from the five interior lattice points

$$
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
-2 & 1 \\
1 & -1
\end{pmatrix} \xrightarrow{1 \to L \cong \mathbb{Z}^2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} \xrightarrow{\mathbb{Z}^5} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} \xrightarrow{\mathbb{Z}^3 \to 1}
$$

The non-trivial stabilizer is $G = \mathbb{Z}/3$ and all other stabilizers are trivial. The corresponding toric 1-skeleton is

![Toric 1-skeleton Diagram](image)

The thick leg is a *gerby leg*, because it has stabilizer $G$. We want to resolve this singularity, to get the full resolution

![Resolved Toric 1-skeleton Diagram](image)

We can also look at partial resolutions, given in the obvious way, e.g.

![Partial Resolution Diagram](image)

**Definition 5.2.** The moduli space is denoted

$$
\mathcal{M} := \mathcal{M}_{(g,h),n}(\mathcal{X}, \mathcal{L}|\beta', \vec{\mu}).
$$

It parametrizes maps $(\Sigma, \partial \Sigma) \xrightarrow{u} (\mathcal{X}, \mathcal{L})$ such that

$$
\beta' = u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L})
$$

$$
\mu_j = u_*[R_j] \in H_1(\mathcal{L})
$$
where $R_j$ is the $j$-th boundary component of $\Sigma$. There are also **evaluation maps**

$$\text{ev}_i : \mathcal{M} \rightarrow I\mathcal{X}$$

landing in the *inertia stack*. Because we have a compact CY torus $T'_R$ acting on $(\mathcal{X}, L)$, we get an action by $T'_R$ on $\mathcal{M}$. **Open-closed invariants** are defined by localization as

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{\mathcal{X}, L, T'_R} := \int_{[F]_{\text{vir}}} \prod_i \text{ev}_i^*(\gamma_i) \in \mathbb{Q}(v'_1, v'_2)$$

where $v'_1, v'_2$ are the weights of $T'_R$. Here $\gamma_i \in H^*_{CR}(\mathcal{X}, \mathbb{Q})$ are elements in orbifold cohomology.

There are two types of contributions to open-closed invariants: from the open disk components (called disk factors), and from the closed components. We computed disk factors by viewing them as relative invariants, for $\mathbb{P}^1$ relative to a point. If we pick a framing $f \in \mathbb{Z}$, we get a 1-dimensional torus $T'_f := \ker(v'_2 - fv'_1) \hookrightarrow T'$. Pulling back to this torus, disk factors basically become number $\cdot v$ where $v$ is the weight of $T'_f$. These numbers are very explicit (Ross ‘14, Fang–Liu–Tseng). Write

$$\beta' = \beta + \sum_{d_j > 0} d_j b + \sum_{d_j < 0} (-d_j)(\alpha - b)$$

where $(b, \alpha - b)$ are the classes the two sides of the curve cut by the Lagrangian $L$. So $\beta \in H_2(\mathcal{X})$. If we restrict the map $u$ to each of the disk components, then we get maps

$$u|_{D_j} : (D_j, \partial D_j) \rightarrow (\mathcal{X}, L)$$

which are accounted for by the disk factors. So the remaining problem is the contribution from the closed component $C$, i.e. the map

$$[u|_C] : (C, x_i, y_j) \rightarrow (\mathcal{X}, L) \in \overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta) =: \tilde{\mathcal{M}}.$$ 

The data of fixed loci in $\tilde{\mathcal{M}}$ consists of decorated graphs $\hat{\Gamma}$, just like the data of fixed loci in $\mathcal{M}$ consists of decorated graphs $\Gamma$. Given $\Gamma$, we get $\hat{\Gamma}$ by requiring that

$$\text{ev}_{n+j}(\hat{\Gamma}) = (p_{\sigma_\pm}, h_1(d_0, \lambda)),$$

i.e. we put constraints on where the extra marked points (corresponding to the disks) map to. Actually, $F_{\Gamma}$ and $F_{\hat{\Gamma}}$ differ by a finite map, i.e.

$$[F_{\Gamma}]_{\text{vir}} = ? \cdot [F_{\hat{\Gamma}}]_{\text{vir}},$$

and we need to know the extra factor. It contains contributions from:

1. automorphisms of $u|_{D_j}$;
2. node smoothing at $y_j$.

The former already shows up in disk factors. The latter is essentially some $\psi$ class insertions. We get

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{\mathcal{X}, L} = \sum_{\Gamma} \gamma \cdot D(d_{0, \lambda}, f) \int_{[F_{\hat{\Gamma}}]_{\text{vir}}} \prod_{i=1}^{n} \text{ev}_i^* \gamma_i \prod_{j=1}^{b} \text{ev}_j^* \phi_j \prod_{j=1}^{b} \left( \frac{u_j}{r_j d_j} - \frac{v_{n+j}}{r_j} \right) e(N_{F_{\hat{\Gamma}}})$$

where $?$ is some combinatorial automorphism factor.
To compute disk invariants, we will compute the whole generating function for \((g, h) = (0, 1)\). Define
\[
F_{(0,1)}^{X, (\mathcal{L}, f)}(\tau_0, \tau_1, \ldots, \tau_k) := \sum_{(d_0, \lambda) \in H_1(\mathcal{L}, \mathbb{Z})} \sum_{\beta, n \geq 0} \cdots \in H^*_\text{CR}(B\mu_m; \mathbb{C}).
\]
One part of this comes from genus-0 closed GW invariants \(J(\tau_1, \ldots, \tau_k, z)\), where \(\tau_1, \ldots, \tau_k\) are formal variables keeping track of coefficients in front of a basis of \(H^2_{\text{CR}}(\mathcal{X})\). The remaining \(\tau_0\) keeps track of the open part. In genus-0, this \(J\) function is related to the \(I\) function
\[
I(q_0, q_1, \ldots, q_k, z) \in H^*_{\text{CR}}(\mathcal{X})[[z]].
\]
The \(q_1, \ldots, q_k\) keep track of curve classes, and \(q_0\) keeps track of the open part. The relation is as follows.

**Theorem 5.3** (Coates–Corti–Iritani–Tseng, equivariant genus-0 mirror symmetry).

\[
e^{\frac{2\pi}{\mathcal{L}}} J(\tau, z) = I(q_0, q, z)
\]
under the closed mirror map.

We can look at \(F_{(0,1)}^{X, (\mathcal{L}, f)}\) under the mirror map, to get
\[
W^{X, (\mathcal{L}, f)}(q_0, q, z).
\]
This \(W\) can be written very explicitly via the work of Fang–Liu–Tseng.

**Example 5.4.** Return to the example \(\mathcal{X} = \left[\mathbb{C}^2/\mathbb{Z}_3\right] \times \mathbb{C}\). Then the rows of the matrix
\[
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
-2 & 1 \\
1 & -1
\end{pmatrix}
\]
are the \(D_i\).

1. The Kähler cone, in terms of the \(D_i\), is described by the picture

\[
\text{[Diagram of the Kähler cone]}
\]
and is given by the interior of the cone spanned by \(D_4\) and \(D_5\). The other chambers correspond to the partial resolutions written earlier. Together, they form a fan covering the whole plane. The whole system is called the secondary fan.

2. The Mori cone parametrizes effective curve classes. This means we take the dual. Let \(\beta_1, \beta_2\) be dual to \(D_4, D_5\).

Then the extended parameter space is
\[
K^\text{eff}(\mathcal{X}, \mathcal{L}) := \{(d, c_1 \beta_1, c_2 \beta_2) : d > 0, c_1, c_2 \geq 0\}.
\]
This is what we sum over in the Fang–Liu–Tseng formula.

This is useful for the crepant resolution conjecture. If \(\mathcal{X}\) is a Gorenstein orbifold and \(Y \overset{\pi}{\rightarrow} \mathcal{X}\) is a crepant resolution, then the enumerative invariants of \(Y\) and \(\mathcal{X}\) should roughly match up.
Conjecture 5.5 (Ruan). If $Y \overset{\pi}{\to} \mathcal{X}$ is crepant, then there is an isomorphism of rings

$$\text{QH}^*(Y, \mathbb{C}) \cong \text{QH}^*_{\text{CR}}(\mathcal{X}, \mathbb{C}).$$

Here $\text{QH}^*_{\text{CR}}$ means the orbifold cohomology with the full quantum product.

This roughly boils down to identifying the Gromov–Witten invariants of $Y$ and $\mathcal{X}$. In our case, if $\mathcal{X}$ is a toric CY3-orbifold, then the partial resolutions are given by subdivision of cones and are all crepant.

Theorem 5.6 (Ke–Zhou). In the case where $L$ is effective, i.e. $L$ intersects a non-gerby leg, for $(\mathcal{X}, \mathcal{L}, \mathcal{f}) \to (\mathcal{X}, L, f)$ we can match up the two disk potentials, i.e.

$$W^{\mathcal{X}, (\mathcal{L}, \mathcal{f})}(\mathcal{x}, \mathcal{q}) = W^{\mathcal{X}, (L, f)}(x, q)$$

with a change of variables.

Example 5.7. Take our previous example. We have the full resolution

```
\begin{center}
\begin{tikzpicture}
  \node (A) {$\mathcal{X}$};
  \node (B) [below left of=A] {$\mathcal{X}$};
  \node (C) [below right of=A] {$\mathcal{X}$};
  \node (D) [below left of=C] {$\mathcal{X}$};
  \draw (A) -- (B) -- (C) -- (D);
\end{tikzpicture}
\end{center}
```

Put the Lagrangian on a non-gerby leg, so that the resolution doesn’t affect the Lagrangian. (This is what makes this case easier.) Then the strategy is to identify

$$K_{\text{eff}}(\mathcal{X}, \mathcal{L}) \leftrightarrow K_{\text{eff}}(\mathcal{X}, L)$$

by doing a change of basis to identify $D_2, D_5$ with $D_4, D_5$.

6 Mar 12 (Clara): Klein TQFTs

We will first recap the complex picture, of local curves. Then we will move to the real Gromov–Witten setting, where we get a Klein TQFT.

Definition 6.1. Consider GW theory for $Y$ a non-singular quasi-projective algebraic 3-fold:

$$\overline{\mathcal{M}}^*_h(Y, \beta).$$

We would like to define invariants via

$$Z'(Y)_{\beta} := \int_{[\overline{\mathcal{M}}^*_h(Y, \beta)]^{\text{vir}}} 1.$$  

Since $Y$ is not compact, we instead need to define invariants via localization:

$$Z'(Y)_{\beta} := \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{\mathcal{M}}_h(Z, \beta)]^{\text{vir}}} \frac{1}{e(\text{Norm}^{\text{vir}})}$$

where $Z \subset Y$ is a compact submanifold.
Definition 6.2 (Local curves). Now specialize to the following setting. Let $X$ be a curve of genus $g$, with two line bundles $L_1, L_2$. Let

$$Y := \text{tot}(L_1 \oplus L_2 \to X)$$

be the total space. There are two tori $\mathbb{C}^\times$ acting on $L_1$ and $L_2$ by scaling fibers. All torus-invariant curves to $Y$ will therefore land in $X$, so

$$\mathcal{M}_h(Y, d[X]) = \mathcal{M}_h(X, d).$$

The difference between the obstruction theories is the term $\text{Norm}^\text{vir} := R\pi_* \text{ev}^*(L_1 \oplus L_2)$.

Definition 6.3 (Relative stable maps). In order to use degeneration, we generalize to relative stable maps, i.e. maps $C \to Y$ with prescribed ramification profiles over points in $X \subset Y$. Degeneration then tells us that if $Y = Y_1 \cup_D Y_2$, then we should reconstruct the GW theory of $Y$ from gluing the GW theories of $Y_1$ and $Y_2$ along $D$. Let $\lambda_1, \ldots, \lambda_r$ be partitions of the degree $d$. Define the moduli of relative stable maps

$$\mathcal{M}_h^*(X, \lambda_1, \ldots, \lambda_r).$$

Denote the invariants by

$$\text{GW}(g|k_1, k_2)^{\lambda_1, \ldots, \lambda_t}_{\mu_1, \ldots, \mu_s}$$

where $k_1, k_2$ are the levels, i.e. the degrees of the line bundles $L_1, L_2$ over $X$.

Theorem 6.4 (Degeneration and gluing). Pick splittings $g = g' + g''$ and $k_i = k'_i + k''_i$. Then

$$\text{GW}(g|k_1, k_2)^{\lambda_1, \ldots, \lambda_t}_{\mu_1, \ldots, \mu_s} = \sum_{\rho \vdash d} \text{GW}(g'|k'_1, k'_2)^{\lambda'_1, \ldots, \lambda'_{t'}} \text{GW}(g''|k''_1, k''_2)^{\mu'_1, \ldots, \mu'_{s'}}.$$

Definition 6.5. The GW partition functions for relative maps into local curves forms a TQFT. A TQFT is a symmetric monoidal functor $F: 2\text{Cob} \to \text{Mod}(R)$.

The category $2\text{Cob}$ has:

1. objects which are compact oriented 1-manifolds, i.e. $\bigcup_{i=1}^n S^1$ for every $n$;
2. morphisms which are oriented cobordisms.

Hence there are generators:

- identity
- multiplication
- comultiplication
- unit
- counit

Example 6.6. We describe the GW local curves TQFT $F$. Take the ring $R := \mathbb{Q}(t_1, t_2)((q))$, where $q$ will keep track of Euler characteristic and $t_1, t_2$ are equivariant weights. Define the functor $F$ as follows.

$$F(S^1) := H := \bigoplus_{\rho \vdash d} R e_\rho, \quad F(S^1 \sqcup \cdots \sqcup S^1)_r := H^\otimes r.$$

On a cobordism between $s$ circles and $t$ circles, the morphism will be

$$e_{\eta^1} \otimes \cdots \otimes e_{\eta^s} \mapsto \sum_{\mu^1, \ldots, \mu^t \vdash d} \text{GW}(g|0, 0)^{\mu^1, \ldots, \mu^t}_{\eta^1, \ldots, \eta^s} e_{\mu^1} \otimes \cdots \otimes e_{\mu^t}.$$
Gluing cobordisms corresponds to gluing GW partition functions via the degeneration formula. To cover all levels \((k_1, k_2)\), i.e. degrees of the line bundles \(L_1, L_2\), we decorate the cobordisms with the levels of the line bundles over them. So for example, we will have a \((0, 0)\) cap and a \((0, -1)\) cap:

\[
(0,0) \quad (0,-1)
\]

Levels are additive with respect to gluing. For example

\[
\begin{array}{c}
(3,4) \circ (-1,2) \circ (0,1) = (2,7)
\end{array}
\]

So now our cobordism category is enriched with the data of labels. Denote it \(2\text{Cob}^{k_1,k_2}\). Now to solve the local curves TQFT, it suffices to compute a few \((0, 0)\) and \((0, -1)\) cobordisms.

Now we will move to the real case. Let \(\Sigma\) be a Riemann surface. The moduli

\[
\overline{\mathcal{M}}_{d,\chi}(\Sigma)
\]

will parametrize real curves \(C \to \Sigma\) with degree \(d\) and Euler characteristic \(\chi\). There is an involution \(\phi\) representing complex conjugation.

**Definition 6.7 (Real GW).** Let \((X, \omega)\) be a symplectic manifold. Let \(\phi\) be an anti-symplectic involution on \(X\), i.e. \(\phi^*\omega = -\omega\). To have a real map, we also need a real domain. A symmetric Riemann surface \((C, \sigma)\) is a Riemann surface \(C\) with an anti-holomorphic involution \(\sigma\). Then \(\sigma\) splits the domain into:

1. real components, invariant under it, and
2. complex-conjugate doublets, which we write as \(C_0 \sqcup \overline{C_0}\).

A real stable map is

\[f: (C, \sigma) \to (X, \phi)\]

compatible with \(\sigma\) and \(\phi\).

**Definition 6.8 (Real local curve).** An R-bundle on \((X, \phi)\) is

\[(V, \varphi) \to (X, \phi)\]

where \(V \to X\) is a vector bundle and \(\varphi\) is an involution lifting \(\phi\) fiber-wise. To get the setting of local curves, take a holomorphic line bundle \(L \to (\Sigma, \sigma)\) and pull back \(\sigma\) to form

\[(L \oplus \sigma^*L, \sigma) \to (\Sigma, \sigma)\].

This is the setting of real local curves. Invariants are denoted

\[RZ^\phi_{d,\chi} := \int_{\overline{\mathcal{M}}_{d,\chi}(\Sigma)} \frac{1}{e(\text{Norm}^{\text{vir}})}\]

Ramification data \(\lambda\) now comes in conjugate pairs \((\lambda, \overline{\lambda})\), via applying the involution \(\sigma\).
Example 6.9 (Doublet target). We want to compute
\[ RZ^d_{\lambda_1}(D\Sigma|D(L_1, L_2))_{\lambda_1, \ldots, \lambda_s}. \]
Here
\[ D\Sigma := (\Sigma \sqcup \Sigma, \sigma) = (\Sigma_1 \sqcup \Sigma_2, \sigma) \]
and \( D(L_1, L_2) \) is a single line bundle given by
\[ D(L_1, L_2)|_{\Sigma_1} = L_1, \quad D(L_1, L_2)|_{\Sigma_2} = \sigma^* L_2. \]
Then we look at the bundle \( D(L_1, L_2) \oplus \sigma^* D(L_1, L_2) \). There is an isomorphism of moduli spaces
\[ D: \overline{M}_{d, \chi}(\Sigma)_{\lambda_1, \ldots, \lambda_s} \leftrightarrow \overline{M}_{d, \chi}(D\Sigma)_{\lambda_1, \ldots, \lambda_s} \]
by doubling. The virtual classes compare as
\[ [\overline{M}_{d, 2\chi}(D\Sigma)_{\lambda_1, \ldots, \lambda_s}]^{vir} = (-1)^{dm_1 + f_2} D_* [\overline{M}_{d, \chi}(D\Sigma)_{\lambda_1, \ldots, \lambda_s}]^{vir}. \]
The normal bundles also differ up to some sign.

7 Mar 26 (Mark): Integral transforms and quantum correspondences

Start with \( \overline{M}_{g, n}(X, d) \) for a smooth projective variety/orbifold \( X \). The general question; if two varieties \( X, Y \) are related in some way, how does their GW theory relate?

Example 7.1 (K-equivalence). If \( X_1, X_2 \) are birational
\[ \begin{array}{ccc}
  \tilde{X} & \xleftarrow{\pi_1} & X_1 \\
  \pi_2 & \downarrow & \pi_2
  \end{array} \]
\[ \begin{array}{ccc}
  & & \pi_1 \downarrow \pi_2 \\
  X_2 & \xrightarrow{\pi_2} & \end{array} \]
such that \( \pi_1^* K_{\tilde{X}} = \pi_2^* K_{X_2} \), we say they are K-equivalent. Then it is known in a lot of generality that the GW theories of \( X_1 \) and \( X_2 \) are related, called the crepant transformation conjecture (CTC). The relationship is complicated, involving analytic continuation of partition functions. This analytic continuation is compatible, in a precise sense, with a Fourier–Mukai transform \( D_b(X_1) \to D_b(X_2) \).

Example 7.2 (Quantum Serre duality). Let \( Z = Z(W) \) be a degree-\( d \) (smooth) hypersurface in \( \mathbb{P}^r \). Let
\[ Y := \text{tot}(\mathcal{O}_{\mathbb{P}^r}(-d)). \]
Then \( Y \) is non-compact of dimension \( r + 1 \), and \( Z \) is compact of dimension \( r - 1 \). There is a relation called quantum Serre duality that says the GW theory of \( Y \) determines the GW theory of \( Z \) in genus 0. (The procedure is to take a derivative of the partition function of \( Y \) and then do a change of variables.) This is not too surprising, because:

1. GW theory of \( Y \) involves \( H^1(C, f^*(\mathcal{O}(-d))(-p_2)) \);
2. GW theory of \( Z \) involves \( H^0(C, f^*(\mathcal{O}(d))(-p_1)) \).

On the smooth locus \( C = \mathbb{P}^1 \), these are related by Serre duality.
Example 7.3. Now let’s look at an example where both are in play. Let $W$ be a degree-$d$ polynomial in $d$ variables, i.e. corresponding to a section $s \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(d))$, so that

$$Z := \{ W = 0 \}$$

is CY. Then we have a diagram of relations

Here the potential $W$ defines a function on $[\mathbb{C}^d/\mathbb{Z}_d]$. The enumerative theory for such singularities with potential is called FJRW theory.

Theorem 7.4 (YP, Nathan, Mark). There exists a correspondence on the left of the above diagram.

Remark. This theorem means we can understand LG/CY in terms of CTC. But how geometric are these correspondences? More precisely, CTC comes from some FM transform at the level of derived categories, but do the rest?

Definition 7.5. Let $K(Y) \to GW_{g=0}(Y)$ be Iritani’s integral structure, then we can draw another square

Iritani’s integral structure map comes from

$$L(t, z): H^*(X) \to H^*(X)[[q]][z^{-1}]$$

$$\alpha \mapsto \alpha + \sum_d q^d \sum_{n \geq 0} \left( \frac{\alpha}{-z - \psi}, t, \ldots, t, \phi_i \right)_{0, n+2} \phi^i.$$}

Then $L(t, z)(\alpha)$ is a solution to the Dubrovin connection

$$\nabla_i = \partial_i + \frac{1}{z} \phi_i X (-).$$

Iritani’s map is given by

$$K(X) \to \text{solutions to the qDE}$$

$$E \mapsto L(t, z)(\text{ch}(E) \Gamma(TX))$$

where $\Gamma(TX)$ is a square root of the Todd class.
**Definition 7.6.** We can add the same square for K-theory of $[\mathbb{C}^d/\mathbb{Z}_d]$ twisted by $W$, and Orlov tells us we have an equivalence $K([\mathbb{C}^d/\mathbb{Z}_d], W) \to K(Z)$. So we now have a diagram

$$
\begin{array}{ccc}
K([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{Fourier–Mukai}} & K(Y) \\
\downarrow & & \downarrow \\
GW_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{crepant transformation}} & GW_{g=0}(Y) \\
\text{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \to \mathbb{C}) & \xrightarrow{\text{LG/CY}} & GW_{g=0}(Z)
\end{array}
$$

If we restrict $K([\mathbb{C}^d/\mathbb{Z}_d])$ to the ones supported only on the origin $[0/\mathbb{Z}_d]$ and the same for $K(Y)$ to the ones supported only on $\mathbb{P}^{d-1}$, then we get arrows

$$
K([\mathbb{C}^d/\mathbb{Z}_d])|_{[0/\mathbb{Z}_d]} \to K([\mathbb{C}^d/\mathbb{Z} - d], W) \\
K(Y)|_{\mathbb{P}^{d-1}} \to K(Z)
$$

The final diagram is

$$
\begin{array}{ccc}
K([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{Orlov}} & K(Z) \\
\downarrow & & \downarrow \\
GW_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{crepant transformation}} & GW_{g=0}(Y) \\
\text{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \to \mathbb{C}) & \xrightarrow{\text{LG/CY}} & GW_{g=0}(Z)
\end{array}
$$

**Theorem 7.7 (YP, Nathan, Mark).** The back square implies the front square.

**Definition 7.8.** What are the arrows in the top square? Look in the simple case of $d = 2$. Let $j: Z \to \mathbb{P}^1$ be the inclusion, and $\pi: Y \to \mathbb{P}^1$ be the projection. Then the diagram is

$$
\begin{array}{ccc}
K([\mathbb{C}^2/\mathbb{Z}_2])|_{[0/\mathbb{Z}_2]} & \xrightarrow{\infty} & K(\text{tot}(\mathcal{O}_{\mathbb{P}^1}(-2)))|_{\mathbb{P}^1} \\
\varepsilon & & \varepsilon \\
\mathcal{O}_{[0/\mathbb{Z}_2]} = [\mathbb{C}[\frac{x_2}{x_1}], \mathbb{C}^2/\mathbb{C} \frac{x_1}{x_2}] & \xrightarrow{\infty} & \mathcal{O}_{Y(2)} \to \mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^1} \\
\varepsilon & & \varepsilon \\
\mathcal{O}_{[0/\mathbb{Z}_2]} = [\mathbb{C}[\frac{x_2}{x_1}], \mathbb{C}^2/\mathbb{C} \frac{x_2}{x_1}] & \xrightarrow{\infty} & \mathcal{O}_{Y(2)} \to \mathcal{O}_Y = \mathcal{O}_Z \\
\varepsilon & & \varepsilon \\
K([\mathbb{C}^2/\mathbb{Z}_2], x_1^2 + x_2^2) & \xrightarrow{\infty} & K(Z)
\end{array}
$$

Here we are viewing $\mathbb{P}^1 = \{\ell = 0\} \subset Y$ and taking the Koszul resolution.

18
1. In general, if \( f : X \to Y \) is a proper map, then there is a pushforward

\[
f_* : K(X, f^*W) \to K(Y, W)
\]

for potentials \( W : Y \to \mathbb{C} \). Using this, the vertical left arrow comes from

\[
[C^2/\mathbb{Z}_2] \leftrightarrow [0/\mathbb{Z}].
\]

by the composition

\[
K([C^2/\mathbb{Z}_2]) \xrightarrow{p} K([0/\mathbb{Z}], 0) \xrightarrow{i} K([C^2/\mathbb{Z}_2], x_1^2 + x_2^2).
\]

2. Under the isomorphism \( K(Z) = K(Y, \tilde{W}) \), the structure sheaf \( \mathcal{O}_Z \) becomes a factorization version of the Koszul complex:

\[
[\mathcal{O}_Y(2) \leftrightarrow \mathcal{O}_Y].
\]

Then the map from \( K(\text{tot}(\mathcal{O}(-2))) \to K(Y, W) \) arises the same way as the vertical left map.

Remark (Connection to \( p \)-fields). Let \( Y = \mathcal{O}_{\mathbb{P}^r}(-d) \). In genus 0, recall that the GW theory of \( Y \) has virtual class

\[
[\mathcal{M}_{0,n}(Y, D)]^{\text{vir}} = e(R^1\pi_*\mathcal{L}^{\otimes -d}) \in A_*(\mathcal{M}_{0,n}(\mathbb{P}^r, d)).
\]

For a hypersurface \( Z \), we can use \( p \)-fields:

\[
[\mathcal{M}_{0,n}(\mathbb{P}^r, D)]^p = e(R^1\pi_*\mathcal{L}^{\otimes -d} \otimes \omega_{\log})) \in A_*(\mathcal{M}_{0,n}(\mathbb{P}^r, D)).
\]

If \( n = 2 \) and \( C \) is smooth, then \( \omega_{\log} = \mathcal{O}_C \). Hence on the smooth locus, the two virtual classes are identical. This perspective generalizes beyond complete intersections.

8 Apr 02 (Clara): Klein TQFTs

Recall that the moduli of stable real maps is

\[
\mathcal{M}^{n,c}_{d,h}(\Sigma)^{\lambda_1, \ldots, \lambda_r}.
\]

We continue with the notation and setup of last time. Start with a symmetric connected Riemann surface \( \Sigma \). Real maps have involutions on domains, so we get a splitting of the domain into components of two types:

1. (doublets) pairs of components swapped by the involution;
2. (connected components) components fixed by the involution.

So we need to compute these individually, and then exponentiate. We can relate both to complex invariants.

Example 8.1 (Doublet domains). For doublet domains, the first step is to pass to the double cover of the domain. The doublet moduli space

\[
\mathcal{D}\mathcal{M}_{d,h}(\Sigma)^{\lambda_1, \ldots}
\]

has domains with fixed first and second components. There is a forgetful map

\[
\varphi : \mathcal{D}\mathcal{M}_{d,h}(\Sigma) \to \mathcal{D}\mathcal{M}_{d,h}(\Sigma)
\]

which gives

\[
\mathcal{D}\mathcal{M}_{d,h}(\Sigma) = \frac{1}{2} \varphi_*[\mathcal{D}\mathcal{M}_{d,h}(\Sigma)]^{\text{vir}}.
\]
Then we need to distribute ramification patterns. Let \( f \) split as \((f_1, f_2)\) over the two components in the domain. If \( f_1 \) has ramification \( \lambda^+ \) over \( x^+ \), then \( f_2 \) has ramification \( \lambda^- \) over \( x^+ \), such that the original is \( \lambda = \lambda^+ \sqcup \lambda^- \). So we have a decomposition of the doublet moduli space into pieces

\[
\tilde{D}\mathcal{M}_{d,h}(\Sigma)^{\lambda^+,\lambda^-}
\]

where \( \tilde{\lambda}^\pm \) are ramifications over \( x^\pm \) in the first component. (This fully determines the ramification over the second component.) These pieces are related to the complex moduli \( \mathcal{M}_{d,h}(\Sigma)^{\lambda^+,\lambda^-} \) by doubling, and their virtual classes agree up to sign.

**Example 8.2** (Sphere relative to a point). Define a shifted version of the partition function such that the level-0 real GW series has no non-zero terms of positive degree. (This is a non-trivial vanishing result.)

1. (Doublet) Upon doubling, this becomes connected GW invariants for a complex tube. Such invariants vanish unless both ramification points are fully ramified.
2. (Connected components) By the vanishing result, we need \( h - 1 + \ell(\lambda) = 0 \). So \( \ell(\lambda) = 1 \) and \( h = 0 \).

(Notes stop here, sorry.)

### 9 Apr 09 (Yunfeng): Vafa–Witten invariants via surface Deligne–Mumford stacks

**Definition 9.1.** Let’s first discuss the background of Vafa–Witten for surfaces. Let \( S \) be a smooth projective surface. If we fix some polarization \( \mathcal{O}(1) \), then we can consider the moduli space \( \mathcal{M}^{r,c_1,c_2}(S) \) of stable torsion-free sheaves of rank \( r \) and Chern classes \( c_1, c_2 \) on \( S \). In Donaldson theory, if we fix \( r, c_1 \) and let \( c_2 \) move, we can write down a generating function

\[
F(q) := \sum_{c_2 \in \mathbb{Z}} \chi(\mathcal{M}^{r,c_1,c_2}) q^{c_2}.
\]

S-duality from physics implies that \( F(q) \) is a modular form (Vafa–Witten, 94).

**Example 9.2.** If \( S = \mathbb{P}^2 \), for \( r = 1 \) and \( c_1 = 0 \), then

\[
F(q) = \sum_{c_2 \in \mathbb{Z}} \chi(\text{Hilb}^{c_2}(S)) q^{c_2} = \frac{1}{\prod_{k \geq 0} (1 - q^k)^3}.
\]

If \( r = 2 \) and \( c_1 = 0 \), then

\[
F(q) = \frac{1}{\prod_{k \geq 1} (1 - q^k)^6} \sum_{m,n \geq 1} \frac{q^{mn+m+n}}{-q^{m+n}}.
\]

This method can be generalized to any smooth toric surface.

**Example 9.3** (Blow-up formula). Let \( S = \mathbb{P}^2 \) and let \( \tilde{S} \to S \) be a blow-up at one point. Then

\[
\sum_n \chi(\mathcal{M}^{H_{\infty,c_1,n}(\tilde{S})}) q^{n-c_1^2/4} = \sum_{n \in \mathbb{Z}} q^{(n+c_1/2)^2} \prod_k (1 - q^k)^2 \sum_n \chi(\mathcal{M}^H(c_1,n)) q^{n-c_1^2/4}.
\]

**Definition 9.4.** Let \( S \) be a 2d DM stack. Many examples are interesting here.

1. Take weighted projective planes \( \mathbb{P}(a,b,c) \).
2. If \( S \) is a smooth surface and \( D \subseteq S \) is a smooth divisor, take the \( r \)-th root stack \( \sqrt[r]{(S,D)} \) of \( S \) along \( D \), where we make \( D \subseteq S \) have multiplicity \( \mathbb{Z}/r \).
3. Quintic surface has deformations into ADE surfaces, which can be viewed as smooth DM surfaces.

F. Nirumi constructed the moduli stack of stable torsion-free sheaves on \( S \) as follows. Let \( p: S \to S \) be the map to the coarse moduli space. Choose a **generating sheaf** \( \Xi \): a locally free sheaf on \( S \) which is relatively \( p \)-very ample. We use this to define the Hilbert polynomial, so we don’t lose any stacky information. For any \( \mathcal{E} \in \text{Coh}(S) \), the **Hilbert polynomial** is

\[
H_{\Xi}(\mathcal{E}, m) := \chi(S, \mathcal{E} \otimes \Xi^\vee \otimes p^*\mathcal{O}_S(m)) = \sum_{i=0}^{\dim} \alpha_{\Xi,i} m^i.
\]

Define the **reduced Hilbert polynomial** as \( h_{\Xi}(\mathcal{E}) := H_{\Xi}(\mathcal{E})/\alpha_{\Xi,\dim} \). We say \( \mathcal{E} \) is **Gieseker stable** if for any subsheaf \( \mathcal{F} \subset \mathcal{E} \),

\[
h_{\Xi}(\mathcal{F}) < h_{\Xi}(\mathcal{E}).
\]

**Remark.** In the root stack case \( S = \sqrt[4]{(S,D)} \), we can choose

\[
\Xi := \mathcal{O}_S \oplus \mathcal{O}_S(D^{1/d}) \oplus \cdots \oplus \mathcal{O}_S(D^{(d-1)/d}).
\]

With this choice, Gieseker stability is equivalent to **parabolic stability** on the pair \((S,D)\). This means we take a parabolic structure, i.e. a filtration

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_d = \mathcal{E}(-D)
\]

along the divisor \( D \), and require stability with respect to this filtration.

**Proposition 9.5** (Nirumi). Fix the Hilbert polynomial \( H \). Then \( \mathcal{M}_H(S) \) exists as a projective scheme.

**Definition 9.6.** Let

\[
\tilde{\text{ch}}: K_*(S) \to H^*(\text{IS})
\]

be the orbifold Chern character. In the decomposition \( \text{IS} = \bigsqcup \iota_S S_i \), write

\[
\tilde{\text{ch}}(\mathcal{E}) := (\tilde{\text{ch}}_i(\mathcal{E}) \in H^*(S_i)), \quad (\tilde{\text{ch}}_i)^k \in H^{\dim S_i-k}(S_i).
\]

Then the generating function we look at is

\[
H_{\alpha,\beta}(q) := \sum_{\tilde{\text{ch}}(C) = \alpha, \tilde{\text{ch}}(C) = \beta} \chi(\mathcal{M}_{\Xi,C}(S)) q^C.
\]

**Example 9.7.** Let \( S = \mathbb{P}(1, 1, 2) \), so that \( \text{IS} = \mathbb{P}(1, 1, 2) \sqcup B\mu_2 \). Let \( \Xi := \mathcal{O}_S \oplus \mathcal{O}_S(-1) \) be the generating sheaf. If we fix \( C \in K(S) \), then

\[
(\tilde{\text{ch}}_1)^2 = 2, \quad (\tilde{\text{ch}}_1)^1 = c_1(S).
\]

But we have other terms:

1. \( q_1 \) will keep track of \( c_2(\mathcal{E}) = (\tilde{\text{ch}}_1)^0 \);
2. \( q_2 \) keeps track of \( (\tilde{\text{ch}}_1)^0 \) where \( \zeta^2 = 1 \) is the generating of \( \mu_2 \).

The generating function is therefore \( F(q_1, q_2) = H_{2,c_1}(S)(q_1, q_2) \). In the case where \( q_1 = q \) and \( q_2 = 1 \), we can compute it to be

\[
\left( \frac{q^{1/6}}{\eta(q)^2} \theta^2(q) \right)^2 \sum_{(w_1, w_2, w_3) \in C_{c_1}} q^{w_1^2/4 + (w_2^2 + w_2^2 + w_3^2)/4}
\]

where \( C_{c_1} \) is the restrictions \( 2 \mid c_1 \) and \( 2 \mid w_2 \) and \( w_i < w_j + w_k \) for \{i, j, k\} = \{1, 2, 3\}. The second term counts contributions from the moduli of bundles, while the first term counts the rest. The moduli of bundles arises because any torsion-free sheaf \( \mathcal{E} \) embeds into its reflexive hull \( \mathcal{E}^\vee \), which is locally free for surfaces. (This is a general phenomenon for surfaces.)
Definition 9.8 (Vafa–Witten). A Higgs bundle \((\mathcal{E}, \phi)\) on \(S\) is a torsion-free coherent sheaf \(\mathcal{E}\) and a section \(\phi: \mathcal{E} \to \mathcal{E} \otimes K_S\). The usual spectral construction relates this to DT invariants on 3-folds. Let \(\mathcal{X} := \text{tot}(K_S)\)

so that \(\mathcal{X}\) is a CY3 DM stack.

Proposition 9.9 (Spectral construction). There exists an equivalence of abelian categories

\[
\text{Higgs}_{K_S}(S) \sim \text{Coh}_c(\mathcal{X}).
\]

Proof. If we let \(\pi: \mathcal{X} \to S\) be the projection, then there is a decomposition

\[
\pi_* \mathcal{O}_\mathcal{X} = \mathcal{O}_S + K_S^{-1} + K_S^{-2} + \cdots.
\]

Then use that \(\mathcal{O}_X\)-modules on \(X\) is equivalent to \(\pi_* \mathcal{O}_\mathcal{X}\)-modules on \(S\).

Remark. We can still define Gieseker invariance for these Higgs bundles, by only imposing the stability condition for \(\phi\)-invariant subsheaves. Using appropriate pullbacks of the polarization, Gieseker stability for \(\text{Higgs}_{K_S}(S)\) matches Gieseker stability on \(\text{Coh}_c(\mathcal{X})\).

Definition 9.10. The symmetric obstruction theory on \(\text{Coh}_c(\mathcal{X})\) now induces a symmetric obstruction theory on \(\text{Higgs}_{K_S}(S)\) which is exactly the usual obstruction theory. Vafa–Witten invariants are defined using this obstruction theory:

\[
\overline{\text{VW}} := \int_{\mathcal{N}_S^{\text{vir}}} \frac{1}{e(N^{\text{vir}})}
\]

where \(\mathcal{N}\) is the moduli. This is not entirely correct: there is a trivial factor in the obstruction theory corresponding to the trace. Actual Vafa–Witten invariants are defined on the moduli \(\mathcal{N}_L^\perp\) of fixed determinant \(L\) and trace-free \((E, \phi)\):

\[
\text{VW} := \int_{\mathcal{N}_L^\perp^{\text{vir}}} \frac{1}{e(N^{\text{vir}})}.
\]

We can also define

\[
\text{vw} := \chi(\mathcal{N}_L^\perp, \nu)
\]

using a Behrend function \(\nu\).

Remark. In general, VW and vw are not the same. However, the invariant vw agrees with the one coming from cosection localization, where the cosection comes from the \(\mathbb{C}^\times\)-action on the symmetric obstruction theory.

Proposition 9.11 (Tanaka–Thomas). \(\text{VW} = \text{vw}\) if \(S\) is Fano.

Proposition 9.12 (Maulik–Thomas). \(\text{VW} = \text{vw}\) if \(S\) is K3.

Definition 9.13. To calculate the invariant, we have to study the \(\mathbb{C}^\times\)-fixed locus on \(\mathcal{N}_L^\perp\). There are two kinds of fixed locus.

1. \((\phi = 0)\) This is the moduli \(\mathcal{M}_L\) of stable sheaves on \(S\). This is called the instanton branch \(\mathcal{M}^1\).

2. \((\phi \neq 0)\) This is the case of \(\phi\) nilpotent and is called the monopole branch \(\mathcal{M}^2\). Gholampour–Thomas proved that \(\mathcal{M}^2\) in general is a nested Hilbert scheme on \(S\).

Remark. S-duality from physics swaps the monopole and instanton branches via the transformation \(\tau \mapsto -1/\tau\). In general, it is easier to compute on the monopole branch, so this gives a lot of conjectural data about Donaldson invariants coming from the instanton branch.

Remark. Yunfeng computed two cases:

1. root stacks of quintic surfaces;

2. quintic surfaces with ADE singularities. These are deformation invariant to smooth quintics, so VW should remain the same. But vw is not deformation invariant, so we expect different answers.
In many mathematical problems, we have a space $\mathcal{M}$ and an associated linear space $V_{\mathcal{M}}$ on which a Hamiltonian $H$ acts, and we care about solutions to the equation

$$Hf = \frac{\partial}{\partial t} f.$$ 

For example: modular forms, wave equation, heat equation. If we diagonalize $H$, we turn the DE problem into a linear algebra problem:

$$\lambda \hat{f} = \frac{\partial}{\partial t} \hat{f}.$$ 

Shuai’s slogans (v1.0):

1. Better universes exist;
2. Representation theory as the exploitation of symmetry;

Concretely, consider the bundle $T^* \mathbb{P}^1 \to \mathbb{P}^1$, where $\mathbb{P}^1$ has weight $a$ and the bundle has linearization $(\hbar + a, \hbar - a)$. Note that

$$e(T^* \mathbb{P}^1)|_0 = -a(h + a), \quad e(T^* \mathbb{P}^1)|_\infty = a(h - a).$$

On the DT side, Hilb($T^* \mathbb{P}^1$) has divisor classes $M(2)$ and $M(1,d)$.

From elementary number theory, we know that a prime $p$ is the sum of two squares iff $p \equiv 1 \mod 4$. One way to prove this is to look at the lattice $\mathbb{Z}[[i]]$. The geometric question to ask is: over which primes $p$ do we have ramification in the map $\text{Spec} \mathbb{Z}[[i]] \to \text{Spec} \mathbb{Z}$. Alternatively, we want to ask which points in the lattice $\mathbb{Z}[[i]]$ have norm $p$.

Now let’s return to $T^* \mathbb{P}^1$ and $\mathbb{P}^1$. We understand the equivariant quantum cohomology $\text{QH}^*_T(\mathbb{P}^1)$ very well. Why would we want to look at $\text{QH}^*_T(T^* \mathbb{P}^1)$? It fits nicely into $\bigwedge^k \text{Gr}(k, 2)$, and the cohomology

$$H^*(\text{pt} \sqcup T^* \mathbb{P}^1 \sqcup \text{pt}) = \mathbb{C} \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus \mathbb{C}$$

is a 4-dimensional rep of $\mathfrak{sl}_2$. The operator of quantum multiplication $u*$ can now be understood in terms of the representation theory of $\mathfrak{sl}_2$. We will find

$$u* = u \cup + \frac{q}{1 - q} [\mathbb{P}^1 \times \mathbb{P}^1].$$

We will see that

$$\text{QH}^*_T(T^* \mathbb{P}^1) = \mathbb{C}[h, a, u] / \langle u^2 - au - h \frac{q}{1 - q} (h + a - 2u) \rangle.$$ 

The projection to $\text{QH}^*_T(\mathbb{P}^1)$ is given by the $h^2 q$ coefficient.

To compute quantum multiplication by $u$, it suffices to compute

$$(u * u, 1) = \langle u, u, 1 \rangle_{0,3,0} + \sum q^k \langle u, u, 1 \rangle_{0,3,n}$$

$$(u * u, u) = \langle u, u, u \rangle_{0,3,0} + \sum q^k \langle u, u, u \rangle_{0,3,n}.$$ 

Let’s compute the first term.

1. The classical part is

$$\langle u, u, 1 \rangle_{0,3,0} = \int_{T^* \mathbb{P}^1} u^2 = a \int_{T^* \mathbb{P}^1} u = a \left( \frac{u}{-a(a + h)} + \frac{a}{a(h - a)} \right) = \frac{a}{h - a}.$$
2. The quantum part is
\[ \langle u, u, 1 \rangle_{0,3,n} = (1)_{0,1,n} = 0 \]
by divisor relation (twice).

Now for the second term.

1. The classical part is
\[ \langle u, u, u \rangle_{0,3,0} = a^2 \int_{T^*P^1} u = \frac{a^2}{h-a} \]
2. The quantum part is
\[ \langle u, u, u \rangle_{0,3,n} = \langle \rangle_{0,0,n} \]
by divisor relation (three times). Then \( \mathcal{M}_{0,0}(T^*\mathbb{P}^1, 1) = \text{pt} \), but the virtual class is \( h \). In general this is true in degree \( n \) as well.

So we find
\[ (u \ast u, 1) = \frac{a}{h-a}, \quad (u \ast u, u) = \frac{a^2}{h-a} + \sum q^K h. \]

What if we want to compute \( c_1 \ast ? \) Look at the Steinberg \( Z \). We want to argue that
\[ c_1 \ast = c_1 \cup + h \frac{q}{1-q} [\mathbb{P}^1 \times \mathbb{P}^1] \]
where \( \mathbb{P}^1 \times \mathbb{P}^1 \subset Z \) is the diagonal. Let \( \pi_1, \pi_2 : Z \to T^*\mathbb{P}^1 \) be the projections. Then we only need to compute the terms
\[ \int_{T^*\mathbb{P}^1} \pi_2 \ast ([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1 \ast (u)) \cdot 1, \quad \int_{T^*\mathbb{P}^1} \pi_2 \ast ([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1 \ast (u)) \cdot u. \]
1. By push-pull, we get
\[ \int_{T^*\mathbb{P}^1} \pi_2 \ast ([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1 \ast (u)) \cdot 1 = \int_{Z} [\mathbb{P}^1 \times \mathbb{P}^1] \pi_1 \ast (u) \]
\[ = \frac{a}{-a^2} + \frac{a}{a^2} = 0. \]
2. Again by push-pull, we get
\[ \int_{T^*\mathbb{P}^1} \pi_2 \ast ([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1 \ast (u)) \cdot u = 1. \]
So we just have to check that \( \int (h + a - 2u) \cdot 1 = 0 \) and \( \int (h + a - 2u) \cdot u = 1 \), which it does. Hence
\[ c_1 \ast = c_1 \cup + h \frac{q}{1-q} (h + a - 2u). \]
Now let’s look at \( \mathcal{M}_{0,3}(T^*\mathbb{P}^1, d) \to \mathcal{M}_{0,3}(\mathbb{P}^1, d) \), i.e. how to take the limit in \( h \). It fits into a diagram
Let’s compare virtual localization on both. In general, localization takes the form
\[
\int \prod \text{ev}^* (\gamma_i) = \sum \int_{\mathcal{M}_g} \prod \text{ev}^* (\gamma_i) N_{Y/X}^{\text{vir}}.
\]
Because the diagram above commutes, the numerators in both cases are the same. The only difference is a factor \( e(f^* \mathcal{O}(-2)) \). By Riemann–Roch, this is \( 1 - 2d \). So downstairs the coefficients we compute are for \( q^d \), and upstairs the coefficients are for \( \hbar^{2d-1}q^d \). It turns out there aren’t many terms like \( \hbar^{2d-1}q^d \) upstairs. In general, there is a complicated change of variables and limit we have to take.

Now look at 2-pointed relative GW invariants for \( X := T^* \mathbb{P}^1 \times \mathbb{P}^1 \) with relative conditions \( \mu, \nu \) at 0 and \( \infty \). We can compare it with \( Y := \mathbb{P}^1 \times \mathbb{P}^1 \):
\[
\int_{\mathcal{M}_{g,1}(X,\mu,\nu)} \text{ev}^* (i^* \omega) =: \langle \mu | M_{(1, \omega)} | \nu \rangle \int_{\mathcal{M}_{g,1}(Y,\mu,\nu)} \text{ev}^* (\text{pt}) =: \langle \mu | M_{(u, Q)} | \nu \rangle.
\]
The difference in virtual normal bundles is still \( e(f^* \mathcal{O}(-2)) \), which again by Riemann–Roch becomes \( (1 - g) + ((dV + mH), -2H) = 1 - g - 2d \).
Hence coefficients of \( u^{2g-2}v^d H^m \) downstairs become coefficients of \( u^{2g-2}v^d H^m \hbar^{2d+g-1} \). Let \( Z_{(\mu, (1, \omega), \nu)} \) denote the generating function. It is given by
\[
u - \ell(\mu) - \ell(\nu) s \frac{d}{ds} \Theta, \quad \Theta = \frac{\hbar}{(\text{Aut } \mu)(\text{Aut } \nu)} \sum (du)_{(\mu) + (\nu) - 2} \prod_{i=1}^{\ell(\mu)} s(du_i u) \prod_{i=1}^{\ell(\nu)} s(du_i u)
\]
where \( s(u) := \sin(u/2)/(u/2) \). For example, if \( \mu = \nu = \Box \), then \( Z = h u^{-2} \sum_{d=1}^{\infty} s^d \), and
\[
\langle \Box | \text{pt} | \Box \rangle = [hu^{-2}s]Z = 1.
\]
If \( |\mu| = |\nu| \), then we will need the \( hu^{-2}s \) coefficient of
\[
u^{-2}h \frac{d^{(\mu) + (\nu) - 2}}{|\text{Aut } \mu||\text{Aut } \nu|} \prod \cdots \prod \cdots.
\]
The \( \cdots \) are of the form \( \sin(u)/u \), and in the limit contribute just 1.

In Maulik’s \( A_n \)-resolutions paper, the quantum multiplication operator is written as
\[
M_{(1, \omega)} = M_{(1, \omega)} \cup s \frac{d}{ds} \sum_{k \in \mathbb{Z}} :f(k)e(-k): \log(1 - q^k s).
\]
We can match
\[
\sum_{k \in \mathbb{Z}} :f(k)e(-k): = \sum_{|\mu|=|\nu|} \alpha_{-\mu} \alpha_{\nu}
\]
by showing the LHS is the zeroth order term in the product \( \Gamma_+ \Gamma_- \) of two vertex operators.