Notes for Topics in AG: Deformation Theory
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Abstract

These are my live-texed notes for the Fall 2017 offering of MATH GR6263 Topics in Algebraic Geometry. Let me know when you find errors or typos. I'm sure there are plenty.

1 Introduction and motivation

Deformation theory is the study of the local structure of a space, e.g. a representable functor, near a point. Later on, a functor will not be sufficiently general, so we will work with categories co-fibered in groupoids. We will use deformation theory to answer the following sorts of questions.

1. Let X be a variety over $\mathbb{C}$, and $x \in X(\mathbb{C})$. Describe $\widehat{\mathcal{O}}_{X,x}$.
2. Prove the (formal) smoothness of a lot of moduli problems, e.g.
   (a) $\overline{\mathcal{M}}_g$ over $\mathbb{Z}$;
   (b) $\mathrm{Hilb}_n(S)$ for S a smooth surface;
   (c) (Tian–Todorov theorem) the "moduli" of Calabi–Yaus;
   (d) the moduli of principally polarized abelian varieties;
   (e) (Deligne) K3 surfaces lift from characteristic p to characteristic 0;
   (f) certain moduli of representations.

   All these examples are atypical: "Murphy's law" (Vakil) says that moduli problems are arbitrarily singular typically.
3. Bound the dimension of moduli spaces, e.g. we can ask for $\dim_{[f:\mathbb{P}^1\to X]} \mathrm{Hom}(\mathbb{P}^1, X)$.
4. Infinitesimal and generic Torelli theorems (e.g. for hypersurfaces), and infinitesimal variation of Hodge structure.

Other topics to be discussed potentially include: the cotangent complex, the generic vanishing theorem and deformations of complexes, dg-algebra techniques, the moduli of sheaves, Galois deformations, and semiregularity (Bloch).

2 Deformation functors

Definition 2.1. Let k be a field, and let Art/k denote the **category of local Artin k-algebras with residue field k**. An important example of such a k-algebra is $k[\epsilon]/\epsilon^2$.

Definition 2.2. A **deformation functor** is a covariant functor $D\colon \mathsf{Art}/k \to \mathsf{Set}$ such that $D(k) = \{*\}$. There is a category of deformation functors, in which morphisms are natural transformations.

Remark. Think of a curve C over $\operatorname{Spec} k$. If $\operatorname{Spec} k$ fits into $\operatorname{Spec} A$, we can ask how the curve C moves as $\operatorname{Spec} k$ moves in $\operatorname{Spec} A$. Here we use $A \in \mathsf{Art}/k$, as a sort of "microlocal neighborhood." This is the setting of deformation theory.

Remark. There are two ways this definition is insufficient: we shouldn't work over a field, and "functor" is too strong of a condition. (There are a lot of complications when deforming objects with automorphisms.)

Example 2.3. Suppose $X_{/k}$ is a scheme, and $x \in X(k)$. Define the deformation functor

$$F_{(X,x)}(A) \coloneqq \left\{ \begin{matrix} \operatorname{Spec} A & \longrightarrow & X \\ \uparrow & & \uparrow \\ \operatorname{Spec} k & \longrightarrow & x \end{matrix} \right\} = \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,x}, A) = \mathrm{Hom}_{k\text{-alg}}(\widehat{\mathcal{O}}_{X,x}, A)$$

where we can complete because the maximal ideal in A is nilpotent. (So this deformation functor can only reveal things about $\widehat{\mathcal{O}}_{X,x}$.)

Definition 2.4. In general, given a ring R, let $h_R(A) \coloneqq \mathrm{Hom}(R, A)$.

Example 2.5. Suppose $Z \hookrightarrow X$ is a closed embedding. Define the deformation functor

$$H_{Z,X}(A) := \left\{ A\text{-flat closed subschemes } \widetilde{Z} \subset X_A : \begin{tikzcd} \widetilde{Z} \arrow[r] & X_A \\ Z \arrow[u] \arrow[r] & X \arrow[u] \end{tikzcd} \text{ Cartesian} \right\}$$

where X_A is the base change to A. The picture to have in mind is that $\widetilde{Z}$ is a deformation of Z specified by A. This is a local version of the Hilbert scheme. Indeed, $H_{Z,X} = F_{(\operatorname{Hilb} X,[Z])}$ if $\operatorname{Hilb} X$ is representable.

Example 2.6. Suppose $C_{/k}$ is a smooth proper geometrically connected curve of genus g. Define

$$F_{[C]}(A) := \left\{ \pi\colon \mathcal{C} \to \operatorname{Spec} A \text{ with } \varphi\colon \mathcal{C}_k \xrightarrow{\sim} C : \pi \text{ is finitely presented and flat of dim } 1 \right\} / \sim .$$

Here, think of $F_{[C]}$ as $F_{(\mathcal{M}_g,[C])}$. This makes sense if $g \geq 2$.

Let's analyze deformation functors. Two questions we can ask about deformation functors D are:

1. are they (pro-)representable?
2. given $B \to A$ a surjection in Art/k, what is the image and fibers of $D(B) \to D(A)$?

2.1 Tangent-obstruction theories

Definition 2.7. Let $B \to A$ be a surjection in Art/k. We say $0 \to M \to B \to A \to 0$ is a **small extension** if $\mathfrak{m}_B M = 0$. Exercise: any such $B \to A$ can be factored as compositions of small extensions.

Let R be a complete local k-algebra with residue field k such that $d := \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ is finite. Exercise: there exists a surjection $k[[x_1,\ldots,x_d]] \to R$. Let $0 \to M \to B \to A \to 0$ be a small extension. We want to analyze the map $h_R(B) \to h_R(A)$.

1. Simple case: $R = k[[x_1,\ldots,x_d]]$. Then the map $R \to A$ always lifts to $R \to B$, by lifting the images of $x_1,\ldots,x_d$. The fibers are torsors for $M^d = \operatorname{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, M)$. So we have an exact sequence of sets
$$(\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \otimes M \to h_R(B) \to h_R(A) \to 0.$$
2. General case: there is a surjection $S = k[[x_1,\ldots,x_d]] \xrightarrow{\pi} R$ with kernel I. Then $h_R(B) = \{f\colon S \to B \in h_S(B) : f(I) = 0\}$. We have the diagram
$$\begin{tikzcd} 0 & R \arrow[l] \arrow[d, "\varphi"'] & S \arrow[l, "\pi"'] \arrow[d, "\widetilde{\varphi}"'] & I \arrow[l] & 0 \arrow[l] \\ 0 & A \arrow[l] & B \arrow[l] & M \arrow[l] & 0 \arrow[l] \end{tikzcd}$$
and we want to know how many lifts $\widetilde{\varphi}$ exist. Suppose we are given $\alpha, \beta\colon S \to B$ lifting $\varphi \circ \pi$. Then $\alpha - \beta\colon S \to M$. Exercise: this is a derivation over k. Hence the set of $\widetilde{\varphi}$ is a torsor for the set of derivations $\operatorname{Der}(S, M) = (\mathfrak{m}_S/\mathfrak{m}_S^2)^\vee \otimes M$. It follows that $(\alpha - \beta)(I) = 0$, because $I \subset \mathfrak{m}_S^2$ and $\alpha - \beta$ is a derivation. So $\widetilde{\varphi}|_I$ does not depend on $\widetilde{\varphi}$. We also know $\widetilde{\varphi}(\mathfrak{m}_S I) = 0$. Hence $\widetilde{\varphi}|_I = 0$ iff $\operatorname{ob}(\varphi) := \widetilde{\varphi}|_I\colon I/\mathfrak{m}_S I \to M$ is zero. This entire argument can be written as an exact sequence
$$(\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \otimes M \to D(B) \to D(A) \xrightarrow{\operatorname{ob}} (I/\mathfrak{m}_S I)^\vee \otimes M.$$

Definition 2.8. A deformation functor D has a **tangent-obstruction theory** if there are finite-dimensional vector spaces T_1 and T_2 and functorial (in the map $B \to A$) exact sequences

$$T_1 \otimes M \to D(B) \to D(A) \xrightarrow{\operatorname{ob}} T_2 \otimes M$$

for any small extension $0 \to M \to B \to A \to 0$ such that:

1. $a \in D(A)$ has $\mathrm{ob}(a) = 0$ iff a lifts to $D(B)$;

2. if α lifts to $D(B)$, then $T_1 \otimes M$ acts transitively on the set of lifts;

3. if $A = k$, then this sequence is left-exact, i.e. the action is actually simply transitive.

In this language, we proved the following theorem.

Theorem 2.9. *Let R be a local k-algebra with $\dim \mathfrak{m}_R/\mathfrak{m}_R^2 = d < \infty$, and a surjection $S \coloneqq k[[x_1, \dots, x_d]] \to R$ with kernel I. Then $h_R(-) \coloneqq \mathrm{Hom}_k(R, -)$ has tangent-obstruction theory given by $T_1 = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$ and $T_2 = (J/\mathfrak{m}_S J)^\vee$.*

Remark. We think of T_1 as the tangent space of the functor. It is determined by D, by taking $B = k[\epsilon]/\epsilon^2$ and $A = k$. However, T_2 is not canonical; this is obvious because we can inject T_2 into any bigger T_2'.

Example 2.10. Let X be a proper smooth scheme and consider

$$\mathrm{Def}_X(A) \coloneqq \{\mathcal{X} \to \operatorname{Spec} A \text{ flat, with } \varphi\colon \mathcal{X}_k \xrightarrow{\sim} X\}.$$

Then the tangent-obstruction theory is $T_i = H^i(X, T_X)$. (Properness is required so these T_i are finite-dimensional.) We can also consider $H_{Z,X}$, for which $T_1 = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$ and $T_2 = \mathrm{Ext}(\mathcal{I}_Z, \mathcal{O}_Z)$.

Example 2.11. Let X be a proper smooth scheme and $\mathcal{E} \in \mathsf{Coh}(X)$. Then

$$\mathrm{Def}_{\mathcal{E}}(A) \coloneqq \{\widetilde{\mathcal{E}} \in \mathsf{Coh}(X_A) \text{ flat over } A \text{ and } \varphi\colon \widetilde{\mathcal{E}}_k \xrightarrow{\sim} \mathcal{E}\}$$

has $T_i = \mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$.

2.2 Application: Lefschetz theorem for Pic

Example 2.12 (Deforming line bundles)**.** Let $X_0 \hookrightarrow X$ be a closed embedding of schemes defined by a square-zero ideal sheaf $\mathcal{I}$. In this situation, we get a map $\operatorname{Pic} X \to \operatorname{Pic} X_0$. To find the kernel and image of this map, we fit it into a long exact sequence:

$$H^1(X, \mathcal{I}) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{O}_{X_0}^*) \xrightarrow{\mathrm{ob}} H^2(X, \mathcal{I})$$

arising from the short exact sequence $1 \to (1 + \mathcal{I}) \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 1$, and the isomorphism $(1 + \mathcal{I}) \cong \mathcal{I}$. Hence a line bundle $\mathcal{L}$ on X_0 lifts to X when $\mathrm{ob}(\mathcal{L}) = 0$, and the set of lifts is acted upon transitively by $H^1(X, \mathcal{I})$. However this is not yet a tangent-obstruction theory; there is no deformation functor and no finite-dimensional T_1 and T_2.

Let's specialize the situation: let X be a k-scheme with k a field. For a given $\mathcal{L} \in \mathrm{Pic}(X)$, we define a deformation functor

$$\widehat{\mathrm{Pic}}_{X,\mathcal{L}}\colon \mathsf{Art}/k \to \mathsf{Set}, \quad A \mapsto \{\text{line bundle } \mathcal{L}' \text{ on } X_A,\, \varphi\colon \mathcal{L}|_X \xrightarrow{\sim} \mathcal{L}\}/\sim.$$

This deformation functor has a tangent-obstruction theory. Let $0 \to M \to B \to A \to 0$ be a small extension in Art/k. The map $\iota\colon X_A \to X_B$ is a closed embedding defined by the square-zero ideal sheaf $\pi^* M$. Hence we get

$$H^1(X, \pi^* M) \to \mathrm{Pic}(X_B) \xrightarrow{\iota^*} \mathrm{Pic}(X_A) \to H^2(X, \pi^* M).$$

By the projection formula, $H^i(X, \pi^* M) = H^i(X, \mathcal{O}_X) \otimes M$. Assume X proper. Then $T_1 \coloneqq H^1(X, \mathcal{O}_X)$ and $T_2 \coloneqq H^2(X, \mathcal{O}_X)$ are finite-dimensional and form a tangent-obstruction theory. We have shown the following.

Corollary 2.13. $T_{[\mathcal{L}]}\underline{\mathrm{Pic}}(X) = H^1(X, \mathcal{O}_X)$ *if* $\mathrm{Pic}(X)$ *is representable.*

Remark. Note that $T_2 \neq 0$ in general here, but nonetheless, if $\operatorname{char} k = 0$ and $\mathrm{Pic}(X)$ is representable, then it is smooth because it is a group scheme. In $\operatorname{char} k = p$, however, $\mathrm{Pic}^0(X)$ is not always smooth.

Theorem 2.14 (Lefschetz hyperplane theorem for Pic)**.** *Let* $\operatorname{char} k = 0$ *and* $X_{/k}$ *be a smooth projective variety of dimension* ≥ 4*. Let* $D \subset X$ *be a smooth ample divisor. Then* $\operatorname{Pic} X \to \operatorname{Pic} D$ *is an isomorphism.*

Remark. If $\dim X = 3$, then it is only injective. The 4 here is sharp because we can take $X = \mathbb{P}^3$ and D a cubic surface. Then $\operatorname{rank} \operatorname{Pic}(D) = 7$.

Proof. We want to show a given line bundle $\mathcal{L}$ on D extends uniquely to all of X. Exercise: over $\mathbb{C}$, we can deduce this from the Lefschetz hyperplane theorem and Lefschetz $(1,1)$ theorem. The first thing to try is to extend $\mathcal{L}$ to some infinitesimal neighborhood of D. Let $\widehat{X}^D$ be the formal scheme of X completed at D (thought of as a tubular neighborhood of D), with

$$\operatorname{Pic} \widehat{X}^D = \varprojlim_n \operatorname{Pic} D_n.$$

The next thing to try is to extend from a tubular neighborhood to an actual neighborhood, by considering $\varinjlim_{U \supset D} \operatorname{Pic} U$. Finally, we extend to all of X. We write these steps as

$$\operatorname{Pic} D \xleftarrow{\text{deformation theory}} \operatorname{Pic} \widehat{X}^D \xleftarrow{\text{algebraization}} \varinjlim_{U \supset D} \operatorname{Pic} U \xleftarrow{\text{extension}} \operatorname{Pic} X.$$

1. (Extension) Note that for any U with $D \subset U$, we have $\dim X \smallsetminus U = 0$. Otherwise the complement contains a curve which is disjoint from D, and therefore D is not ample. Now use Hartog's theorem: line bundles extend uniquely along subsets of codimension ≥ 2 when X is smooth.

2. (Deformation theory) Write $D = V(f)$ with $f \in H^0(X, \mathcal{O}(D))$, and let $D_n \coloneqq V(f^n)$. (This is the n-th infinitesimal neighborhood of D.) We want to show that the natural map $\varprojlim_n \operatorname{Pic} D_n \to \operatorname{Pic} D$ is an isomorphism. It is enough to show $\operatorname{Pic} D_n \to \operatorname{Pic} D_{n-1}$ are all isomorphisms. These fit into sequences

$$H^1(D, \mathcal{O}(-(n-1)D)|_D) \to \operatorname{Pic} D_n \to \operatorname{Pic} D_{n-1} \to H^2(D, \mathcal{O}(-(n-1)D)|_D).$$

   (Equivalently, $\mathcal{O}(-(n-1)D)|_D \cong \mathcal{I}_D^{n-1}/\mathcal{I}_D^n$.) But these H^i vanish by Kodaira vanishing, which is where we need $\operatorname{char} k = 0$.

3. (Algebraization) We want to show $\varprojlim_n \operatorname{Pic} D_n \to \varinjlim_{U \supset D} \operatorname{Pic} U$ is an isomorphism. Suppose $\mathcal{E}$ is a vector bundle on X. Then $H^i(X, \mathcal{E}) \xrightarrow{\sim} \varprojlim H^i(D_n, \mathcal{E}|_{D_n})$ for $i \leq \dim X - 2$. This is because there is a short exact sequence

$$0 \to \mathcal{E}(-nD) \to \mathcal{E} \to \mathcal{E}|_{D_n} \to 0$$

   and moreover, $H^i(\mathcal{E}(-nD)) = 0$ for $0 \leq i \leq \dim X - 1$ and $n \gg 0$ by Serre vanishing (and Serre duality). Hence for $i \leq \dim X - 2$, we have $H^i(\mathcal{E}) \xrightarrow{\sim} H^i(\mathcal{E}|_{D_n})$ for $n \gg 0$.

   Now suppose $\mathcal{E}$ is a vector bundle on $\widehat{X}^D$. Then there exists an open $U \supset D$ and a vector bundle $\widetilde{\mathcal{E}}$ on U such that $\mathcal{E} = \widetilde{\mathcal{E}}|_{\widehat{X}^D}$. This is because of the following. On the formal scheme $\widehat{X}^D$, we can resolve $\mathcal{E}$,

$$\mathcal{O}_{\widehat{X}^D}(a_1)^{n_1} \xrightarrow{f} \mathcal{O}_{\widehat{X}^D}(a_2)^{n_2} \to \mathcal{E} \to 0,$$

   so $\mathcal{E} = \operatorname{coker}(f)$. But $f \in H^0(\widehat{X}^D, \mathcal{H}om(\mathcal{O}(-a_1)^{n_1}, \mathcal{O}(a_2)^{n_2}))$, so we can apply the above isomorphism to get $\widetilde{f} \in H^0(X, \mathcal{H}om(\mathcal{O}(-a_1)^{n_1}, \mathcal{O}(a_2)^{n_2}))$ with $f = \widetilde{f}|_{\widehat{X}^D}$. Hence define $\widetilde{\mathcal{E}} \coloneqq \operatorname{coker}(\widetilde{f})$, which is locally free on some open set U containing D. This proves surjectivity.

   Injectivity goes as follows. Given $\mathcal{L}_1$ and $\mathcal{L}_2$ on U isomorphic on D via $\varphi \colon \mathcal{L}_1|_D \xrightarrow{\sim} \mathcal{L}_2|_D$. Then $\varphi \in \Gamma(\widehat{X}^D, \mathcal{H}om_{\widehat{X}^D}(\mathcal{L}_1, \mathcal{L}_2))$ and we can apply the isomorphism again to lift φ to an isomorphism on some open set $U' \supset D$. Hence $\mathcal{L}_1 \cong \mathcal{L}_2$ in $\varinjlim_{U \supset D} \operatorname{Pic} U$. $\square$

2.3 Pro-representability and Schlessinger's criterion

Definition 2.15. A deformation functor $D\colon \mathsf{Art}/k \to \mathsf{Set}$ is **pro-representable** if there exists a complete local k-algebra $R \in \mathsf{Loc}/k$ (with residue field k and finite-dimensional tangent space) and an isomorphism $h_R \coloneqq \mathrm{Hom}_{\mathsf{Loc}/k}(R,-) \xrightarrow{\sim} D$.

Example 2.16. Suppose $X_{/k}$ is a smooth projective variety and consider the deformation functor

$$\mathrm{Def}_X\colon \mathsf{Art}/k \to \mathsf{Set}, \quad A \mapsto \{\mathcal{X} \to \operatorname{Spec} A \text{ flat with } \varphi\colon \mathcal{X}_0 \xrightarrow{\sim} X\}/\sim.$$

Then Def_X is pro-representable iff $H^0(X, T_X) = 0$. We will go through this example carefully later.

Definition 2.17. Let $\alpha\colon F \to G$ be a morphism of deformation functors. We say α is **smooth** if it satisfies the lifting property that defines formal smoothness, i.e. if $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective for all small extensions $B \to A$. Exercise: $h_R \to h_S$ is smooth iff $S \to R$ is formally smooth.

Definition 2.18. $\alpha\colon h_R \to G$ is a **hull** if $h_R(k[\epsilon]/\epsilon^2) \to G(k[\epsilon]/\epsilon^2)$ is an isomorphism and α is smooth.

Remark. Exercise: any two hulls are isomorphic, but not canonically.

Theorem 2.19 (Baby Schlessinger)**.**

1. *A hull for a deformation functor D exists iff D admits a tangent-obstruction theory.*
2. *D is pro-representable iff the $(T_1 \otimes M)$-action on the set of lifts is simply-transitive, i.e. the tangent-obstruction sequence is exact on the left*

$$0 \to T_1 \otimes M \to D(B) \to D(A) \to T_2 \otimes M.$$

Remark. We will work in the following more general setting: Λ is a Noetherian complete local ring with residue field k, and $\mathcal{C}$ is the category of Artinian (as Λ-modules) Λ-algebras with residue field k.

Theorem 2.20 (Schlessinger)**.** *Let $F\colon \mathcal{C} \to \mathsf{Set}$ be a deformation functor. Given maps $R \to A$ and $S \to A$ in $\mathcal{C}$, consider the natural map*

$$(*)\colon F(R \times_A S) \to F(R) \times_{F(A)} F(S).$$

Then F has a hull (resp. is pro-representable) iff conditions H1-H3 (resp. H1-H4) are satisfied:

- *H1. (gluing) if $R \to A$ is small, $(*)$ is surjective;*
- *H2. (tangent spaces make sense) if $R = k[\epsilon]/\epsilon^2$ and $A = k$, then $(*)$ is a bijection;*
- *H3. (finite-dimensionality of tangent spaces) $\dim_k F(k[\epsilon]/\epsilon^2) < \infty$;*
- *H4. (separatedness) if $(R \to A) = (S \to A)$, then $(*)$ is a bijection.*

Proposition 2.21. *If $\Lambda = k$ and the hypotheses of baby Schlessinger hold, so do the hypotheses of Schlessinger.*

Proof. Suppose we have a tangent-obstruction theory; we want to check H1 holds. Let $0 \to M \to R \to A \to 0$ be a small extension. Apply F to the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & R \times_A S \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & R \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

to get

$$\begin{tikzcd} T_1 \otimes M \arrow[r] \arrow[d, equal] & F(R \times_A S) \arrow[r] \arrow[d] & F(S) \arrow[r] \arrow[d, "\beta \mapsto \gamma"] & T_2 \otimes M \arrow[d, equal] \\ T_1 \otimes M \arrow[r] & F(R) \arrow[r, "\alpha \mapsto \gamma"] & F(A) \arrow[r] & T_2 \otimes M. \end{tikzcd}$$

Given $\beta \mapsto \gamma$ and $\alpha \mapsto \gamma$, a small diagram chase gives a lift to $\eta \in F(R \times_A S)$.

Suppose the T_1-action is simply-transitive; we want to check H4 holds. Look at the diagram

$$\begin{tikzcd} 0 \arrow[r] & T_1 \otimes M \arrow[r] \arrow[d, equal] & F(R \times_A R) \arrow[r] \arrow[d] & F(R) \arrow[r] \arrow[d] & T_2 \otimes M \arrow[d, equal] \\ 0 \arrow[r] & T_1 \otimes M \arrow[r] & F(R) \arrow[r] & F(A) \arrow[r] & T_2 \otimes M. \end{tikzcd}$$

Now we run through the same argument as before, which will also show the lift is unique. ☐

Theorem 2.22 (Grothendieck)**.** *F is pro-representable iff F satisfies H2 and H3 and preserves all finite limits.*

Proof of Schlessinger. Exercise: hull implies H1-H3 and pro-representable implies H1-H4. Suppose F satisfes H1-H3; we want to make a hull R. We want $R/\mathfrak{m}_R = k =: R_1$. Let $t_F := F(k[\epsilon]/\epsilon^2)$; the condition H2 implies this is a k-vector space. Let $x_1, \ldots, x_r$ be a basis of t_F, with dual basis $x_1^*, \ldots, x_r^*$. Set $S := \Lambda[[T_1, \ldots, T_r]]$. Then $R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda R) = S/(\mathfrak{m}_S^2 + \mathfrak{m}_\lambda S) =: R_2$. More explicitly, $R_2 = k[\epsilon] \times_k \cdots \times_k k[\epsilon]$. Hence by H2,

$$F(R_2) = F(k[\epsilon]) \times \cdots \times F(k[\epsilon]) = t_F \times \cdots \times t_F = t_F \otimes t_F^\vee.$$

Let $\xi_2 := \mathrm{id}_{t_F \otimes t_F^\vee} = \sum x_i \otimes x_i^*$. In general, we want R_q and $\xi_q \in F(R_q)$ with $R_q = S/J_q$ such that:

1. $R_q/J_{q-1} = R_{q-1}$;
2. $\xi_q \mapsto \xi_{q-1}$ under the induced map $F(R_q) \to F(R_{q-1})$;
3. $\varprojlim(R_q, \xi_q)$ is a hull.

By Yoneda's lemma, $\xi := \varprojlim \xi$ will define the map $h_R \to F$. The idea is to let J_q be minimal such that $\mathfrak{m}_S J_{q-1} \subset J_q \subset J_{q-1}$, and ξ_{q-1} lifts to $F(R_q)$ under the map $F(R_q) \to F(R_{q-1})$. Minimality will eventually give smoothness of the hull. Observe there is a bijection

$$\{m_S J_{q-1} \subset J \subset J_{q-1}\} \xrightarrow{\sim} \{\text{vector subspaces of } J_{q-1}/\mathfrak{m}_S J_{q-1}\}.$$

We want to show that if J and K satisfy these properties, so does $J \cap K$; then we can just let J_q be the intersection of all J satisfying these properties. Wlog assume $J + K = J_{q-1}$. This implies $S/J \times_{S/J_{q-1}} S/K = S/(J \cap K)$. Then H1 implies the map

$$F(S/(J \cap K)) = F(S/J \times_{S/J_{q-1}} S/K) \to F(S/J) \times_{F(S/J_{q-1})} F(S/K)$$

is surjective. Hence given $\xi_q' \in F(S/J)$ and $\xi_q'' \in F(S/K)$, they lift to an element of $S/(J \cap K)$. So a minimal J_q exists. We want to check:

1. $t_R \to t_F$ is an isomorphism (this we checked already, by the construction of R);
2. $h_R \to F$ is smooth.

Let $0 \to M \to B \to A \to 0$ be a small extension, so that for (2) we must check $h_R(B) \to h_R(A) \times_{F(A)} F(B)$ is surjective. Wlog, by taking a filtration of M, assume $\dim_k M = 1$. Then $B \times_A B \cong B \times_k k[\epsilon]/\epsilon^2$, by $(x, y) \mapsto (x, x \bmod \mathfrak{m}_B + y - x)$ (exercise). Then H2 implies

$$F(B) \times t_F = F(B) \times_{F(k)} F(k[\epsilon]/\epsilon^2) \xrightarrow{\sim} F(B \times_k k[\epsilon]/\epsilon^2) = F(B \times_A B) \twoheadrightarrow F(B) \times_{F(A)} F(B).$$

(If we assume H4, then the last surjection is an isomorphism; this is the origin of H4.) The composition gives an action $(x, \delta) \mapsto (x, \delta \cdot x)$, and therefore $F(B) \to F(A)$ becomes a torsor under t_F.

Let $f \in h_R(A)$ and $\eta \in F(B)$ such that $\xi(f) = \overline{\eta}$. By the transitivity of the action we just defined, it suffices to find *any* lift of f to $h_R(B)$. Note that f factors through $S/J_q = R_q$ for some q. In the diagram

$$\begin{tikzcd} S \arrow[r, "w"] \arrow[d] & R_q \times_A B \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d] \\ R_{q+1} \arrow[r] & R_q \arrow[r, "f"] & A, \end{tikzcd}$$

we want a lift $\ell\colon R_{q+1} \to R_q \times_A B$. Exercise: either pr_1 splits, or w is surjective. (This uses that $B \to A$ has 1-dimensional kernel.) Wlog assume w is surjective. By H1, the map $F(R_q \times_A B) \to F(R_q) \times_{F(A)} F(B)$ is surjective. So we find a lift $\widetilde{\xi_q}$ of $\xi_q \in F(R_q)$. This implies $\ker w$ is an ideal in S such that:

1. by the smallness of the extension, $\mathfrak{m}_S J_q \subset \ker w \subset J_q$;

2. ξ_q lifts to $S/\ker w$.

By the minimality of J_{q+1}, we get $J_{q+1} \subset \ker w$. Hence w descends to J_{q+1} and we are done proving the statement for hulls.

Now assume H4; we want to show $h_R \to F$ is an isomorphism. By Yoneda, it suffices to show $h_R(B) \to F(B)$ is an isomorphism for all B. We prove this by induction on the length of B. Choose a small extension $0 \to M \to B \to A \to 0$ with $\dim_k M = 1$. We get

$$\begin{tikzcd} 0 \arrow[r] & t_R \otimes M \arrow[r] \arrow[d, equal] & h_R(B) \arrow[r] \arrow[d] & h_R(A) \arrow[d, equal] \\ 0 \arrow[r] & t_F \otimes M \arrow[r] & F(B) \arrow[r] & F(A), \end{tikzcd}$$

which shows $h_R(B) \cong F(B)$. □

Remark. If T_1, T_2 are a tangent-obstruction theory for F, then it is also a tangent-obstruction theory for any hull R. This is because by smoothness, there is no obstruction to lifting, and the tangent spaces are the same (exercise). Question: what is the dimension of R?

Theorem 2.23. *If R is a hull,* $\dim R \geq \dim T_1 - \dim T_2$.

Lemma 2.24. *Let $R \in \mathsf{Loc}/k$ with $S \coloneqq k[[x_1, \dots, x_r]] \twoheadrightarrow R$ with $t_R \cong t_S$ with kernel J. Set $T_1 \coloneqq (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$ and $T_2 \coloneqq (J/\mathfrak{m}_S J)^\vee$. Let T_1', T_2' be another tangent-obstruction theory for R. Then:*

1. *$T_1 \cong T_1'$;*

2. *there exists a functorial injection $T_2 \hookrightarrow T_2'$.*

Proof of theorem. We know $\dim R \geq \dim S - (\min \#\text{ generators of } J)$, and $\dim S = \dim T_1$. By Nakayama, the number of generators of J is at least $\dim T_2$. By the lemma, $\dim T_1 - \dim T_2 \geq \dim T_1' - \dim T_2'$. We need this because a priori we have some random tangent-obstruction theory, not the one that comes from R. □

Proof of lemma. We already showed that for any tangent-obstruction theory, $T_1 \cong h_R(k[\epsilon]/\epsilon^2) \cong T_1'$. By Artin–Rees, there exists $i > 0$ such that $\mathfrak{m}_S^i \cap J \subset \mathfrak{m}_S \cdot J$. Let

$$\begin{aligned} M &\coloneqq (J + \mathfrak{m}_S^i)/(\mathfrak{m}_S J + \mathfrak{m}_S^i) = J/\mathfrak{m}_S J \\ B &\coloneqq S/(\mathfrak{m}_S J + \mathfrak{m}_S^i), \end{aligned}$$

so that $A \coloneqq B/M = R/\mathfrak{m}_S^i R$ makes a small extension $0 \to M \to B \to A \to 0$. Consider lifting the quotient map $\pi \in h_R(R/\mathfrak{m}_S^i R)$ to $h_R(B)$. The obstruction here is

$$h_R(B) \to h_R(R/\mathfrak{m}_S^i R) \xrightarrow{\mathrm{ob}} T_2' \otimes M = T_2' \otimes T_2^\vee.$$

Then $\mathrm{ob}(\pi)$ can be viewed as an element in $\mathrm{Hom}(T_2, T_2')$, giving the desired map $T_2 \to T_2'$. Claim: $\mathrm{ob}(\pi)$ is injective. Assume not, so that $M^\vee \xrightarrow{\mathrm{ob}(\pi)} T_2'$ has a non-trivial kernel $(M/V)^\vee \subset M^\vee$. Now draw the diagram

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\pi"'] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & R/\mathfrak{m}_S^i R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M/V \arrow[r] & B/V \arrow[r] & R/\mathfrak{m}_S^i R \arrow[r] & 0,
\end{tikzcd}$$

where $\mathrm{ob}(\pi)$ is the obstruction to the existence of $\ell_1 : R \to B$. But we purposefully chose V so that the following diagram holds:

$$\begin{tikzcd}
h_R(B) \arrow[r] \arrow[d] & h_R(R/\mathfrak{m}_S^i R) \arrow[r, "\pi \mapsto \mathrm{ob}(\pi)"] \arrow[d, equal] & T_2 \otimes M \arrow[d, "\mathrm{ob}(\pi) \mapsto 0"] \\
h_R(B/V) \arrow[r] & h_R(R/\mathfrak{m}_S^i R) \arrow[r] & T_2 \otimes (M/V),
\end{tikzcd}$$

i.e. by functoriality, using the T_2' obstruction theory, there is no obstruction to a lift $\ell_2 : R \to B/V$ of $\pi : R \to R/\mathfrak{m}_S^i$. But using the T_2 obstruction theory, the obstruction to such a lift is the map $M \to M/V$. This is a quotient map, and is therefore non-zero. But whether the obstruction is zero must be independent of the obstruction theory, a contradiction. Hence $\mathrm{ob}(\pi)$ is injective. $\square$

2.4 Application: smoothness of Hilbert scheme

Definition 2.25. Let S be a scheme. Define the functor

$$S^{[n]}(T) \coloneqq \{Z \subset S \times T \text{ flat over } T : \mathrm{length}(\mathcal{O}_{Z_t}) = n \text{ for all geom. pts. } t \in T\}.$$

The **Hilbert scheme of n points** $S^{[n]}$ is the moduli space representing this functor.

Remark. Note that $S^{[n]}$ has an open subscheme which consists of sets of n distinct points on S.

Theorem 2.26. *Let S be a smooth (projective) surface over a field k. Then $S^{[n]}$ is smooth.*

Remark. Earlier we said there is a global deformation theory for $S^{[n]}$; we will go through this carefully. The obstruction space is non-zero, so it is not a priori obvious that this deformation problem is smooth. The proof idea is that the following two properties will imply $S^{[n]}$ smooth:

1. $S^{[n]}$ is connected (this is the global step);
2. $\dim T_{[Z]} = 2n$ for all $[Z]$.

Lemma 2.27. *Let S be a connected variety over k. Then $S^{[n]}$ is connected.*

Proof. Clearly $S^{[0]}$ and $S^{[1]}$ are connected; we will induct on n. Let $Z_n \subset S \times S^{[n]}$ be the universal subscheme, with ideal sheaf $\mathcal{I}_{Z_n}$. Recall that the Quot scheme lets us parametrize quotients of a coherent sheaf; let $\mathbb{P}(\mathcal{I}_{Z_n}) \coloneqq \mathrm{Quot}^1_{S \times S^{[n]}}(\mathcal{I}_{Z_n})$ parametrize 1-dimensional quotients of $\mathcal{I}_{Z_n}$. In other words, it parametrizes, universally, the operation of "adding one more point to the subscheme $[Z] \in S^{[n]}$." We will show $\mathbb{P}(\mathcal{I}_{Z_n})$ is connected, make a map $\mathbb{P}(\mathcal{I}_{Z_n}) \to S^{[n+1]}$, and show it is dominant. Hence $S^{[n+1]}$ will be connected as well.

1. Firstly, the fiber of $\mathbb{P}(\mathcal{I}_{Z_n}) \to S \times S^{[n]}$ at $t \in S \times S^{[n]}$ is $\mathbb{P}(\mathcal{I}_{Z_n,t})$. By induction, $S \times S^{[n]}$ is connected. Hence $\mathbb{P}(\mathcal{I}_{Z_n})$ is connected, as the domain of a surjective proper morphism whose fibers and codomain are connected.

2. We want a map $\mathbb{P}(\mathcal{I}_{Z_n}) \to S^{[n+1]}$. On $S \times \mathbb{P}(\mathcal{I}_{Z_n})$, there is a universal exact sequence

$$0 \to \mathcal{I} \to \pi^* \mathcal{I}_{Z_n} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is the universal quotient line bundle on $S \times \mathbb{P}(\mathcal{I}_{Z_n})$, and $\mathcal{I}$ is the kernel. It is an ideal sheaf, and therefore defines a subscheme $Z \subset S \times \mathbb{P}(\mathcal{I}_{Z_n})$. Hence we get an exact sequence

$$0 \to \mathcal{Q} \to \mathcal{O}_Z \to \pi^* \mathcal{O}_{Z_n} \to 0.$$

Since $\mathcal{Q}$ is a line bundle, this is an extension by length 1. Both $\mathcal{Q}$ and $\pi^*\mathcal{O}_{Z_n}$ are flat over $\mathbb{P}(\mathcal{I}_{Z_n})$, so $\mathcal{O}_Z$ is as well. Hence Z gives a map

$$\mathbb{P}(\mathcal{I}_{Z_n}) \to S^{[n+1]}, \quad [\varphi: \mathcal{I}_{Z_n} \to k(x)] \mapsto V(\ker(\varphi)) \subset S.$$

3. Now we check $\mathbb{P}(\mathcal{I}_Z) \to S^{[n+1]}$ is surjective. Take $[W] \in S^{[n+1]}$. It suffices to find a closed subscheme of W of length n. Pick $p \in \operatorname{supp} W$ and let $\mathfrak{m}$ be its maximal ideal in $\mathcal{O}_W$. Take any $f \in \operatorname{Ann} \mathfrak{m}$. Set $W' \coloneqq \operatorname{Spec}(\mathcal{O}_W/(f))$. Since f is annihilated by $\mathfrak{m}$, this has length n. Take $\mathcal{I}_{Z_n} \to \mathcal{I}_W/\mathfrak{m}\mathcal{I}_W \xrightarrow{\gamma} k$ such that $\gamma(f) = 0$ and $\gamma \neq 0$. This is a point in $\mathbb{P}(\mathcal{I}_{Z_n})$, and it maps to W. ☐

Lemma 2.28. *Let $[\mathcal{F} \to \mathcal{Q} \to 0] \in \operatorname{Quot}(\mathcal{F})$. Then its deformation functor $D_{[\mathcal{F}\to\mathcal{Q}\to 0]}$ has tangent-obstruction theory given by*

$$T_1 = \operatorname{Hom}(\mathcal{S}, \mathcal{Q}), \quad T_2 = \operatorname{Ext}^1(\mathcal{S}, \mathcal{Q})$$

where $\mathcal{S} \coloneqq \ker(\mathcal{F} \to \mathcal{Q})$.

Proof. Let $0 \to M \to B \to A \to 0$ be a small extension. Then we have the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & M \otimes_A \mathcal{S} \arrow[d] \arrow[dr, "\alpha"] & & \widetilde{\mathcal{S}} \arrow[d] & \\
0 \arrow[r] & M \otimes \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_B \arrow[r] \arrow[dr, "\beta"] & \mathcal{F}_A \arrow[r] \arrow[d] & 0 \\
 & M \otimes \mathcal{Q} \arrow[d] & & \widetilde{\mathcal{Q}} \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}.$$

Note that $\beta \circ \alpha = 0$. Denote $\widetilde{\mathcal{F}} \coloneqq \ker \beta / \operatorname{im} \alpha$, so that there is a short exact sequence

$$0 \to Q \otimes M \to \widetilde{\mathcal{F}} \to \widetilde{\mathcal{S}} \to 0.$$

Claim: deformations of $\mathcal{F}_A \to \widetilde{\mathcal{Q}}$ to B are in bijection with splittings of the above SES. In other words, the obstruction to extending from A to B lies in $\operatorname{Ext}^1(\widetilde{\mathcal{S}}, \mathcal{Q} \otimes M)$. This is because such a deformation is a sequence $0 \to \mathcal{S}' \to \mathcal{F}_B \to \mathcal{Q}' \to 0$ making the entire diagram commute. Given such a deformation, $\mathcal{S}'/\operatorname{im}(\alpha) \to \widetilde{\mathcal{S}} \subset \mathcal{F}_A$. This map is actually an isomorphism. But $\mathcal{S}'/\operatorname{im}(\alpha)$ evidently maps into $\widetilde{\mathcal{F}}$, giving a splitting of the SES. Conversely, given a splitting $\xi: \widetilde{\mathcal{S}} \to \widetilde{\mathcal{F}}$, set

$$\mathcal{S}' \coloneqq \{x \in \mathcal{F} \otimes B : (s \bmod \operatorname{im} \alpha) \in \operatorname{im} \xi\}.$$

This finishes the proof modulo the claim that $\operatorname{Ext}^1(\widetilde{S}, Q \otimes M) = \operatorname{Ext}^1(\mathcal{S}, \mathcal{Q})$. ☐

Corollary 2.29. *Setting $\mathcal{Q} = \mathcal{O}$, the deformation functor $D_{Z,X}$ for the Hilbert scheme $S^{[n]}$ has tangent-obstruction theory*

$$T_1 = \operatorname{Hom}(\mathcal{I}_Z, \mathcal{O}_Z), \quad T_2 = \operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z).$$

Lemma 2.30. *Let A be a 2-dimensional regular local ring, e.g. $A = k[[x,y]]$. Let $\mathcal{I} \subset A$ be an ideal of co-length n. Then* $\operatorname{length} \operatorname{Hom}(I, A/I) \leq 2n$.

Proof of theorem. We have already shown $S^{[n]}$ is connected, so it suffices to show $\dim T_{[Z]} S^{[n]} = 2n$. The open subset of $S^{[n]}$ corresponding to n distinct points clearly already has tangent space of dimension $2n$. By the lemma,

$$T_{[Z]} S^{[n]} = \operatorname{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) = \prod_{p \in \operatorname{supp} Z} \operatorname{Hom}(\mathcal{I}_Z|_p, \mathcal{O}_Z|_p) \leq 2n$$

for any $[Z] \in S^{[n]}$. We are done because then there can be no other bad components in $S^{[n]}$. □

Proof of lemma. Let $0 \to R \to A^{r+1} \to I \to 0$. Claim: R is free of rank r. It suffices to check $\operatorname{Tor}_1(R, k) = 0$ (by the local criterion of flatness). By the SES and the Koszul complex of k,

$$\operatorname{Tor}_1(R, k) = \operatorname{Tor}_2(I, k) = \{x \in I : x\mathfrak{m} = 0\} = 0.$$

Hence R is free. It has rank r because $\operatorname{rank} I = 1$. So we can rewrite the SES as $0 \to A^r \to A^{r+1} \to I \to 0$. We get a long exact sequence

$$0 \to \operatorname{Hom}(I, A/I) \to (A/I)^{r+1} \to (A/I)^r \to \operatorname{Ext}^1(I, A/I) \to 0.$$

We know the length of $(A/I)^r$ is rn. So it suffices to show $\operatorname{length} \operatorname{Ext}^1(I, A/I) \leq n$. Take the SES $0 \to I \to A \to A/I \to 0$, giving an isomorphism

$$\operatorname{Ext}^1(I, A/I) \xrightarrow{\sim} \operatorname{Ext}^2(A/I, A/I).$$

Applying the sequence in the other factor, we also get a surjection

$$\operatorname{Ext}^2(A/I, A) \twoheadrightarrow \operatorname{Ext}^2(A/I, A/I) \to \operatorname{Ext}^3(A/I, I) = 0.$$

(Here Ext^3 vanishes by finite global dimension of regular local rings, or by explicitly taking a three-term resolution of A/I.) So now it is enough to show $\operatorname{length} \operatorname{Ext}^2(A/I, A) \leq n$. Let $E(k)$ denote the injective envelope of k. Then by local duality (cf. Matlis duality),

$$\operatorname{Ext}^2(A/I, A) = \operatorname{Hom}(H_{\mathrm{loc}}(A/I), E(k)) = \operatorname{Hom}(A/I, E(k)).$$

Local duality preserves length, and $\operatorname{Hom}(-, E(k))$ preserves length. Since A/I has length n by hypothesis, $\operatorname{Hom}(A/I, E(k))$ has length n as well. □

Second proof of lemma. We will show $\operatorname{Hom}(I, A/I) \leq 2n$ using a global proof, via Serre duality instead of local duality. There is an exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{I}, \mathcal{O}_Z) \to \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = 0.$$

The first map is an isomorphism, because a map $\mathcal{O}_X \to \mathcal{O}_Z$ factors through $\mathcal{O}_Z$ anyway. Hence the map $\operatorname{Hom}(\mathcal{I}, \mathcal{O}_Z) \hookrightarrow \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$ is an injection, and it suffices to show $\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \leq 2n$.

We will first show $\dim \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \leq n$. There is an exact sequence

$$\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X) \to \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^3(\mathcal{O}_Z, \mathcal{I}) = 0$$

where Ext^3 vanishes by the local-to-global Ext spectral sequence. Hence it is enough to show the bound $\dim \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X) \leq n$. But by Serre duality,

$$\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_Z \otimes K_X)^\vee = H^0(\mathcal{O}_Z \otimes K_X)^\vee,$$

which has length n. Now we have

$$\dim \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) = \dim H^0(\mathcal{O}_Z, \mathcal{O}_Z) + \dim H^2(\mathcal{O}_Z, \mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z) \le 2n - \chi(\mathcal{O}_Z, \mathcal{O}_Z).$$

So to finish the proof, it suffices to show $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 0$. Take a locally free resolution $0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{O}_Z \to 0$. Then

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum (-1)^i \chi(\mathcal{E}_i, \mathcal{O}_Z) = \sum (-1)^i \operatorname{rank}(\mathcal{E}_i) n = 0$$

and we are done. □

Remark. This statement is false for 3-folds X. It is true that $X^{[1]}$ and $X^{[2]} = \mathrm{Bl}_\Delta((X \times X)/(\mathbb{Z}/2))$ are smooth, but in general $X^{[n]}$ is not.

Theorem 2.31. $\operatorname{Hilb}^N X$ *is not smooth for* $N \gg 0$ *when* $\dim X > 2$.

Proof. There is an open in $\operatorname{Hilb}^N X$ with dimension $3N$. Our proof of connectedness worked in all dimensions, so being singular is the same as having a point where the tangent space has dimension $> 3N$. In fact we can find a point where the *dimension* is $> 3N$. Take $\mathfrak{m} \subset R \coloneqq k[[x_1, x_2, x_3]]$ and consider ideal sheaves supported only at $\mathfrak{m}$. For example, look at $(R/\mathfrak{m}^s)/V$ where $V \subset \mathfrak{m}^{s+1}/\mathfrak{m}^s$. This is a big Grassmannian $\mathrm{Gr}(V, \mathfrak{m}^{s+1}/\mathfrak{m}^s)$ whose dimension is $> 3\dim((R/\mathfrak{m}^s)/V)$. So we have found a component in $\operatorname{Hilb}^N X$ of dimension $> 3N$. □

3 Tian–Todorov theorem

Let k be a field of characteristic 0.

Definition 3.1. A **weak Calabi–Yau variety** over k is a smooth projective variety X over k such that $K_X \cong \mathcal{O}_X$. A **Calabi–Yau variety** is a weak Calabi–Yau variety X with $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Theorem 3.2 (Tian–Todorov)**.** *Let X be a weak Calabi–Yau variety with $H^0(X, T_X) = 0$. Then the functor* $\mathrm{Def}_X \colon \mathsf{Art}/k \to \mathsf{Set}$ *is pro-representable by a power series ring in n variables where* $n \coloneqq \dim H^1(X, T_X)$.

Remark. If X is Calabi–Yau and $\dim X \ge 2$, then $H^0(X, T_X) = 0$ automatically. This is because by Serre duality and Hodge symmetry (which requires $\operatorname{char} k = 0$),

$$h^0(X, T_X) = h^{\dim X}(X, \Omega^1) = h^1(X, \Omega^{\dim X}) = h^1(X, \mathcal{O}_X) = 0.$$

Example 3.3. Here are some weak Calabi–Yaus:

1. abelian varieties;
2. $\operatorname{Hilb}^n(\mathrm{K3})$ for $n > 1$, where $H^2(X, \mathcal{O}_X) \ne 0$;
3. $\operatorname{Hilb}^n(A)$ where A is an abelian surface,

Here are some Calabi–Yaus:

1. a K3 surface, e.g. quartic in $\mathbb{P}^3$;
2. degree $n+1$ (smooth) hypersurface in $\mathbb{P}^n$;
3. anti-canonical sections of Fano varieties.

Tian–Todorov will tell us that locally, the moduli spaces of such objects are beautiful.

Proof sketch for Tian–Todorov. Here are the steps we will follow for this proof.

1. Prove that $T_1 = H^1(X, T_X)$ and $T_2 = H^2(X, T_X)$. This will be true for any smooth projective variety.

1'. Prove that if in addition $H^0(X, T_X) = 0$, then Def_X is pro-representable.

2. Prove the T_1-lifting theorem, which gives (in characteristic 0) a criterion for when a deformation functor is smooth basically in terms of just T_1.
3. Check the T_1-lifting theorem for the functor Def_X. □

3.1 Deformations of a smooth variety

Let X be a smooth variety over a field k. Recall that the deformation functor associated to X is

$$\mathrm{Def}_X(A) \coloneqq \{\mathcal{X} \to \operatorname{Spec} A \text{ flat, with } \varphi\colon \mathcal{X}_k \xrightarrow{\sim} X\}/\sim.$$

The following theorem implies that Def_X is pro-representable for X Calabi–Yau, where $H^0(X, T_X) = 0$ automatically.

Theorem 3.4. *1. If $H^1(X, T_X)$ and $H^2(X, T_X)$ are finite-dimensional, then Def_X has a hull.*

2. If in addition $H^0(X, T_X) = 0$, then Def_X is pro-representable.

Remark. The point will be that $H^i(X, T_X)$ for $i = 1, 2$ form a tangent-obstruction theory for Def_X, and we will basically obtain the result from Schlessinger. However, we will actually prove the following more general result.

Theorem 3.5. *Let $\pi\colon X_0 \to S_0$ be smooth and separated, with $S_0 \xrightarrow{g} S$ an extension with square-zero ideal $\mathcal{J}$.*

1. *There exists a canonical class $\mathrm{ob}(\pi) \in \mathrm{Ext}^2_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J})$ such that $\mathrm{ob}(\pi) = 0$ iff there exists a deformation X of X_0 to S.*
2. *The set of such deformations (up to isomorphism) is a torsor for $\mathrm{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J})$.*
3. *The set of automorphisms of a given deformation is $\mathrm{Hom}_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J})$.*

Proof. First consider the case where $S_0 = \operatorname{Spec} R_0$, $S_1 = \operatorname{Spec} R$ and $X_0 = \operatorname{Spec} R_0[x_1, \dots, x_n]/(f_1, \dots, f_r)$ are all affine, and X_0 is a global complete intersection over S_0.

1. Everything in sight is affine, so $\mathrm{Ext}^i_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J}) = 0$ for $i = 1, 2$. Take $\mathrm{ob}(\pi) \coloneqq 0$.
2. The content here is that there exists a unique deformation of X_0 to X. Take the equations $f_1, \dots, f_r$ and lift them to $\widetilde{f}_1, \dots, \widetilde{f}_r \in R[x_1, \dots, x_n]$, and set $X \coloneqq \operatorname{Spec} R[x_1, \dots, x_n]/(\widetilde{f}_1, \dots, \widetilde{f}_r)$. This gives existence. For uniqueness, consider two lifts X, X' of $X_0 \to S_0$:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[r, dashed] \arrow[dr] & X' \arrow[d] \\ S_0 \arrow[r, hook] & S. & \end{tikzcd}$$

   There is a map between these lifts from (formal) smoothness of $X_0 \to S_0$. This map is an isomorphism because it is an isomorphism on the central fiber.

3. Consider an automorphism $\varphi\colon X \to X$ as an extension of X_0. Then $\varphi - \mathrm{id}\colon \mathcal{O}_X \to \mathcal{O}_X$ kills the ideal sheaf of X_0, and therefore descends to a map $\varphi - \mathrm{id}\colon \mathcal{O}_{X_0} \to \pi^*\mathcal{J}$. Check that this is a derivation.

In general, we cover X_0 by open affines that are global complete intersections. This is possible by the assumption that $X_0 \to S_0$ is smooth, and hence a local complete intersection. Choose $X_0 = \bigcup_i U_i$ where each U_i is a complete intersection. Then by the local case, there exists unique lifts U_i' of U_i over S. For each i, j,

$$\varphi_{ij}\colon U_i'|_{U_i \cap U_j} \xrightarrow{\sim} U_j'|_{U_i \cap U_j}$$

is an isomorphism by (2) in the affine case. (Note that by separatedness, intersections are affine.) However it is not necessarily the case that these glue together; we require the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. So set

$$\mathrm{ob}(\pi) \coloneqq \varphi_{jk} \circ \varphi_{ij} \circ \varphi_{ik}^{-1} - \mathrm{id}\colon U_k'|_{ijk} \to U_k'|_{ijk}.$$

Check that this is indeed a Čech cocycle and independent of choices (exercise using (3) from the affine case). Since each $\varphi - \mathrm{id}$ is a derivation $\mathcal{O}_{X_0} \to \pi^*\mathcal{J}$, it follows that we get a class in $\mathrm{Ext}^2_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J})$. Clearly $\mathrm{ob}(\pi) = 0$ iff the φ_{ij} glue, iff there exists a global lift X of X_0.

Now suppose we have two deformations X', X'' of X_0. We need to construct a unique $\alpha(X', X'') \in \mathrm{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J})$. Consider the open cover U_i as before. By the affine case, there exists a unique isomorphism $\psi'_i : U'_i \to U''_i$, and these form the diagram

$$\begin{tikzcd} U'_i|_{ij} \arrow[r, "\varphi'_{ij}"] \arrow[d, "\psi'_i"'] & U'_j|_{ij} \arrow[d, "\psi'_j"'] \\ U''_i|_{ij} \arrow[r, "\varphi''_{ij}"] & U''_j|_{ij}. \end{tikzcd}$$

However, the diagram commutes iff $X' \cong X''$ as deformations of X_0. Hence we define the class

$$\alpha(X', X'') \coloneqq \varphi'_{ij} - \varphi''_{ij} \in \mathrm{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, \pi^*\mathcal{J}).$$

Finally, the automorphism statement is the same as for the affine case. □

Corollary 3.6. *If X_0 and S_0 are both affine with $X_0 \to S_0$ smooth, then there exists a unique deformation to $X \to S$.*

Corollary 3.7. *If $X_0 \to S_0$ has cohomological dimension 1, i.e. R^2 and above vanish, then there exists a deformation (because the obstruction space vanishes).*

Proof of Theorem 3.4. Set $T_1 \coloneqq H^1(X, T_X)$ and $T_2 \coloneqq H^2(X, T_X)$. To check this is a tangent-obstruction theory, for every small extension $0 \to M \to B \to A \to 0$ we need an exact sequence

$$T_1 \otimes M \to \mathrm{Def}_X(B) \to \mathrm{Def}_X(A) \to T_2 \otimes M.$$

Take $S_0 \coloneqq \mathrm{Spec}\, A$ and $S \coloneqq \mathrm{Spec}\, B$ and $\mathcal{J} \coloneqq \widetilde{M}$. Then the theorem applies: given $[\pi : \mathcal{X} \to S_0] \in \mathrm{Def}_X(A)$, we get $\mathrm{ob}(\pi)$ in $\mathrm{Ext}^2(\Omega^1_{\mathcal{X}/A}, \pi^*\mathcal{J})$. But

$$\mathrm{Ext}^2(\Omega^1_{\mathcal{X}/A}, \pi^*\mathcal{J}) \cong H^2(\mathcal{X}, T_{\mathcal{X}/A} \otimes \pi^*\widetilde{M}) = H^2(\mathcal{X}, T_{\mathcal{X}/A}) \otimes M = H^2(X, T_X) \otimes M$$

where the last equality uses that the extension is small in order to return to the central fiber X in $\widetilde{X}$. This gives the part $\mathrm{Def}_X(B) \to \mathrm{Def}_X(A) \to T_2 \otimes M$.

Now we study the fibers of $\mathrm{Def}_X(B) \to \mathrm{Def}_X(A)$. There is a surjection from the space of all deformations of $\mathcal{X}$ over B (which only must preserve the central fiber X) to the ones which preserve a given deformation $\mathcal{X} \in \mathrm{Def}_X(A)$. Since $T_1 \otimes M$ acts transitively on the former, it also acts transitively on the latter.

By baby Schlessinger, it follows that Def_X has a hull. For pro-representability, we can prove the following more general statement. It finishes the proof because if $H^0(X, T_X) = 0$, then by the theorem, $\mathrm{Aut}(\mathcal{X}_A) = 0$ and we vacuously satisfy the condition below. □

Theorem 3.8. *Let $\mathcal{X}_A \in \mathrm{Def}_X(A)$. The following are equivalent:*

1. *Def_X is pro-representable;*
2. *for every small extension $B \to A$ and every $\mathcal{X} \in \mathrm{Def}_X(B)$, any automorphism of $\mathcal{X}_A$ extends to an automorphism of $\mathcal{X}$.*

Proof. There is a long exact sequence of sets

$$0 \to H^0(X, T_X) \otimes M \to \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\mathcal{X}_A) \to H^1(X, T_X) \otimes M \to \mathrm{Def}_X(B) \to \mathrm{Def}_X(A).$$

We check exactness at $H^1(X, T_X) \otimes M$. Suppose $\mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Def}_X(B)$ with an isomorphism $\varphi : \mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ restricting to the identity on X. Then $\varphi|_{\mathcal{X}_A}$ is an automorphism of $\mathcal{X}_A$. By hypothesis, it extends to $\widetilde{\varphi} : \mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_1$. The map $\varphi \circ \widetilde{\varphi}^{-1}$ therefore exhibits $\mathcal{X}_1$ and $\mathcal{X}_2$ as the same deformation of $\mathcal{X}_A$ over B. Exercise: check the above carefully. □

Corollary 3.9. *Let X be a proper smooth curve of genus g. Then:*

1. Def_X *is pro-representable;*
2. Def_X *is smooth;*
3. $\dim(\mathrm{Def}_X) = 3g - 3$ *for* $g \geq 2$.

Proof. If $g \geq 2$, then $H^0(X, T_X) = 0$ and we are done by the theorem. If $g = 0$, then there are no deformations of genus 0 curves, so $\mathrm{Def}_X = \mathrm{pt}$. The $g = 1$ case requires some more work.

Smoothness follows from cohomological dimension ≤ 1, which implies obstruction spaces vanish and therefore formal smoothness. By smoothness, $\dim(\mathrm{Def}_X) = \dim(T_1)$, which is $3g - 3$ by Riemann–Roch. $\square$

Example 3.10 (Non-pro-representable deformation problem)**.** Consider

$$X \coloneqq \mathrm{Bl}_Z(\mathbb{P}^2), \quad Z \coloneqq \{100 \text{ pts of } \mathbb{P}^2 \text{ lying on a line } \ell\}.$$

Exercise: the space of infinitesimal automorphisms of X is $\{g \in \mathfrak{pgl}_3 : g \text{ preserves } \ell\}$. Hence Def_X is not pro-representable: if we deform points off the line, the automorphism will not extend.

3.2 T_1-lifting theorem

Definition 3.11. Let $A \in \mathsf{Art}/k$ and $M \in \mathsf{Mod}(A)$. Define $A \oplus M$ with underlying abelian group $A \oplus M$, with multiplication

$$(a, m) \cdot (a', m') \coloneqq (aa', am' + a'm).$$

Warning: $A \oplus M \to A$ is *not* a small extension, since M is only an A-module and not a k-module.

Theorem 3.12 (T_1-lifting theorem)**.** *Let $R \in \mathsf{CLoc}/k$. Then $R \cong k[[x_1, \ldots, x_n]]$ with $n \coloneqq \dim \mathfrak{m}_R/\mathfrak{m}_R^2$ iff:*

t1. for all $A \in \mathsf{Art}/k$ and all surjections $M' \twoheadrightarrow M$ in $\mathsf{Mod}(A)$, the induced map

$$\mathrm{Hom}_{k\text{-}alg}(R, A \oplus M') \to \mathrm{Hom}_{k\text{-}alg}(R, A \oplus M)$$

is surjective.

Remark. The T_1-lifting theorem is false in characteristic p. For example, set $R = k[x]/(x^p)$. The key fact making R a counterexample is that $\Omega_{R/k}$ is free of rank 1.

Proof 1. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{k\text{-alg}}(R, A \oplus M') \arrow[rr] \arrow[dr, "\pi_{M'}"'] & & \mathrm{Hom}_{k\text{-alg}}(R, A \oplus M) \arrow[dl, "\pi_M"] \\
& \mathrm{Hom}(R, A). &
\end{tikzcd}$$

Then the t1 condition is equivalent to the condition:

t2. For A, M, M' as in the theorem and a map $g : R \to A$, the natural map $\pi_{M'}^{-1}(g) \to \pi_M^{-1}(g)$ is surjective.

Observe that $\pi_M^{-1}(g)$ is a torsor for $\mathrm{Der}_k(R, M) = \mathrm{Hom}_k(\widehat{\Omega}_R^1, M)$, and the same is true for $\pi_{M'}^{-1}(g)$. These torsors are actually the trivial torsor, because $A \oplus M \to A$ splits. Hence t2 is equivalent to the condition: $\mathrm{Hom}(\widehat{\Omega}_R^1, M') \to \mathrm{Hom}(\widehat{\Omega}_R^1, M)$ is surjective for any surjection $M' \to M$ of finite-length R-modules. This is equivalent to $\widehat{\Omega}_R^1$ being free.

Choose a surjection $P \coloneqq k[[x_1, \ldots, x_n]] \twoheadrightarrow R$, with $n \coloneqq \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ and kernel I. The conormal exact sequence is

$$I/I^2 \xrightarrow{d} \widehat{\Omega}_P^1 \otimes_P R \to \widehat{\Omega}_R^1 \to 0.$$

But the surjection is of free modules of the same rank, so $df = 0$ for $f \in I/I^2$. In characteristic 0, this implies $f = 0$, and hence $I = 0$. $\square$

Proof 2. Take $P \coloneqq k[[x_1,\dots,x_n]]$ with $n = \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ and a surjection $\pi\colon P \twoheadrightarrow R$, and let $I \coloneqq \ker\pi$. Note that $I \subset \mathfrak{m}_P^2$, because π is an isomorphism on tangent spaces. Assume $I \subset \mathfrak{m}_P^{r-1}$; we will show $I \subset \mathfrak{m}_P^r$. It suffices to show that the dashed arrow exists:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\ A_{r+1} \coloneqq P/\mathfrak{m}_P^r. & \end{tikzcd}$$

Let $M_r \coloneqq \bigoplus_{i=1}^n A_{r-1}\epsilon_i = \bigoplus_{i=1}^n P/\mathfrak{m}_P^{r-2}$, which is an A_r-module. There is a map $M_{r+1} \to M_r$ given by quotienting on each coordinate. Set

$$P \to A_r \oplus M_r = P/\mathfrak{m}_P^{r-1} \oplus \bigoplus_{i=1}^n P/\mathfrak{m}_P^{r-2}\epsilon_i, \quad x_i \mapsto x_i + \epsilon_i.$$

This sends a polynomial $f \in P$ to $f + \sum_i (\partial f/\partial x_i)\epsilon_i$. It descends to a map $R \to A_r \oplus M_r$ because $I \subset \mathfrak{m}_R^{r-1}$, so for $f \in I$, we have $\partial f/\partial x_i \in \mathfrak{m}_P^{r-2}$. We want to lift to $A_r \oplus M_{r+1}$. The existence of the lift implies $\partial f/\partial x_i \in \mathfrak{m}_P^{r-1}$ for $f \in I$. By the same argument as before (in characteristic 0), we get $I \subset \mathfrak{m}_P^r$. ☐

3.3 Proof of Tian–Todorov

Theorem 3.13 (Deligne–Illusie)**.** *Let S be a scheme of characteristic 0, and $X \to S$ be smooth and proper. Then:*

1. *$R^q f_* \Omega^p_{X/S}$ is locally free and its formation commutes with base change;*
2. *the spectral sequence $E_2^{p,q} = H^q(X, \Omega^p_{X/S}) \Rightarrow H^{p+q}_{dR}(X/S)$ degenerates at E_2.*

Remark. We need this for Artin schemes. If S were a reduced scheme (think smooth complex manifold), this follows directly from Hodge theory as follows. In this setting, $X \to S$ is a submersion, and topologically it is locally trivial. So $H^n(X_s, \mathbb{C})$ is locally constant as a function of s. Hodge theory says $H^n(X_s,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X_s, \Omega^p_{X_s})$. But these are coherent cohomology groups, and their ranks can only jump up. Hence $H^q(X_s, \Omega^p_{X_s})$ is locally constant as a function of s. By proper base change (in the reduced case) and descending induction, we are done.

Lemma 3.14 (Deformations of Calabi–Yaus are Calabi–Yau)**.** *If $\mathcal{X} \to \operatorname{Spec} A$ is a smooth proper morphism with $A \in \mathsf{Art}/k$ with $\operatorname{char} k = 0$ and X_k weak CY, then $\omega_{\mathcal{X}/A}$ is trivial.*

Proof. Consider $H^0(X, \omega_{\mathcal{X}/A})$. We know it is locally free and its formation commutes with base change. Hence it surjects onto $H^0(\mathcal{X}, \omega_{\mathcal{X}/A}) \otimes k = H^0(X_k, \omega_{X_k})$, which has a nowhere-vanishing section. It lifts back to $\omega_{\mathcal{X}/A}$, which is therefore trivial. ☐

Proof of Tian–Todorov. We showed pro-representability. Now we want $\operatorname{Def}_X(A \oplus M') \to \operatorname{Def}_X(A \oplus M)$ to be surjective, to apply the T1 lifting theorem. We'll actually use the t2 condition: pick $\mathcal{X} \in \operatorname{Def}_X(A)$, so we want $\pi^{-1}_{M'}(\mathcal{X}) \to \pi^{-1}_M(\mathcal{X})$ is surjective. Both are torsors, so equivalently we want to show

$$H^1(\mathcal{X}, T_{\mathcal{X}/A} \otimes f^*M') \twoheadrightarrow H^1(\mathcal{X}, T_{\mathcal{X}/A} \otimes f^*M)$$

is surjective. By the projection formula,

$$R\Gamma(T_{\mathcal{X}/A} \otimes^L Lf^*M') = R\Gamma(T_{\mathcal{X}/A} \otimes f^*M') \cong R\Gamma(T_{\mathcal{X}/A}) \otimes^L M.$$

We can't directly erase the L because M is not locally free. In order for this to work for R^1, it turns out we need the following. Exercise:

$$R^1\Gamma(T_{\mathcal{X}/A} \otimes f^*M) \cong R^1\Gamma(T_{\mathcal{X}/A}) \otimes M$$

if $R^1\Gamma(T_{\mathcal{X}/A})$ is locally free for $i > 1$.

$$R^i\Gamma(T_{\mathcal{X}/A}) = H^i(\mathcal{X}, T_{\mathcal{X}/A}) = H^{\dim \mathcal{X}-i}(\mathcal{X}, \Omega^1_{\mathcal{X}/A} \otimes \omega_{\mathcal{X}/A})^\vee.$$

By the lemma, $\omega_{\mathcal{X}/A}$ is trivial. By Deligne–Illusie, $H^{\dim \mathcal{X}-i}(\mathcal{X}, \Omega^1_{\mathcal{X}/A})$ is locally free. So we have reduced the map to

$$H^1(\mathcal{X}, T_{\mathcal{X}/A}) \otimes M' \to H^1(\mathcal{X}, T_{\mathcal{X}/A}) \otimes M,$$

which is clearly surjective. $\square$

Proof of Deligne–Illusie. Exercise: reduce to the case $S = \operatorname{Spec} A$ for an Artinian local ring $\mathbb{C}$-algebra A. Claim: $\Omega^{\bullet \mathrm{an}}_{X/S}$ is a resolution for $\underline{A}$ on X^{an}. There is a map $\underline{A} \to \Omega^{\bullet \mathrm{an}}_{X/S}$ given by pullback, which will give our quasi-isomorphism. It suffices to check

$$\mathrm{Gr}^\bullet_{\mathfrak{m}} \underline{A} \to \mathrm{Gr}_{\mathfrak{m}} \Omega^{\bullet \mathrm{an}}_{X/S} = \Omega^{\bullet \mathrm{an}}_{X^{\mathrm{red}}} \otimes_{\mathbb{C}} \mathrm{Gr}_{\mathfrak{m}} A$$

is a quasi-isomorphism, and this is obvious since $\Omega^{\bullet \mathrm{an}}_{X^{\mathrm{red}}} \cong \underline{\mathbb{C}}$. By GAGA, it suffices to prove the theorem for $\Omega^{\bullet \mathrm{an}}_{X/S}$. In addition, $R^q f_* \Omega^{\bullet \mathrm{an}}_{X/S} = R^q f_* \Omega_{X/S}$, by constructing an isomorphism of spectral sequences.

Since we have a resolution $\Omega^{\bullet \mathrm{an}}_{X/S} \to \underline{A}$, there are equalities

$$H^i(X^{\mathrm{an}}, \Omega^{\bullet \mathrm{an}}_{X/S}) = H^i(X^{\mathrm{an}}, \underline{A}) = H^i(X_{\mathrm{red}}, \underline{\mathbb{C}}) \otimes A = H^i(X_{\mathrm{red}}, \Omega^\bullet_{X^{\mathrm{red}}}) \otimes A.$$

Hence, taking lengths and noting these are just $\mathbb{C}$-vector spaces,

$$\operatorname{length} H^i(X^{\mathrm{an}}, \Omega^{\bullet \mathrm{an}}_{X/S}) = \operatorname{length}(A) \operatorname{length}(H^i(X_{\mathrm{red}}, \Omega^\bullet_{X/S})).$$

This is the analogue of the constancy of Betti numbers in the non-reduced setting. A general fact:

$$\operatorname{length} H^q(X, \Omega^p_{X/S}) \le \operatorname{length}(A) \operatorname{length}(H^q(X^{\mathrm{red}}, \Omega^p_{X^{\mathrm{red}}})),$$

with equality iff $H^q(X, \Omega^p_{X/S})$ is locally free. Also,

$$\sum_{p+q=n} \operatorname{length} H^q(X, \Omega^p_{X/S}) \ge \operatorname{length} H^n(X, \Omega^p_{X/S})$$

with equality iff the spectral sequence $H^q(X, \Omega^p_{X/S}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X)$ degenerates. These two inequalities give

$$\sum_{p+q=n} \operatorname{length}(A) \operatorname{length} H^q(X_{\mathrm{red}}, \Omega^p_{X^{\mathrm{red}}}) \ge \operatorname{length} H^n(X, \Omega_{X/S}) = \operatorname{length}(A) \operatorname{length} H^n_{\mathrm{dR}}(X_{\mathrm{red}}),$$

with equality iff the spectral sequence degenerates and $H^q(X, \Omega^p)$ is locally free. But this is in fact an equality, because dividing by $\operatorname{length}(A)$ we get the usual statement of Hodge theory. $\square$

4 Generic vanishing

Let X be a compact Kähler manifold (but think projective variety). Let $\operatorname{Pic}^0 X = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ be the identity component of $\operatorname{Pic} X$. In general, this is a complex torus. Our goal in generic vanishing is to study $H^i(X, \mathcal{L})$ for $\mathcal{L} \in \operatorname{Pic}^0 X$. In contrast, if $\mathcal{L}$ is ample, then Kodaira vanishing says $H^i(X, K_X \otimes \mathcal{L}) = 0$ for $i > 0$; equivalently, $H^i(X, \mathcal{L}^\vee) = 0$ for $i < \dim X$. More generally, Nakano's generalization says for $\mathcal{L}$ ample, $H^q(X, \Omega^p \otimes \mathcal{L}) = 0$ for $p + q \ge \dim X$. We want analogues for $\mathcal{L} \in \operatorname{Pic}^0 X$.

Example 4.1. We show that it is too much to hope for global vanishing.

1. Take $H^0(X, \mathcal{L})$ for $\mathcal{L} \in \operatorname{Pic}^0 X$. Then this is non-zero iff $\mathcal{L} \cong \mathcal{O}_X$.

2. Suppose X is a genus-2 curve and consider $H^1(X,\mathcal{L})$ for $\mathcal{L} \in \operatorname{Pic}^0 X$. Either $\mathcal{L} = \mathcal{O}_X$ and this is 2-dimensional, or $\mathcal{L}$ is something else and $\chi(\mathcal{L}) = -1$, so that $H^1(X,\mathcal{L})$ is 1-dimensional.

3. Let $f\colon Y \to X$ be proper with geometrically connected fibers. So we get $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 Y$. Take $\mathcal{L} \in \operatorname{Pic}^0 Y$ and consider $H^i(Y,\mathcal{L})$. For $\mathcal{L} \in \operatorname{Pic}^0 X$, there is an injection $H^i(X,\mathcal{L}) \to H^i(Y, f^*\mathcal{L})$. So $H^i(Y,\mathcal{L}) \neq 0$ if $\mathcal{L}$ comes from X.

The picture is that the torus $\operatorname{Pic}^0 Y$ has a sub-torus $\operatorname{Pic}^0 X$ where cohomology jumps, and we hope that outside of $\operatorname{Pic}^0 X$ there is vanishing.

Definition 4.2. Define the subset $S^i(X) \subset \operatorname{Pic}^0 X$ by

$$S^i(X) \coloneqq \{\mathcal{L} : H^i(X,\mathcal{L}) \neq 0\}.$$

(We will give it the structure of an analytic subvariety soon.) Fix $x_0 \in X$. Let

$$\operatorname{Alb} X \coloneqq H^0(X,\Omega^1_X)^\vee / H_1(X,\mathbb{Z})$$

be the **Albanese**. Let $\alpha\colon X \to \operatorname{Alb} X$ be the **Albanese map**, given by $x \mapsto (\alpha \mapsto \int_{x_0}^x \alpha)$. (This is universal for maps from X to a complex torus with $x_0 \mapsto 0$.)

Theorem 4.3 (Green–Lazarsfeld)**.** $\operatorname{codim}(S^i(X), \operatorname{Pic}^0 X) \geq \dim \alpha(X) - i$.

Corollary 4.4. *For $\mathcal{L} \in \operatorname{Pic}^0 X$ generic, $H^i(X,\mathcal{L}) = 0$ for $i < \dim \alpha(X)$.*

Corollary 4.5. *If $\dim \alpha(X) = \dim X$, then $(-1)^{\dim X} \chi(X,\mathcal{O}_X) \geq 0$.*

Proof. Since χ is deformation-invariant, move $\mathcal{O}_X$ to a generic $\mathcal{L} \in \operatorname{Pic}^0 X$. By assumption, $H^i(X,\mathcal{L}) = 0$ for $i \neq \dim X$. ☐

Theorem 4.6 (Green–Lazarsfeld)**.** *Let $w(X) \coloneqq \max\{\operatorname{codim}_X V(\omega) : 0 \neq \omega \in H^1(X,\Omega^1_X)\}$. For generic $\mathcal{L} \in \operatorname{Pic}^0 X$, $H^q(X,\Omega^p \otimes \mathcal{L}) = 0$ for $p + q < w(X)$.*

4.1 Analytic structure on $S^i(X)$

Since $\operatorname{Pic}^0 X$ is a moduli space, it has a universal bundle $\mathcal{P}$, which is a line bundle on $X \times \operatorname{Pic}^0 X$ called the **Poincaré line bundle**. Let $\pi\colon X \times \operatorname{Pic}^0 X \to \operatorname{Pic}^0 X$ be the projection. Is $S^i(X) = \operatorname{supp}(R^i\pi_*\mathcal{P})$? This is not true in general, because cohomology does not commute with base change. We want a complex $\mathcal{E}^\bullet$ which actually computes the cohomology of $\mathcal{P} \otimes k(s)$ for $s \in \operatorname{Pic}^0 X$.

Lemma 4.7. *Let $f\colon X \to S$ be a qcqs morphism of schemes. Let $\mathcal{F} \in \mathsf{Coh}(X)$ be flat over S. Then locally on S, there exists a bounded complex $\mathcal{E}^\bullet$ of locally free sheaves such that for all $\mathcal{G} \in \mathsf{Coh}(S)$, there exists a functorial isomorphism*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{G}) = \mathcal{H}^i(\mathcal{E}^\bullet \otimes \mathcal{G}),$$

i.e. $R^i f_*$ is perfect.

Definition 4.8. Let $\mathcal{E}^\bullet$ be the complex obtained from the lemma applied to $\pi\colon X \times \operatorname{Pic}^0 X \to \operatorname{Pic}^0 X$ and the Poincaré line bundle $\mathcal{P}$. Observe that $H^i(\mathcal{E}^\bullet \otimes k(s)) = H^i(X,\mathcal{L}_s)$ for $s \in \operatorname{Pic}^0 X$. Let

$$S^i_m(X) \coloneqq \{\mathcal{L} \in \operatorname{Pic}^0 X : \dim H^i(X,\mathcal{L}) \geq m\}$$
$$S^i_m(\mathcal{E}^\bullet) \coloneqq \{x \in X : \dim H^i(\mathcal{E}^\bullet \otimes k(x)) \geq m\}.$$

Proposition 4.9. *These $S^i_m(X)$ are analytic sub-varieties of $\operatorname{Pic}^0 X$.*

Proof. This is a local statement, so locally $\mathcal{E}^\bullet$ exists and $\mathcal{E}^i$ are actually free. For $s \in \operatorname{Pic}^0 X$, $s \in S^i_m(X)$ iff $H^i(\mathcal{E}^\bullet \otimes k(s))$ has dimension $\geq m$. This locus is given by some matrix minors of the differentials of $\mathcal{E}^\bullet$. Explicitly, if $\mathcal{I}_k$ denotes $k \times k$ minors, then the ideal sheaf defining $S^i_m(X)$ is (exercise)

$$I(S^i_m(X)) = \sum_{a+b=\operatorname{rank}\mathcal{E}^i - m+1} \mathcal{I}_a(d^{i-1})\mathcal{I}_b(d^i). \qquad \square$$

Our goal is to understand the tangent cone to $S^i_m(\mathcal{E})$. Recall that the dimension of the tangent cone of V is the dimension of V itself. Pick $\mathcal{L} \in \operatorname{Pic}^0 X$ and set $m = \dim H^i(X, \mathcal{L})$. Suppose $\dim_L S^i_m(X) < \dim \operatorname{Pic}^0 X$. Then there exists some $\mathcal{L}'$ such that $\dim H^i(X, \mathcal{L}') < m$. Replacing $\mathcal{L}$ with $\mathcal{L}'$, we can keep reducing all the way down to $m = 0$. To analyze the local structure of $S^i_m(\mathcal{E}^\bullet)$, we need the derivative complex.

Definition 4.10. Consider the $\mathfrak{m}$-adic filtration

$$0 \to \mathfrak{m}_s\mathcal{E}^\bullet/\mathfrak{m}_s^2\mathcal{E}^\bullet \to \mathcal{E}^\bullet/\mathfrak{m}_s^2\mathcal{E}^\bullet \to \mathcal{E}^\bullet \otimes k(s) \to 0.$$

The boundary map on cohomology is

$$D(d^i, s)\colon H^i(\mathcal{E}^\bullet \otimes k(s)) \to H^{i+1}(\mathcal{E}^\bullet \otimes k(s)) \otimes T_s^\vee.$$

Given $v \in T_sS$, we get a map $D_v(d^i, s)\colon H^i(\mathcal{E}^\bullet \otimes k(s)) \to H^{i+1}(\mathcal{E}^\bullet \otimes k(s))$, and we will show this forms a complex. More generally, set $\mathcal{O}_{T_s} \coloneqq \operatorname{Sym}^* T_s^\vee = \bigoplus_n \mathfrak{m}_s^n/\mathfrak{m}_s^{n+1}$. Extend $D(d^i, s)$ to an $\mathcal{O}_{T_s}$-linear map to get

$$D(\mathcal{E}^\bullet)\colon \cdots \to H^i(\mathcal{E}^\bullet \otimes k(s)) \otimes \mathcal{O}_{T_s} \to H^{i+1}(\mathcal{E}^\bullet \otimes k(s)) \otimes \mathcal{O}_{T_s} \to \cdots.$$

(This recovers the previous complex by taking a section.)

Remark. $S^i_m(D(\mathcal{E}^\bullet))$ is always a cone; it lives in T_s, which has a natural scaling action. In the setting of the generic vanishing theorem, each irreducible component is linear.

Lemma 4.11. *$D(\mathcal{E}^\bullet)$ is a complex on* $\operatorname{Spec}\mathcal{O}_{T_s}$.

Proof. It is enough to show that the composition

$$H^i(\mathcal{E} \otimes k(s)) \to H^{i+1}(\mathcal{E} \otimes k(s)) \otimes T_s^\vee \to H^{i+2}(\mathcal{E} \otimes k(s)) \otimes \operatorname{Sym}^2 T_s^\vee$$

is zero. There is a short exact sequence

$$0 \to \mathfrak{m}\mathcal{E}/\mathfrak{m}^3\mathcal{E} \to \mathcal{E}/\mathfrak{m}^3\mathcal{E} \to \mathcal{E}/\mathfrak{m}\mathcal{E} \to 0.$$

There is a diagram

$$\begin{tikzcd}
H^i(\mathcal{E} \otimes k(s)) \arrow[d, "\partial"] \arrow[dr, "{D(d^i,s)}"] & & \\
H^{i+1}(\mathfrak{m}\mathcal{E}/\mathfrak{m}^3\mathcal{E}) \arrow[r] & H^{i+1}(\mathfrak{m}\mathcal{E}/\mathfrak{m}^3\mathcal{E}) \arrow[r, "{D(d^{i+1},s)}"] \arrow[d, "="] & H^{i+2}(\mathfrak{m}^2\mathcal{E}/\mathfrak{m}^3\mathcal{E}) \arrow[d, "="] \\
& H^{i+1}(\mathcal{E} \otimes k(s)) \otimes T^\vee \arrow[r] & H^{i+2}(\mathcal{E} \otimes k(s)) \otimes \operatorname{Sym}^2 T_s^\vee.
\end{tikzcd}$$

But the composition in the bottom row is zero, from the LES associated to $0 \to \mathfrak{m}^2\mathcal{E}/\mathfrak{m}^3\mathcal{E} \to \mathfrak{m}\mathcal{E}/\mathfrak{m}^3\mathcal{E} \to \mathfrak{m}\mathcal{E}/\mathfrak{m}^2\mathcal{E} \to 0$. $\square$

Theorem 4.12 (Tangent cone theorem)**.** $TC_sS^i_m(\mathcal{E}^\bullet) \subset S^i_m(D(\mathcal{E}^\bullet))$.

Remark. Recall that if $Z \subset X$ is an analytic subvariety of a smooth X defined by an ideal sheaf $\mathcal{I}$ and $z \in Z$, then $TC_z Z \subset T_z X = \operatorname{Spec}\operatorname{Sym}^* T_z^\vee$ is defined by

$$TC_z Z \coloneqq V\left(\bigoplus(\mathcal{I}\cap \mathfrak{m}_s^n)/(\mathcal{I}\cap\mathfrak{m}_s^{n+1})\right).$$

Corollary 4.13. $\dim_s S^i_m(\mathcal{E}^\bullet) \le \dim S^i_m(D(\mathcal{E}^\bullet, s))$.

Proof. The dimension of the tangent cone $TC_s S^i_m$ is the dimension of S^i_m. □

Example 4.14. If $m = \dim H^i(X, \mathcal{E}^\bullet \otimes k(s))$, then $S^i_m(D(\mathcal{E}^\bullet, s)) = \{v \in T_s : D(d^i, v), D(d^{i-1}, v) = 0\}$. This is because

$$\begin{aligned} v \in S^i_m(D(\mathcal{E}^\bullet, s)) &\iff \dim H^i(D(\mathcal{E}^\bullet, s)\otimes k(v)) \ge m \\ &\iff \dim H^i(D_v(\mathcal{E}^\bullet, s)) \ge m \\ &\iff \cdots \to H^{i-1} \xrightarrow{0} H^i(\mathcal{E}\otimes k(s)) \xrightarrow{0} H^{i+1} \to \cdots. \end{aligned}$$

Corollary 4.15. *If $H^i(D_v(\mathcal{E}^\bullet, s)) = 0$ for some i and $v \in T_s$, then $S^i(\mathcal{E}^\bullet)$ is a proper subset of S.*

Proof. $TC_s(S^i_1(\mathcal{E}^\bullet)) \subset S^i(D(\mathcal{E}^\bullet, s)) \subsetneq T_s$. So $S^i(\mathcal{E}^\bullet) \subsetneq S$. □

Corollary 4.16. *Suppose $s \in S^i_m(\mathcal{E}^\bullet)$ and $H^i(D_v(\mathcal{E}^\bullet, s)) = 0$ for all $v \neq 0$. Then s is an isolated point of $S^i_m(\mathcal{E}^\bullet)$.*

Lemma 4.17. *Locally near s, $\mathcal{E}^\bullet$ is quasi-isomorphic to a minimal perfect complex $\mathcal{E}^\bullet_0$ such that $S^i_m(\mathcal{E}^\bullet) \cong S^i_m(\mathcal{E}^\bullet_0)$ as complex analytic spaces. Here, minimal means the $d^k_0 = 0 \bmod \mathfrak{m}_s$.*

Remark. Let $\mathcal{E}^\bullet$ be a bounded complex of free modules over a local ring. Then $\mathcal{E}^\bullet$ contains a minimal sub-complex $\mathcal{E}^\bullet_0$ such that the complex $\mathcal{E}^\bullet_0 \to \mathcal{E}^\bullet$ is a quasi-isomorphism. (This is an auxiliary step to make the differentials in the derivative complex easier to understand.)

Proof. We do backward induction. Consider $\mathcal{E}^{n-1} \xrightarrow{d^{n-1}} \mathcal{E}^n \xrightarrow{d^n} \mathcal{E}^{n+1} \to 0$. Locally near s, write $\mathcal{E}^{n+1} = \operatorname{im} d^n + \operatorname{coker} d^n$. Call these two terms F^n and $\mathcal{E}^{n+1}_0$ respectively. Then locally near s, we get $\mathcal{E}^n = F^n \oplus \operatorname{im} d^{n-1} \oplus \mathcal{E}^n_0$, and we proceed backward. □

Proof of tangent cone theorem. Wlog assume $\mathcal{E}^\bullet = \mathcal{E}^\bullet_0$. We want an inclusion

$$\mathcal{I}_{S^i_m(D(\mathcal{E}^\bullet))} \subset \mathcal{I}_{TC_s(S^i_m(\mathcal{E}^\bullet))}$$

of ideal sheaves. Let $\mathcal{I}_m(\mathcal{E}^\bullet)$ be the ideal sheaf of $S^i_m(\mathcal{E}^\bullet)$. We have an explicit description for this:

$$\sum_{a+b=\operatorname{rank}\mathcal{E}^i - m+1} \mathcal{I}_a(d^{i-1})\mathcal{I}_b(d^i).$$

By construction of $\mathcal{E}^\bullet_0$, this lives in $\mathfrak{m}_s^{\operatorname{rank}\mathcal{E}^i - m+1}$. Also, write

$$\mathcal{I}_{TC} = \bigoplus_n \frac{\mathcal{I}_m(\mathcal{E}^\bullet)\cap\mathfrak{m}_s^n}{\mathcal{I}_m(\mathcal{E}^\bullet)\cap\mathfrak{m}_s^{n+1}} \eqqcolon \bigoplus_\ell J_\ell \subset \operatorname{Sym}^* T_s^\vee.$$

Easy observation: $J_\ell 0$ for $\ell < \operatorname{rank}\mathcal{E}^i - m + 1$. Similarly,

$$\mathcal{I}_m(D(\mathcal{E}^\bullet)) = \sum_{a+b=\operatorname{rank} H^i(\mathcal{E}^\bullet\otimes k(s))-m+1} \mathcal{I}_a(D(d^{i-1}))\mathcal{I}_b(D(d^i)).$$

These ideals are generated in rank $H^i(\mathcal{E}^\bullet \otimes k(s)) - m+1$. Moreover, because $\mathcal{E}^\bullet_0$ is minimal, $\operatorname{rank} H^i(\mathcal{E}^\bullet\otimes k(s)) = \operatorname{rank}\mathcal{E}^i$. Hence it is enough to show $J_{e_i - m+1} = \mathcal{I}_m(D(\mathcal{E}^\bullet))_{e_i-m+1}$. This is just a first-order computation of $D(d^i)$ (exercise). □

4.2 The derivative complex via Hodge theory

Proposition 4.18. *Let $[\mathcal{L}] \in \operatorname{Pic}^0 X$. Let $v \in T_{[\mathcal{L}]} \operatorname{Pic}^0 X$ be a tangent vector. By choosing a complex computing the cohomology of $R\pi_*\mathcal{P}$, write*

$$D_v(R\pi_*\mathcal{P}): \cdots \to H^i(X, \mathcal{L}) \xrightarrow{\delta} H^{i+1}(X, \mathcal{L}) \to \cdots.$$

There are two explicit descriptions of δ:

1. *using $T_{[\mathcal{L}]} \operatorname{Pic}^0 X = H^1(X, \mathcal{O}_X)$, write $\delta = - \cup v$ for $v \in H^1(X, \mathcal{O}_X)$;*
2. *using the Dolbeault resolution $0 \to \mathcal{E} \to \mathcal{A}^{0,1}(\mathcal{E}) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,2}(\mathcal{E}) \xrightarrow{\overline{\partial}} \cdots$, view $v \in H^1(X, \mathcal{O}_X)$ as an element $\widetilde{v} \in \mathcal{A}^{0,1}(\mathcal{O}_X)$, so that $\delta = - \wedge \widetilde{v}$.*

Proof. Clearly (1) implies (2), because wedging represents cup product when we use Dolbeault cohomology. Given a SES of sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, it is an element $\alpha \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{E})$ and there is a boundary map $H^i(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{E})$. Claim (exercise): the boundary map is cupping with α. Claim: v corresponds to the extension $0 \to \epsilon\widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}} \to \mathcal{L} \to 0$ in $\operatorname{Ext}^1(\mathcal{L}, \mathcal{L})$, where $\widetilde{\mathcal{L}} \in \operatorname{Pic}(X \times k[\epsilon]/\epsilon^2)$. But the differential in $D_v(R\pi_*\mathcal{P})$ is precisely the boundary map in the LES associated to this SES. ☐

Remark. If $\mathcal{F} \in \mathsf{Coh}(X)$, then first-order deformations of $\mathcal{F}$ are given by $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$ by the same construction. The differential in $D_v(\mathcal{F}): \cdots \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \cdots$ is given by cupping with v.

This description of the differential in $D_v(R\pi_*\mathcal{P})$ is still not explicit enough. To obtain a more explicit description we use Hodge theory of unitary vector bundles.

Theorem 4.19 (Riemann–Hilbert correspondence)**.** *Let X be a complex manifold. There is an equivalence of categories*

$$\begin{pmatrix} \text{complex rep} \\ \text{of } \pi_1(X, x_0) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{locally constant sheaves} \\ \text{of } \mathbb{C}\text{-vector spaces} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{flat holomorphic} \\ \text{vector bundle on } X \end{pmatrix}$$
$$\rho \mapsto \mathbb{V}_\rho \mapsto \mathbb{V}_\rho \otimes_{\mathbb{C}} \mathcal{O}.$$

We want to compute the cohomology of $\mathbb{V}$. Naively we can do

$$0 \to \mathbb{V} \to (\mathcal{E} \coloneqq \mathbb{V} \otimes \mathcal{O}) \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1 \xrightarrow{\nabla} \cdots$$

but these terms are not acyclic. Instead, we use the double complex $\mathcal{A}^{p,q}(\mathcal{E})$ of (p,q)-forms. Then Hodge theory says the following.

1. Let X be compact. There are canonical representatives for classes in $H^i(X, \mathbb{C})$ or $H^q(X, \Omega^p)$. There is a resolution $\mathbb{C} \to \mathcal{A}^0(\mathcal{O}) \xrightarrow{d^0} \mathcal{A}^1(\mathcal{O}) \xrightarrow{d^1} \cdots$, so that $H^i(X, \mathbb{C}) = \ker d^i / \operatorname{im} d^{i-1}$. Hodge theory provides an operator (the Hodge laplacian) whose kernel gives the "orthogonal complement" of $\operatorname{im} d^{i-1} \subset \ker d^i$.
2. Let X be compact complex. Then we can repeat the story for $\mathcal{O}_X \to \mathcal{A}^{0,0} \to \mathcal{A}^{0,1} \to \cdots$. If X is Kähler, then the inclusions $\mathcal{A}^{i,j} \to \mathcal{A}^{i+j}$ is compatible with the inner product, i.e. a canonical representative for Dolbeault cohomology is also a representative for singular cohomology. Hence $\bigoplus_{p+q=n} H^q(X, \Omega^p) = H^{p+q}(X, \mathbb{C})$. Moreover, because we can apply complex conjugation to the double complex, $H^{p,q} = \overline{H^{q,p}}$ as subspaces of $H^{p+q}(X, \mathbb{C})$.

Let $\rho: \pi_1(X) \to U(n) \subset \mathrm{GL}(n, \mathbb{C})$ be a representation. Then Hodge theory works for the local system $\mathbb{V}_\rho$. Specifically, let X be compact Kähler and $\mathbb{V}_\rho$ be a unitary local system. Then

$$H^i(X, \mathbb{V}_\rho) = \bigoplus_{p+q=i} H^q(X, \mathbb{V}_\rho \otimes_{\mathbb{C}} \Omega^p_X) = \bigoplus_{p+q=i} H^q(X, \mathcal{E}_\rho \otimes_{\mathcal{O}} \Omega^p_X)$$

where $\mathcal{E}_\rho \coloneqq \mathbb{V}_\rho \otimes_{\mathbb{C}} \mathcal{O}$, and there is a symmetry $H^q(X, \mathbb{V}_\rho \otimes \Omega^p) = \overline{H^p(X, \mathbb{V}_\rho^\vee \otimes \Omega^q)}$.

Proposition 4.20. *There is a functor*

$$(\textit{representations } \pi_1(X) \to U(1)) \to (\textit{flat unitary line bundles})$$

and all $\mathcal{L} \in \operatorname{Pic}^0 X$ *are in the essential image.*

Proof. We want to understand $H^1(X, \mathcal{O}_X^*)$, so we write the sequence

$$\begin{tikzcd} & & H^1(X,U(1)) \arrow[r, "c_1"] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[d] \\ 0 \arrow[r] & \operatorname{Pic}^0 X \arrow[r] & H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}). \end{tikzcd}$$

Since the upper c_1 is just zero, there is a lift $H^1(X, U(1)) \to \operatorname{Pic}^0 X$. We want to show this is surjective. To see this, write out the entire sequence

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^1(X,\mathbb{R}) \arrow[r] \arrow[d] & H^1(X,U(1)) \arrow[r, "c_1"] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[d] \\ H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[r] & H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}). \end{tikzcd}$$

By Hodge theory, $H^1(X, \mathbb{R}) \cong H^1(X, \mathcal{O})$. By the 4-lemma, the map $H^1(X, U(1)) \to H^1(X, \mathcal{O}_X^*)$ is surjective. ☐

Corollary 4.21. *Given* $[\mathcal{L}] \in \operatorname{Pic}^0 X$, *there exists a unitary local system* $\mathbb{V}_{\mathcal{L}}$ *(of rank* 1*) on* X *such that:*

1. $\mathcal{L} \cong \mathbb{V}_{\mathcal{L}} \otimes_{\mathbb{C}} \mathcal{O}$;
2. $H^i(X, \mathcal{L}) = H^{0,i}(X, \mathbb{V}_{\mathcal{L}})$;
3. $H^i(X, \mathbb{V}_{\mathcal{L}}) = \bigoplus_{p+q=i} H^{p,q}(X, \mathcal{L})$;
4. $H^{p,q}(\mathcal{L}) = \overline{H^{q,p}(\mathcal{L}^\vee)}$.

Corollary 4.22. $\overline{D_v(R\pi_*\mathcal{P})}$ *is the complex* $H^0(X, \mathcal{L}^\vee) \to H^0(X, \Omega_X^1 \otimes \mathcal{L}^\vee) \to H^0(X, \Omega_X^2 \otimes \mathcal{L}^\vee) \to \cdots$ *with differentials* $- \wedge \overline{v}$ *where* $\overline{v} \in H^0(X, \Omega_X^1)$.

Corollary 4.23. *Let* $[\mathcal{L}] \in \operatorname{Pic}^0 X$ *and let* $m \coloneqq \dim H^i(X, \mathcal{L})$. *Then*

$$\dim_{[\mathcal{L}]} S_m^i(X) \le \dim \left\{ \omega \in H^0(X, \Omega_X^1) : \omega \wedge - \textit{ annihilates } \begin{array}{l} H^0(X, \Omega^{i-1} \otimes \mathcal{L}^\vee) \\ H^0(X, \Omega^i \otimes \mathcal{L}^\vee) \end{array} \right\}.$$

Proof. We have

$$\dim_{[\mathcal{L}]} S_m^i(X) = \dim TC_{[\mathcal{L}]}(S_m^i(X)) \le \dim S_m^i(D_{[\mathcal{L}]}(R\pi_*\mathcal{P})) = \dim S_m^i(\overline{D_{[\mathcal{L}]}(R\pi_*\mathcal{P})}).$$

By the choice of m, $H^0(X, \Omega^i \otimes \mathcal{L}^\vee)$ is exactly m-dimensional. For cohomology to still be m-dimensional, both differentials around it must vanish. ☐

Corollary 4.24. *If the sequence* $H^0(X, \Omega^{i-1} \otimes \mathcal{L}^\vee) \xrightarrow{\omega\wedge-} H^0(X, \Omega^i \otimes \mathcal{L}^\vee) \xrightarrow{\omega\wedge-} H^0(X, \Omega^{i+1} \otimes \mathcal{L}^\vee)$ *is exact for some* ω, *then* $S^i(X) \neq \operatorname{Pic}^0 X$.

Corollary 4.25. *If the sequence above is exact for all* $\omega \neq 0$, *then* $[\mathcal{L}]$ *is isolated in* $S^i(X)$.

4.3 Proof of generic vanishing

Example 4.26 (Generic vanishing on a complex torus)**.** Let T be a complex torus $\mathbb{C}^g/\Lambda$. The Albanese map $\mathrm{alb}: T \to T$ is just the identity, and $\operatorname{Pic}^0 X = \{\chi: \mathbb{Z}^{2g} \to U(1)\}$. Then

$$S^i(T) = \{\chi : H^i(T, \mathcal{L}_\chi) \neq 0\}$$

where $\mathcal{L}_\chi$ is the flat unitary line bundle associated to χ. By Hodge theory, we get an inclusion

$$H^i(T, \mathcal{L}_\chi) \subset H^i(T, \chi) = H^i(\mathbb{Z}^{2g}, \chi) = \bigoplus_{\sum n_j = i} \bigotimes_j H^{n_j}(\mathbb{Z}, \chi|_{\mathbb{Z}}).$$

Since Euler characteristic is zero for the trivial local system, we know $h^0 = h^1$ for any χ on $\mathbb{Z}$. Hence $S^i(T) = \{\mathcal{O}\}$ for all i.

Proof of generic vanishing. Let $Z \subset S^i(X)$ be an irreducible component and choose $[\mathcal{L}] \in Z$ such that $h^i(X, \mathcal{L})$ is minimal. It suffices to show $\operatorname{codim}_{[\mathcal{L}]} S^i_m(X) \geq \dim \mathrm{alb}(X) - i$; equivalently, $\dim_{[\mathcal{L}]} S^i_m(X) \leq \dim \operatorname{Pic}^0 X - \dim \mathrm{alb}(X) + i$. Choose $\beta \neq 0$ in $H^0(X, \Omega^i \otimes \mathcal{L}^\vee)$, which is non-zero by choice of $\mathcal{L}$. By a previous corollary, clearly

$$\dim_{[\mathcal{L}]} S^i_m(X) \leq \dim\{\omega \in H^0(X, \Omega^1_X) : \omega \wedge \beta = 0\}.$$

Fix a point $x \in X$. Then clearly

$$\dim\{\omega \in H^0(X, \Omega^1_X) : \omega \wedge \beta = 0\} \leq \dim\{\omega \in H^0(X, \Omega^1_X) : (\omega \wedge \beta)(x) = 0\}.$$

Let $e(x): H^0(X, \Omega^1_X) \to T^*_x X$ be the evaluation map. Then

$$\dim\{\omega \in H^0(X, \Omega^1_X) : (\omega \wedge \beta)(x) = 0\} \leq \dim \ker e(x) + \dim\{\varphi \in T^*_x X : (\phi \wedge \beta)(x) \neq 0\}.$$

But $e(x)$ is the dual to the differential of alb, by the fundamental theorem of calculus (exercise). At a general point of X, then, $\operatorname{rank} e(x) = \dim \mathrm{alb}(X)$. Hence

$$\dim \ker e(x) = \dim \operatorname{Pic}^0 X - \dim \mathrm{alb}(X),$$

and it remains to show

$$\dim\{\varphi \in T^*_x X : (\phi \wedge \beta)(x) \neq 0\} \leq i$$

for general $x \in X$. This follows by the following linear algebra lemma (exercise). ☐

Lemma 4.27. *Let V be a finite-dimensional vector space and $\beta \in \wedge^i V$ non-zero. Then $\{v \in V : v \wedge \beta = 0\}$ has dimension $\leq i$.*

Theorem 4.28 (Green–Lazarsfeld)**.** *Suppose there exists $\omega \in H^0(X, \Omega^1_X)$ such that $\operatorname{codim}_X V(\omega) \geq k$. Then for generic $[\mathcal{L}] \in \operatorname{Pic}^0 X$,*

$$H^q(X, \mathcal{L} \otimes \Omega^p_X) = 0 \quad \forall p + q < k.$$

Proof. Let $\pi_1: X \times \operatorname{Pic}^0 X \to X$ and $\pi_2: X \times \operatorname{Pic}^0 X \to \operatorname{Pic}^0 X$. Take $\mathcal{E}^\bullet \coloneqq R\pi_{2*}(\pi_1^* \Omega^p_X \otimes \mathcal{P}) \in D^b(\operatorname{Pic}^0 X)$. The fibers of this complex compute the cohomology $H^q(X, \mathcal{L} \otimes \Omega^p_X)$, and we will study $S^i(\mathcal{E}^\bullet)$.

Choose $[\mathcal{O}_X] \in \operatorname{Pic}^0 X$ and $v \in H^1(X, \mathcal{O}_X)$. Then the complex $D_{[\mathcal{O}_X], v}(\mathcal{E}^\bullet)$ is

$$H^0(X, \Omega^p) \xrightarrow{v\wedge -} H^1(X, \Omega^p) \xrightarrow{v\wedge -} H^2(X, \Omega^p) \xrightarrow{v\wedge -} \cdots$$

Let $\omega \in H^0(X, \Omega^1)$ be the conjugate of v. The conjugate $\overline{D_{[\mathcal{O}_X], v}(\mathcal{E}^\bullet)}$ is

$$H^p(X, \mathcal{O}) \xrightarrow{\omega\wedge -} H^p(X, \Omega^1) \xrightarrow{\omega\wedge -} H^p(X, \Omega^2) \xrightarrow{\omega\wedge -} \cdots.$$

If we pick ω as in the statement of the theorem, it turns out the complex $H^q(X, \Omega^{p-1}) \to H^q(X, \Omega^p) \to H^q(X, \Omega^{p+1})$ is exact for $p + q < k$. Then ω is not in $S^i(\overline{D(\mathcal{E}^\bullet)})$ and we are done. So it suffices to prove the following proposition. ☐

Proposition 4.29. *Take ω as in the theorem. Then the complex $H^q(X,\Omega^{p-1}) \to H^q(X,\Omega^p) \to H^q(X,\Omega^{p+1})$ is exact for $p+q<k$.*

Proof. Consider the complex $K^\bullet := (0 \to \mathcal{O} \xrightarrow{\omega\wedge-} \Omega^1 \xrightarrow{\omega\wedge-} \Omega^2 \xrightarrow{\omega\wedge-} \cdots)$; this is some kind of Koszul complex for ω, and the conjugate derivative complex follows from taking global sections. Claims:

1. $E_1^{p,q} = H^q(X,\Omega^p) \Rightarrow \mathbb{H}^{p+q}(K^\bullet)$ degenerates at E_2;

2. $E_2^{p,q} = H^p(X,\mathcal{H}^q(K^\bullet)) \Rightarrow \mathbb{H}^{p+q}(K^\bullet)$ and $\mathcal{H}^q(K^\bullet) = 0$ for $q<k$.

These imply the proposition because of the following. (2) implies $\mathbb{H}^{p+q}(K^\bullet) = 0$ for $p+q<k$. (1) implies $E_2^{p,q} = 0$ for $p+q<k$. But $E_2^{p,q}$ are the cohomology groups of the sequence we are interested in.

Let's prove (2) first. General fact about Koszul complexes: if X is regular and $\mathcal{E}$ is a vector bundle on X with section $s \in \Gamma(X,\mathcal{E})$ vanishing with codim k, then the complex $0 \to \mathcal{O} \xrightarrow{\wedge s} \mathcal{E} \xrightarrow{\wedge s} \wedge^2\mathcal{E} \xrightarrow{\wedge s} \cdots$ is exact in degree $\leq k$. This is a local statement, so let A be a regular local ring with sections $f_1,\dots,f_n$, in which case the sequence becomes

$$K(f_1,\dots,f_n) = \bigotimes_i [0 \to A \xrightarrow{f_i} A \to 0] = [0 \to A \xrightarrow{(f_1,\dots,f_n)} A^n \xrightarrow{(f_1,\dots,f_n)} \wedge^2 A^n \to \cdots].$$

By commutative algebra, there is a subsequence of length k in $f_1,\dots,f_n$ which is regular. Hence it is enough to show the desired claim when we replace $(f_1,\dots,f_n)$ with the subsequence. Then we can replace $0 \to A \xrightarrow{f_i} A \to 0$ by A/f_i in degree 1, which is quasi-isomorphic, and induct on k.

Now we prove (1). Pick an explicit resolution of $K^\bullet$ so that we get a bicomplex

$$\begin{tikzcd}
{} & {} & {} \\
\mathcal{A}^{0,1} \arrow[u, "\bar\partial"] \arrow[r, "\wedge\omega"] & \mathcal{A}^{1,1} \arrow[u, "\bar\partial"] \arrow[r, "\wedge\omega"] & \cdots \\
\mathcal{A}^{0,0} \arrow[u, "\bar\partial"] \arrow[r, "\wedge\omega"] & \mathcal{A}^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\wedge\omega"] & \mathcal{A}^{2,0} \arrow[u, "\bar\partial"] \arrow[r, "\wedge\omega"] & \cdots,
\end{tikzcd}$$

from which we get the spectral sequence. Now compute that $d_2 = 0$ via a diagram chase. Pick $\widetilde{\alpha} \in E_2^{p,q}$, where $\alpha \in H^q(X,\Omega^p)$ is harmonic with $\alpha\wedge\omega = \bar\partial\beta$ for some β. Now observe that $\partial(\alpha\wedge\omega) = 0$, because ω is a global 1-form and α is harmonic. By the $\partial\bar\partial$-lemma, $\omega\wedge\alpha = \bar\partial\partial\gamma$ for some γ. Then $d_2\alpha = [\omega\wedge\partial\gamma]$, which we want to be zero, i.e. that $\omega\wedge\partial\gamma = \bar\partial\xi$. It is easy to see this is both ∂ and $\bar\partial$-closed, and $\partial(\omega\wedge\gamma) = \omega\wedge\partial\gamma$, so again by the $\partial\bar\partial$-lemma, we are done. ◻

5 Unobstructedness of abelian varieties

Definition 5.1. Let k be a field. An **abelian variety** over k is a proper smooth geometrically connected k-group scheme. ("Smooth" here rules out things like $\operatorname{Spec} k[t]/(t^p-1) = \operatorname{Spec} k[t]/(t-1)^p$.)

Definition 5.2. Let S be a scheme. An **abelian S-scheme** is an S-group scheme proper and flat over S whose geometric fibers are integral.

Example 5.3. Some examples of abelian varieties:

1. take a complex torus with an ample line bundle (by GAGA);

2. Jacobian of a curve;

3. $\operatorname{Alb}(X)$ for X smooth and proper.

Lemma 5.4. *If S is normal, any abelian S-scheme is projective over S.*

Corollary 5.5. *Any abelian variety A over k contains a curve such that the map $\operatorname{Jac} X \to A$ is surjective.*

Proof. By the lemma, A is projective and therefore has an ample line bundle. Try to take a generic section of $\mathcal{O}(N)^{\oplus(\dim A-1)}$ for $N \gg 0$. We want $T_0 \operatorname{Jac} \to T_0 A$ to be surjective. This is dual to the map $H^0(A, \Omega^1) \to H^0(C, \Omega^1)$, which we want to be injective. Taking $N \gg 0$, by Serre vanishing, the kernel will be H^0 of some very negative thing, so we are done. □

Theorem 5.6. *Let k be a perfect field, and A be an abelian variety over k. Then:*

1. Def_A *is pro-representable and smooth;*
2. $\operatorname{Def}_A^{grp}$ *is pro-representable and smooth, where* $\operatorname{Def}_A^{grp}(B)$ *consists of abelian schemes over B with an isomorphism $\varphi: B_k \to A$.*

Furthermore, $\operatorname{Def}_A = \operatorname{Def}_A^{grp}$*, and both are pro-representable by a polynomial ring* $k_{W(k)}[x_1, \dots, x_{g^2}]$.

Theorem 5.7 (Grothendieck)**.** *Let S be an affine scheme and $S \hookrightarrow S'$ be a closed embedding defined by a square-zero ideal sheaf $\mathcal{I}$. Let $A_{/S}$ be an abelian scheme. Then:*

1. *A admits a flat deformation $A' \to S'$;*
2. *for any lift of the identity section to S', there is a unique group structure on A' extending the group structure on A.*

Corollary 5.8. *Let R be a complete local ring with residue field k, and let A be a k-scheme. Then there exists an abelian scheme $\widetilde{A} \to R$ and an isomorphism $\widetilde{A}_k \xrightarrow{\sim} A$.*

Proof sketch. The theorem allows us to lift:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & \cdots \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r] & \cdots.
\end{tikzcd}$$

But this only gives a formal scheme over $\operatorname{Spec} R$. There are two issues: we need to lift an ample line bundle, and then we need to use formal GAGA (see below) to conclude we get an actual abelian scheme. (This only works when the special fiber is projective.) □

Proof of theorem. Assume we are in the special case where 2 is invertible on S. Recall there is a canonical class $\mathrm{ob} \in H^2(A, T_{A/S} \otimes \mathcal{I})$ such that $\mathrm{ob} = 0$ iff there exists a flat deformation A' of A. Hence $[-1]^* \mathrm{ob} = \mathrm{ob}$. But $[-1]$ acts by -1 on $T_{A,0}$, and, by functoriality, by -1 on $H^2(A, T_{A/S} \otimes \mathcal{I})$. (Proof follows below.) Since 2 is invertible, $\mathrm{ob} = 0$.

When $S = \operatorname{Spec} k$, then $H^0(A, T_A) = T_A$ and $T_A = \mathcal{O}$. Lemma: $H^i(A, \mathcal{O}) = \wedge^i H^1(A, \mathcal{O})$. This implies $H^2(A, T_A) = T_{A,0} \otimes \wedge^2 H^1(A, \mathcal{O})$, and $[-1]$ acts by -1 on each term.

Now we do the general case. Consider the map $\psi: A \times A \to A \times A$ given by $(x, y) \mapsto (x, x+y)$. Lemma: if X and Y are smooth S-schemes, then $T_{X \times_S Y} = \pi_1^* T_X \oplus \pi_2^* T_Y$, and

$$\mathrm{ob}(X \times Y) = \pi_1^* \mathrm{ob}(X) + \pi_2^* \mathrm{ob}(Y) \in H^2(X \times Y, T_{X \times Y} \otimes \mathcal{I}).$$

This arises from unwinding the construction of the ob cocycle. The theorem will follow by using that $\psi^* \mathrm{ob} = \mathrm{ob}$. □

Example 5.9. Examples of group schemes:

1. $\mu_p \coloneqq \ker(\mathbb{G}_m \xrightarrow{(-)^p} \mathbb{G}_m) = \operatorname{Spec} k[t]/(t^p - 1)$, with group law $x + y + xy$;

2. $\alpha_p \coloneqq \ker(\mathbb{G}_a \xrightarrow{(-)^p} \mathbb{G}_a) = \operatorname{Spec} k[t]/t^p$, with group law $x+y$.

Exercise: these lift from characteristic p to characteristic 0. Claim: $G \coloneqq \mu_p \ltimes \alpha_p$ does not lift. Observe that all abstract groups of order p^2 are commutative. So if G lifts, then the central fiber would be commutative. Since commutativity is a closed condition, we get a contradiction.

Theorem 5.10 (Smooth curves lift)**.** *Let X be a smooth proper curve over a perfect field k of characteristic $p>0$. Then there exists $\mathcal{X}$ over $\operatorname{Spec} W(k)$ such that the base change $\mathcal{X}_k = X$.*

Proof. There is a sequence $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} W_2(k) \hookrightarrow \cdots$. The obstruction to lifting X lies in $H^2(X, T_{X/k})$, but X is a curve so this is automatically zero. Let the lifts be X_i. Hence $\varprojlim X_i$ is a formal scheme over $\operatorname{Spf} W(k)$.

We want a line bundle $\mathcal{L}$ on $\widehat{X} \coloneqq \varprojlim X_i$ such that $\mathcal{L}|_X$ is ample. Let D be a rational point of X over some finite extension of k. By formal smoothness, there exists a Cartier divisor $\widetilde{D}$ on $\widehat{X}$ specializing to D. Set $\mathcal{L} \coloneqq \mathcal{O}(\widetilde{D})$ and then apply formal GAGA. ☐

Theorem 5.11 (Formal GAGA)**.** *Let A be an adic Noetherian ring with ideal of definition I. Suppose X is a finite type scheme over $\operatorname{Spec} A$, and $\widehat{X} \coloneqq \varprojlim (X \otimes A/I^n)$ is the associated formal scheme. Then there is a restriction map*

$$\mathsf{Coh}_{perf}(X) \to \mathsf{Coh}_{perfl}(\widehat{X}) \quad \mathcal{F} \mapsto \widehat{\mathcal{F}} \coloneqq \varprojlim \mathcal{F} \otimes A/I^n$$

is an equivalence of categories, where perfl means flat over A and support is proper over I.

Corollary 5.12. *Let A be an adic Noetherian ring with ideal of definition I. Let Y be a proper formal scheme over $\operatorname{Spf} A$. Suppose there exists a line bundle $\mathcal{L}$ on Y such that $\mathcal{L}|_{Y \otimes A/I}$ is ample over $\operatorname{Spec} A/I$. Then there exists an X over $\operatorname{Spec} A$ such that $Y = \widehat{X}$ at I.*

Theorem 5.13. *Suppose X_0 is a g-dimensional abelian variety over a perfect field k. Then $\operatorname{Def}_{X_0} \cong \operatorname{Def}^{grp}_{X_0}$ and both are pro-represented by $W(k)[[x_1, \dots, x_{g^2}]]$.*

Proof. Last time we showed Def_A is smooth. By Cohen structure theorem, $W(k)[[x_1, \dots, x_d]]$ is a hull. Now we show $\operatorname{Def}_A = \operatorname{Def}^{grp}_A$ and is pro-representable. First define $\operatorname{Def}^{grp}_A \to \operatorname{Def}_A$ by forgetting the group structure. This is injective on T-points (exercise). It is also surjective on T-points, as follows. ☐

Proposition 5.14. *Suppose $0 \to I \to B \to A \to 0$ is a small extension, and $X \in \operatorname{Def}_{X_0}(B)$ and X_A is an abelian A-scheme and $\epsilon \colon \operatorname{Spec} B \to X$ is a section. Then there exists a unique structure of an abelian B-scheme on X such that:*

1. *ϵ is the identity section extending the identity section on X_A;*
2. *the abelian B-scheme structure on X extends the abelian A-scheme structure on X_0.*

Proof. Let $\mu \colon X_A \times X_A \to X_A$ be the subtraction map $(x,y) \mapsto x - y$, and let $\mu_0 \colon X_0 \times X_0 \to X_0$ be the same thing. It suffices to extend μ to X in a way such that $\mu(\epsilon \times \epsilon) = \epsilon$. For this, we need to understand the deformation theory of maps. By the following proposition, the obstruction to extending μ lives in $H^1(X_0 \times X_0, \mu_0^* T_{X_0} \otimes \mathcal{I})$ where $\mathcal{I}$ is the sheaf on $X_0 \times X_0$ corresponding to I.

1. Let $\Delta \colon X_A \to X_A \times X_A$ be the diagonal, and note that $\mu \circ \Delta = \epsilon_A$ has trivial obstruction class because ϵ_A extends to X.
2. Let $\mathrm{id} \times \epsilon_A \colon X_A \to X_A \times X_A$, and note that $\mu \circ (\mathrm{id} \times \epsilon_A) = \mathrm{id}$ has trivial obstruction class.

Observe that $T_{X_0} = \mathcal{O}_{X_0} \otimes \Gamma(X_0, T_{X_0})$ because all global sections are translation-invariant. By Künneth,

$$H^1(X_0 \times X_0, \mu^* T_{X_0}) = (\pi_1^* H^1(X, \mathcal{O}_X) \oplus \pi_2^* H^1(X, \mathcal{O}_X)) \otimes H^0(X, T_X) \otimes I.$$

Hence $\mathrm{ob} = 0$ (exercise). This shows that μ lifts.

Now we show there is a unique lift $\widetilde{\mu}$ of μ such that $\widetilde{\mu}(\epsilon \times \epsilon) = \underline{\epsilon}$. The lifts are a torsor for $H^0(X_0 \times X_0, \mu^* T_{X_0} \otimes I) = H^0(X_0, T_{X_0}) \otimes I$ (exercise with projection formula + that fibers of μ are geometrically connected). Extensions of μ restricted to $\epsilon \times \epsilon$ are a torsor for $H^0(\operatorname{Spec} k, \mu_0|^*_{\epsilon_0 \times \epsilon_0} T_{X_0} \otimes I)$, which is the same thing. So we can fix $\widetilde{\mu}$ using $H^0(X_0, T_{X_0}) \otimes I$.

Finally, $\widetilde{\mu}$ defines a group structure on X. In fact, we have almost proved injectivity as well! □

Corollary 5.15. $\operatorname{Def}_A = \operatorname{Def}_A^{grp}$ *are pro-representable.*

Proof. The content is that any infinitesimal automorphism of X_A as a group scheme lifts to an infinitesimal automorphism of X as a group scheme, but the tangent computation showed there are no infinitesimal automorphisms. □

Proposition 5.16. *Suppose X and Y are smooth and $f: X \to Y$ is a qcqs morphism over* $\operatorname{Spec} A$. *Let* $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ *be a square-zero embedding defined by an ideal I, and let X' and Y' be lifts of X and Y over* $\operatorname{Spec} B$.

1. *There exists a canonical class* $\operatorname{ob}(f) \in H^1(X, f^* T_Y \otimes I)$ *such that f extends iff* $\operatorname{ob}(f) = 0$.
2. *If* $\operatorname{ob}(f) = 0$*, then the set of extensions is a torsor for* $H^0(X, f^* T_Y \otimes I)$.

Proof sketch (exercise). By smoothness, cover Y with global complete intersections, and cover their pre-images with global complete intersections. Locally, we can therefore extend the map. On intersections, we get derivations $f^* \Omega_Y \to I$, giving Čech cocycles. □

5.1 Deformations of polarized AVs

Definition 5.17. Let X be an abelian variety over k. Define $X^\vee := \operatorname{Pic}^0 X$, the identity component of $\operatorname{Pic} X$. Here $\operatorname{Pic} X = \{T \to \operatorname{Pic}(X \times T)\}^\#$. If there is a rational point (which there is),

$$\operatorname{Pic} X = \{T \to \text{line bundles on } X \times T \text{ with trivialization on } \{0\} \times T\}/\sim.$$

Remark. For a general variety V, $\operatorname{Pic}^0(V)$ is not always an abelian variety. But if V is an abelian variety, so is $\operatorname{Pic}^0(V)$.

Definition 5.18. There is a map $\Lambda: \operatorname{Pic} X \to \operatorname{Hom}_{\mathrm{AV}}(X, X^\vee) \subset \operatorname{Pic}(X \times X)$ given by

$$L \mapsto (m - p_1 - p_2)^* L.$$

Complex-analytically, given a line bundle $L \in \operatorname{Pic}(A)$ for an abelian variety A, the first Chern class gives $c_1(L) \in H^{1,1}(A) \cap H^2(A, \mathbb{Z})$, and we know $H^2(A, \mathbb{Z}) = \wedge^2 H^1(A, \mathbb{Z})$. So $c_1(L)$ is a bilinear form on $H^1(A, \mathbb{C})$, and therefore a map $H^1(A, \mathbb{C})^\vee \to H^1(A, \mathbb{C})$ which preserves Hodge structure.

Proposition 5.19. *There is an exact sequence*

$$0 \to X^\vee \to \operatorname{Pic}(X) \xrightarrow{\Lambda} \operatorname{Hom}(X, X^\vee).$$

Proof. Use $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ and "think a little bit". Explicitly,

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[r] \arrow[d] & H^1(X,\mathcal{O}^*) \arrow[r] \arrow[d, equal] & H^2(X,\mathbb{Z}) \\
0 \arrow[r] & \operatorname{Pic}^0 X \arrow[r] & \operatorname{Pic} X \arrow[r] & \operatorname{Hom}(X,X^\vee). \arrow[u, "\subset"]
\end{tikzcd}$$

□

Definition 5.20. A map $\lambda: X \to X^\vee$ is a **quasi-polarization** (resp. **polarization** if $\lambda = \Lambda L$ for some $L \in \operatorname{Pic} X$ (resp. ample L).

Proposition 5.21 (Oort)**.** *Suppose (X,λ) lives over $\operatorname{Spec} R$ with a small extension $\operatorname{Spec} R \to \operatorname{Spec} R'$ of Artin $W(k)$-algebras. Suppose X lifts to X' and $\lambda = \Lambda L$ is a quasi-polarization. Then λ lifts iff L lifts.*

Definition 5.22. Let (A,λ) be a quasi-polarized AV. Define

$$\operatorname{Def}_{(A,\lambda)}(R) \coloneqq \{(\widetilde{A},\widetilde{\lambda},\varphi) : (\widetilde{A},\widetilde{\lambda}) \text{ is quasi-polarized AV over } R,\ \varphi\colon (\widetilde{A},\widetilde{\lambda})|_1 \to (A,\lambda)\}/\sim .$$

Theorem 5.23 (Mumford)**.** *The functor $\operatorname{Def}_{(A,\lambda)}$ is pro-representable and is a closed sub-functor of $\operatorname{Def}_A = \operatorname{Def}_A^{grp} = h_{W(k)[x_1,\dots,x_{g^2}]}$ cut out by an ideal in $W(k)[x_1,\dots,x_{g^2}]$ generated by $g(g-1)/2$ elements.*

Remark. The $g(g-1)/2$ is the dimension of $H^2(X,\mathcal{O}_X)$.

Proof. That it is a closed subfunctor is in (GIT, 6.2). This is easy if $\lambda\colon X \to X^\vee$ is separable, hence étale, hence $\Omega^1_{X/X^\vee} = 0$. So given a deformation of $X^\vee$, there exists exactly one deformation of X over it.

For pro-representability, use Schlessinger's criterion. (Property H2) Clearly $\operatorname{Def}_{(A,\lambda)}(k[\epsilon]/\epsilon^2)$ is finite-dimensional; it is a sub-functor of Def_A. Properties H1, H3, H4 boil down to $\operatorname{Def}_{(A,\lambda)}$ being left exact. Suppose

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "r"'] & R \arrow[d, "\pi"'] \\ T \arrow[r] & R' \end{tikzcd}$$

is Cartesian, with π a small surjection with kernel I. We want $\operatorname{Def}_{(A,\lambda)}(Q) \to \operatorname{Def}_{(A,\lambda)}(T) \times_{\operatorname{Def}_{(A,\lambda)}(R')} \operatorname{Def}_{(A,\lambda)}(R)$ to be an isomorphism. Put this into a square w.r.t. Def_A:

$$\begin{tikzcd} \operatorname{Def}_{(A,\lambda)}(Q) \arrow[r] \arrow[d] & \operatorname{Def}_{(A,\lambda)}(T) \times_{\operatorname{Def}_{(A,\lambda)}(R')} \operatorname{Def}_{(A,\lambda)}(R) \arrow[d] \\ \operatorname{Def}_A(Q) \arrow[r, equal] & \operatorname{Def}_A(T) \times_{\operatorname{Def}_A(R')} \operatorname{Def}_A(R). \end{tikzcd}$$

The vertical arrows are injective because $\operatorname{Def}_{(A,\lambda)}$ is a sub-functor. Hence the horizontal arrow is injective. Now do surjectivity. Choose $(Y,\mu,\psi) \in \operatorname{Def}_{(A,\lambda)}(T)$ and $(X,\lambda,\phi) \in \operatorname{Def}_{(A,\lambda)}(R)$ both mapping to the same thing in $\operatorname{Def}_{(A,\lambda)}(R')$. Let $Z \in \operatorname{Def}_A(Q)$ to be a lift of X and Y. Choose K and L such that $\Lambda K = \mu$ and $\Lambda L = \lambda$. The obstruction to lifting line bundles on Y to line bundles on Z is

$$\operatorname{Pic}(Z) \to \operatorname{Pic}(Y) \to H^2(A,\mathcal{O}) \otimes I.$$

So we would like $\operatorname{ob}(K) = 0$. Draw the diagram

$$\begin{tikzcd} \operatorname{Pic}(Z) \arrow[r] \arrow[d] & \operatorname{Pic}(Y) \arrow[r] \arrow[d] & H^2(A,\mathcal{O}) \otimes I \arrow[d, equal] \\ \operatorname{Pic} X \arrow[r] & \operatorname{Pic} X' \arrow[r] & H^2(A,\mathcal{O}) \otimes I \end{tikzcd}$$

where $X' \coloneqq X|_{R'}$. But the image of K under this map is L, and L lifts. Hence $\operatorname{ob}(K) = \operatorname{ob}(L) = 0$, and K lifts as well.

Recall that if T_1, T_2 is a tangent-obstruction theory for F, then there exists a hull for F which is the quotient of a power series ring on $\dim T_1$ generators by an ideal with at most $\dim T_2$ generators. So the ideal cutting out $\operatorname{Def}_{(A,\lambda)}$ has at most $\dim H^2(A,\mathcal{O}) = g(g-1)/2$ generators. ☐

Theorem 5.24 (Grothendieck)**.** *λ is separable iff $\operatorname{Def}_{(A,\lambda)}$ is formally smooth over $W(k)$.*

Corollary 5.25. *If X is a principally polarized AV, then X lifts to $\operatorname{Spec} W(k)$.*

Proof. Lift X to $\operatorname{Spf} W(k)$, and then apply formal GAGA using the ample line bundle on X. ☐

6 Formal GAGA

References are EGA 3, or FGA Explained chapter 8.

Theorem 6.1. *Let A be a Noetherian ring with an ideal I, and let X be finite type over A. Then for $\mathcal{F} \in \mathsf{Coh}(X)$ with proper support over A, the natural maps*

$$H^q(X, \mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim H^q(X, \mathcal{F} \otimes A/I^n)$$
$$H^q(\widehat{X}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \varprojlim H^q(X, \mathcal{F} \otimes A/I^n)$$

are isomorphisms.

Proposition 6.2. *Let X be a scheme, and $(\mathcal{F}_n)$ be an inverse system of coherent sheaves on X with surjective maps. If for all i, the systems $(H^i(X, \mathcal{F}_n))$ satisfy the Mittag–Leffler condition, then*

$$\varprojlim H^i(X, \mathcal{F}_n) \to H^i(X, \varprojlim \mathcal{F}_n)$$

is an isomorphism.

Remark. Recall that the Mittag–Leffler condition says $\mathrm{im}(H^i(X, \mathcal{F}_{n+1}) \to H^i(X, \mathcal{F}_n))$ stabilizes for $n \gg 0$.

Proposition 6.3. *Suppose $f: X \to Y$ is proper and $S = \bigoplus_{n \geq 0} S_n$ is a graded $\mathcal{O}_Y$-algebra which is quasi-coherent, S_0, S_1 coherent, and such that S_1 generates S over S_0. Let $M = \bigoplus M_n$ be a finitely generated f^*S-module. Then for all q, $R^q f_* M$ is a graded S-module of finite type, and there exists an n_0 such that $R^q f_* M_n = S_{n-n_0} R^q f_* M_{n_0}$ for $n \geq n_0$.*

Proof. Replace X and Y by $\mathrm{Spec}_Y S$ and $\mathrm{Spec}_X f^*S$, and then use coherence properties of proper pushforward. ☐

Corollary 6.4. *In the setting of the theorem, $B \coloneqq \bigoplus_{n \geq 0} I^n$, the module $\bigoplus H^q(X, I^n \mathcal{F})$ is finitely generated over B.*

Proof of theorem, EGA III.1.4. We have a short exact sequence $0 \to I^n \mathcal{F} \to \mathcal{F} \to \mathcal{F}_n \to 0$, giving a long exact sequence

$$0 \to R_n \to H^q(\mathcal{F}) \to H^q(\mathcal{F}_n) \to Q_n \to 0.$$

Claims:

1. $(R_n)_n$ is an "I-good" filtration of $H^q(\mathcal{F})$, i.e. $R_0 = H^q(\mathcal{F})$ and $I R_n \subset R_{n+1}$ with equality for $n \gg 0$;
2. $(Q_n)_n$ is Artin–Rees zero, i.e. there exists m such that for all n, $Q_n \to Q_{n+m}$ is zero;
3. $(H^q(\mathcal{F}_n))_n$ is Artin–Rees Mittag–Leffler, i.e. there exists r such that for $m' \geq m + r$, the image of $H^q(\mathcal{F}_{m'}) \to H^q(\mathcal{F}_m)$ is equal to the image of $H^q(\mathcal{F}_{m+r}) \to H^q(\mathcal{F}_m)$.

(Here, the adjective Artin–Rees means that there is some uniform statement, as opposed to a non-uniform one.) Assuming these claims, we prove the theorem. There is an exact sequence

$$0 \to H^q(\mathcal{F})/R_n \to H^q(\mathcal{F}_n) \to Q_n \to 0.$$

(2) implies that $\varprojlim Q_n = 0$. The left-exactness of $\varprojlim$ then implies $\varprojlim H^q(\mathcal{F})/R_n \xrightarrow{\sim} \varprojlim H^q(\mathcal{F}_n)$ is an isomorphism. (1) implies $\varprojlim H^q(\mathcal{F})/R_n \cong H^q(\mathcal{F})^\wedge$. These prove the first isomorphism. For the second isomorphism, use (3) to move inverse limits around: by the proposition, Mittag–Leffler implies $\varprojlim H^q(\mathcal{F}_n) = H^q(\widehat{\mathcal{F}})$.

Now we sketch the proof of the claims. Claim (1) is easy: the first part is straightforward, the second part follows from functoriality of H^q (with respect to multiplication by $a \in I$), and the third part follows from the corollary, which implies that $\bigoplus R_n$, which is a quotient of $\bigoplus H^q(X, I^n \mathcal{F})$, is finite type over B.

For claim (2), set $N \coloneqq \bigoplus_n H^{q+1}(I^n\mathcal{F})$, which is finitely generated by the corollary. But B is Noetherian, so $\bigoplus Q_n$ is finitely generated over B as a sub-module of N. Hence $Q_{n+1} = IQ_n$ for $n \gg 0$. This is because Q_k is a quotient of $H^q(\mathcal{F}_k)$, and hence is killed by I^k. Hence there exists some r such that Q_n is killed by I^r for all n. Suppose we are given $a \in I^p$. Then the composition

$$H^{q+1}(I^{n+1}\mathcal{F}) \xrightarrow{\cdot a} H^{q+1}(I^{n+p+1}\mathcal{F}) \to H^{q+1}(I^{n+1}\mathcal{F})$$

is actually multiplication by a as an A-module. For $p > r$, the composition is zero. For $n \gg 0$, every element in Q_{n+p+1} arises from Q_{n+1}. Hence $Q_{n+r+1} \to Q_{n+1}$ is zero.

Claim (3) is a diagram-chase. □

Corollary 6.5. *Let $\mathcal{F}^\bullet \in D^+_{coh}(X)$. Suppose $\mathcal{H}^i(\mathcal{F}^\bullet)$ has support proper over A. Then $\mathbb{H}^q(\mathcal{F}^\bullet)^\wedge \to H^q(\widehat{X}, \widehat{\mathcal{F}}^\bullet)$ is an isomorphism. (Here $\widehat{\mathcal{F}}^\bullet \coloneqq i^*\mathcal{F}^\bullet$ is the pullback to $\widehat{X}$.)*

Proof. "Some spectral sequence thing". □

Corollary 6.6. *Suppose $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(X)$ and $X_{/A}$ is proper. Then*

$$R\Gamma(X, R\operatorname{Hom}(\mathcal{F}, \mathcal{G}))^\wedge \xrightarrow{\sim} R\Gamma(\widehat{X}, R\operatorname{Hom}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}}))$$

is an isomorphism.

Proof. From the previous corollary,

$$R\Gamma(X, R\operatorname{Hom}(\mathcal{F}, \mathcal{G}))^\wedge = R\Gamma(X, R\operatorname{Hom}(\mathcal{F}, \mathcal{G})^\wedge).$$

To move the hat inside, we want the natural map $R\operatorname{Hom}(\mathcal{F}, \mathcal{G})^\wedge \to R\operatorname{Hom}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$ to be a quasi-isomorphism. To do so, use that completion is exact, and that it is true locally. □

Theorem 6.7 (Formal GAGA)**.** *Suppose A is an adic Noetherian ring with ideal of definition I, and X is finite type over A. Then there is an equivalence of categories*

$$\mathsf{Coh}_{pr}(X) \to \mathsf{Coh}_{pr}(\widehat{X}), \quad \mathcal{F} \mapsto \widehat{\mathcal{F}}$$

where pr means "proper support over A".

Proof. We want to show this is fully faithful, i.e. given $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(X)$, we want $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$ to be an isomorphism. First replace X with $\operatorname{supp}(\mathcal{F}) \cap \operatorname{supp}(\mathcal{G})$, so that X is proper. By the corollary,

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G})^\wedge \cong \operatorname{Hom}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}}).$$

But $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is a finite-type A-module, since $\mathcal{F}$ and $\mathcal{G}$ have proper support. Hence it is complete (finite type over a complete ring), i.e. $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{F}, \mathcal{G})^\wedge$.

Now we need essentially surjective. Given $\mathcal{F} \in \mathsf{Coh}_{\mathrm{pr}}(\widehat{X})$, we want $\mathcal{G} \in \mathsf{Coh}_{\mathrm{pr}}(X)$ such that $\mathcal{F} \cong \widehat{\mathcal{G}}$. We do the special case where $X_{/A}$ is projective, with $\mathcal{L} \in \operatorname{Pic}(X)$ relatively ample. Choose m such that $\mathcal{F}|_{X \otimes A/I} \otimes \mathcal{L}^{\otimes m}$ is globally generated. Then $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated (exercise). So there is a surjection $(\mathcal{L}^{-m})^{\oplus s} \to \mathcal{F} \to 0$. The same argument applies to the kernel of this surjection, giving

$$(\mathcal{L}^{-m})^{\oplus s'} \xrightarrow{\gamma} (\mathcal{L}^{-m})^{\oplus s} \to \mathcal{F} \to 0.$$

By full faithfulness, $\gamma = \widehat{\beta}$ for some β. Set $\mathcal{G} \coloneqq \operatorname{coker} \beta$, and then note right-exactness of completion. □

Corollary 6.8. *Let $X_{/A}$ be finite over A adic Noetherian. Then*

$$\begin{pmatrix} \textit{closed subschemes of } X \\ \textit{proper over } A \end{pmatrix} \to \begin{pmatrix} \textit{closed formal subschemes of } \widehat{X} \\ \textit{proper over } A \end{pmatrix}$$

is a bijection.

Proof. $\mathcal{O}_Z$ algebraizes, and the map $\mathcal{O}_X \to \mathcal{O}_Z$ also algebraizes. □

Corollary 6.9. *Suppose $X_{/A}$ is proper. Then there is an equivalence of categories*

$$\{\textit{finite } X\textit{-schemes}\} \xrightarrow{\sim} \{\textit{finite } \widehat{X}\textit{-schemes}\}.$$

Proof. Given $\pi: Y \to \widehat{X}$ finite, $\pi_* \mathcal{O}_Y$ algebraizes. The structure maps algebraize, so we get a sheaf of algebras $\widetilde{\mathcal{O}}_Y$ on X. Set $\widetilde{Y} \coloneqq \operatorname{Spec}_X \widetilde{\mathcal{O}}_Y$. □

Theorem 6.10. *Suppose $\mathcal{X}_{/A}$ is a formal scheme proper over A adic Noetherian with ideal of definition I. Then if there exists $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X} \otimes A/I}$ is relatively ample, then there exists an A-scheme X such that $\widehat{X} \cong \mathcal{X}$.*

Proof. Apply the first corollary to $\mathbb{P}(\Gamma(\mathcal{X}, \mathcal{L}^{\otimes N}))$. □

Example 6.11. Suppose A is a complete DVR and X is proper over A. Then $\pi_1^{\text{ét}}(X) \cong \pi_1^{\text{ét}}(X_k)$, i.e. there is an equivalence of categories between finite étale covers of X and X_k. In fact, both are isomorphic to finite étale covers of $\widehat{X}$, by formal GAGA. (The natural map from covers of $\widehat{X}$ to covers of X_k is essentially surjective.)

7 Bend and break

Theorem 7.1 (Mori, Hartshorne's conjecture)**.** *If X is smooth projective and geometrically connected with ample tangent bundle, then X is isomorphic to $\mathbb{P}^n$.*

Definition 7.2. A vector bundle $\mathcal{E}$ is **ample** if $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E}) \coloneqq \operatorname{Proj} \operatorname{Sym}^* \mathcal{E}$ is ample. Equivalently,

1. for any coherent $\mathcal{F}$ and $n \gg 0$, $\operatorname{Sym}^n \mathcal{E} \otimes \mathcal{F}$ is globally generated, or
2. for any coherent $\mathcal{F}$ and $n \gg 0$, $H^i(\operatorname{Sym}^n \mathcal{E} \otimes \mathcal{F})$ for $i > 0$.

Definition 7.3. X is Fano means X is smooth projective and $-K_X$ is ample. (Here $-K_X$ means the numerical class of $K_X^\vee$.)

Theorem 7.4 (Mori)**.** *Suppose X is Fano of dimension n. Then through any point $x \in X$, there exists a rational curve C with $(-K_X) \cdot C \le n + 1$.*

Remark. More is true: Fano varieties are rationally connected.

Definition 7.5. Let $N_1(X)$ denote $\{\text{1-cycles}\}/\sim_{\text{num}}$, which is dual to $N^1(X)$. Let $\operatorname{NE}(X)$ be the cone spanned by effective 1-cycles. Let $\overline{\operatorname{NE}}(X)$ denote the closure.

Theorem 7.6 (Mori, Cone theorem)**.** *Let X be smooth projective. The set $\mathcal{R}$ of K_X-negative extremal rays of $\overline{\operatorname{NE}}(X)$ is countable, and*

$$\overline{\operatorname{NE}}(X) = \overline{\operatorname{NE}}(X)_{K_X \ge 0} + \sum_{R \in \mathcal{R}} R.$$

Moreover, the set $\mathcal{R}$ is locally discrete in $N_1(X)_{K_X < 0}$. Each $R \in \mathcal{R}$ is $\mathbb{R}^+\Gamma$ where Γ is a rational curve with $0 \le -K_X \cdot \Gamma \le \dim X + 1$.

Corollary 7.7. *Let X be Fano. Then $\overline{\operatorname{NE}}(X)$ is polyhedral. Therefore* $\operatorname{Nef}(X)$ *is also polyhedral (as the dual cone).*

Example 7.8 (Simple varieties which are not Fano)**.** The blow-up $\operatorname{Bl}_{C_1 \cap C_2} \mathbb{P}^2$ at the nine points at an intersection of two cubics does not have polyhedral $\overline{\operatorname{NE}}(X)$. If A is an abelian surface, then $\overline{\operatorname{NE}}(X)$ has curved walls.

Example 7.9 (Fano varieties)**.** $\mathbb{P}^n$, hypersurface of degree $\le n$ in $\mathbb{P}^n$, Grassmannian. There is a complete classification in dimensions 2 and 3.

Idea behind bend-and-break: take any curve in the surface and use deformation theory to move it. If we move it a lot, it will break up into pieces, and the pieces want to be rational curves. To move curves, we will have to use characteristic p techniques.

7.1 Moving curves in varieties

Suppose X is smooth and Y is proper, and they are finite type over S. We will study the parameter space $\mathrm{Mor}(Y,X)$ of maps from Y to X, given by

$$\mathrm{Mor}(Y,X)(T) \coloneqq \mathrm{Hom}(T \times Y, X).$$

Proposition 7.10 (Grothendieck)**.** $\mathrm{Mor}(Y,X)$ *is representable by an S-scheme locally of finite type.*

Corollary 7.11. *The locus of $[f] \in \mathrm{Mor}(Y,X)$ such that $f^*\mathcal{L}$ has bounded degree for $\mathcal{L}$ ample is finite type.*

Definition 7.12. Let $f: Y \to X$. Define

$$\mathrm{Def}_f(A) \coloneqq \{\widetilde{f}: Y_A \to X : \widetilde{f}|_Y = f\}.$$

Theorem 7.13. *$T_1 = H^0(Y, f^*T_X)$ and $T_2 = H^1(Y, f^*T_X)$ is a tangent-obstruction theory for Def_f.*

Proof. Choose a square-zero extension $Y \to Y'$ with ideal sheaf $\mathcal{I}$. We want to show there exists $\mathrm{ob} \in H^1(Y, f^*T_X \otimes \mathcal{I})$ such that $\widetilde{f}: Y' \to X$ exists iff $\mathrm{ob} = 0$, and the set of such $\widetilde{f}$ is a torsor for $H^0(Y, f^*T_X)$. Cover Y by affines U_i. For each U_i, there exists a lift of f_i to U_i' by formal smoothness of X. Check that $f_i - f_j: f^{-1}\mathcal{O}_X \to \mathcal{I}$ is a derivation on $U_i \cap U_j$. Hence we get a cocycle $\mathrm{ob} \in \mathrm{Ext}^1(f^*\Omega_X^1, \mathcal{I}) = H^1(Y, f^*T_X \otimes \mathcal{I})$. The difference of two such global lifts $\widetilde{f}_1, \widetilde{f}_2$, the difference $\widetilde{f}_1 - \widetilde{f}_2$ is a derivation $f^{-1}\mathcal{O}_X \to \mathcal{I}$, so we are done. ☐

Corollary 7.14. *Given $f: Y \to X$,*

$$\dim_{[f]} \mathrm{Mor}(Y,X) \geq h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

Definition 7.15. Given $B \subset Y$ and a map $g: B \to X$, define

$$\mathrm{Mor}_B(Y,X)(T) \coloneqq \{f: T \times Y \to X : f|_{T \times B} = g \circ \pi_2\}.$$

Given $f: Y \to X$ with $f|_B = g$, define

$$\mathrm{Def}_{f,B}(A) \coloneqq \{f: Y_A \to X : f|_B = g \circ \pi_B\}.$$

Theorem 7.16. *Let $\mathcal{I}_B$ be the ideal sheaf of B. Then $T_1 = H^0(Y, f^*T_X \otimes \mathcal{I}_B)$ and $T_2 = H^1(Y, f^*T_X \otimes \mathcal{I}_B)$ is a tangent-obstruction theory for $\mathrm{Def}_{f,B}$. Hence*

$$\dim_{[f]} \mathrm{Mor}_{f,B}(Y,X) = h^0(f^*T_X \otimes \mathcal{I}_B) - h^1(f^*T_X \otimes \mathcal{I}_B).$$

Example 7.17. Let C be a smooth curve of genus g, and X be smooth and projective of dimension n. Let $B \subset C$ be an effective Cartier divisor. Then

$$\dim_{[f]} \mathrm{Mor}(C,X) \geq \chi(f^*T_X) = \mathrm{rank}(f^*T_X)(1-g) + \deg(f^*T_X) = n(1-g) + (-K_X)\cdot(f_*C).$$

Similarly,

$$\dim_{[f]} \mathrm{Mor}_B(C,X) \geq (-K_X)\cdot f_*C + n(1-g-\mathrm{length}\, B).$$

Idea behind bend-and-break: if $(-K_X)\cdot C$ is big, then $\dim_{[f]} \mathrm{Mor}(C,X)$ is also big.

7.2 Bend and break

Proposition 7.18 (Bend-and-break 1). *Over an algebraically closed field k, suppose X is projective and smooth, C is a smooth projective curve, $c \in C$ is a closed point, and $f\colon C \to X$. If $\operatorname{Mor}_c(C,X)$ has dimension ≥ 1, there exists a rational curve in X passing through $f(c)$.*

Remark. The hypothesis is satisfied if $(-K_X) \cdot f_*C - g \dim X \geq 1$.

Lemma 7.19 (Rigidity). *Let X, Y, Y' be smooth varieties, and $\pi\colon X \to Y$ and $\pi'\colon X \to Y'$ be proper maps such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. If π' contracts $\pi^{-1}(y_0)$, then there exists a neighborhood Y_0 of y_0 such that π' factors as $\pi'|_{\pi^{-1}(Y_0)}\colon \pi^{-1}(Y_0) \xrightarrow{\pi} Y_0 \to Y$. If π' contracts all fibers of π, then it factors through π.*

Proof of bend-and-break 1. Choose a curve $T \in \operatorname{Mor}_c(C,X)$. Let $\widetilde{T}$ be its normalization and $\overline{T}$ be its compactification. Then we get a rational map $\varphi\colon \overline{T} \times C \dashrightarrow X$, defined over T. Let $S \to X$ be a resolution of this rational map.

Claim: φ is not defined along $\overline{T} \times \{c\}$. Suppose it is. Then φ contracts $\overline{T} \times \{c\}$ to a point $\operatorname{im}(c)$. By the rigidity lemma with $Y = C$ and $\pi = \pi_C$ and $X = Y'$, we see that φ contracts a neighborhood of $\overline{T} \times \{c\}$. This is a contradiction, because this is equivalent to the existence of $U \subset C$ such that φ contracts $\pi_C^{-1}(c')$ for $c' \in U$, and hence $T \to \operatorname{Mor}_c(C,X)$ is constant.

The exceptional divisor in S of the blow-up is a rational curve which passes through $\operatorname{im}(c)$. $\square$

Proposition 7.20 (Bend-and-break 2). *Let X be projective and $f\colon \mathbb{P}^1 \to X$ generically injective. If*

$$\dim_{[f]} \operatorname{Mor}_{\{0,\infty\}}(\mathbb{P}^1, X) \geq 2,$$

then $f_*[\mathbb{P}^1]$ is numerically equivalent to a non-integral connected rational 1-cycle passing through $f(0)$ and $f(\infty)$.

Proof. Choose a smooth curve $T \to \operatorname{Mor}_{\{0,\infty\}}(\mathbb{P}^1, X)$ such that the image is not contained within a $\mathbb{G}_m$-orbit of a map $\mathbb{P}^1 \to X$. This means $T \times \mathbb{P}^1 \to X \times T$ is finite. Let $\overline{T}$ be the smooth compactification of T. Then write

$$\begin{tikzcd}
T \times \mathbb{P}^1 \arrow[r] \arrow[d] & X \times \overline{T} \\
\overline{T} \times \mathbb{P}^1 \arrow[ur, dashed] & S \arrow[u] \\
S' \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where S' is the resolution of the rational map $\overline{T} \times \mathbb{P}^1 \to X \times \overline{T}$, and S is the Stein factorization of the map. The map $\pi\colon S \to \overline{T}$ is flat because S is integral and $\overline{T}$ is a curve. Hence the fibers have arithmetic genus 0. So each fiber is a tree of $\mathbb{P}^1$'s.

We want at least one singular fiber. (Over $T \subset \overline{T}$, the fibers are $\mathbb{P}^1$'s.) Assume otherwise, so that $S \to \overline{T}$ is a ruled surface. Let T_0 and T_∞ be two sections contracted by π_X. Take H ample on X and consider π_X^*H. Then $(\pi_X^*H)^2 > 0$. Also, $\pi_X^*H \cdot T_0 = 0$ by the projection formula, and the same for T_∞. Hence $T_0^2, T_\infty^2 < 0$ by Hodge index theorem. This is a contradiction: $(T_0 - T_\infty)^2 = 0$ because both are pulled back from the base. $\square$

Theorem 7.21. *Let X be Fano of dimension n. Then X is covered by rational curves Γ with $\Gamma \cdot (-K_X) \leq n+1$.*

Proof. Let $\operatorname{char} k = p > 0$. Choose any map $f\colon C \to X$ passing through $x \in X$. Note that

$$\begin{aligned}
\dim_{[f \circ \operatorname{Frob}_C^N]} \operatorname{Mor}_x(C,X) &\geq n(-g(C))(-K_X) \cdot (f \circ \operatorname{Frob}_C^N)_*C \\
&= n(-g(C))p^N(-K_X) \cdot f_*C > 1
\end{aligned}$$

for $N \gg 0$, because $-K_X$ is ample. By bend-and-break 1, there exists a rational curve through x. Now we want a rational curve Γ through x such that $\Gamma \cdot (-K_X) \le n+1$. Call the curve we just constructed $\widetilde{\Gamma}$, and set $f(0) = x$ and $f(\infty) = y$. If $\widetilde{\Gamma} \cdot (-K_X) \le n+1$, then we are done. Otherwise $\dim_{[f]} \operatorname{Mor}_{\{0,\infty\}}(\mathbb{P}^1, X) \ge 2$. By bend-and-break 2, $f_* \mathbb{P}^1$ is numerically equivalent to $g_* \mathbb{P}^1 + h_* \mathbb{P}^1 + \cdots$. Take the component $f' \colon \mathbb{P}^1 \to X$ passing through x. Then $f'_* \mathbb{P}^1 \cdot (-K_X) < f_* \mathbb{P}^1 \cdot (-K_X)$. Hence by induction we are done in characteristic p.

Let $\operatorname{char} k = 0$. Claim: there exists a finite-type $\mathbb{Z}$-algebra R and an R-scheme $\widetilde{X}$ with an R-point $\widetilde{x}$ and an embedding $R \hookrightarrow k$ such that $(\widetilde{X}, \widetilde{x})_k \cong (X, x)$. Choose a model; we may assume it is smooth proper and Fano. Now for each $\mathfrak{m} \in \operatorname{Spec} R$ closed, there exists a rational curve of degree $\le n+1$ passing through $\widetilde{X}$ mod $\mathfrak{m}$. Consider $\operatorname{Mor}_{\widetilde{x}}(\mathbb{P}^1, \widetilde{X})^{\le n+1}$, which is finite type over R and non-empty over every closed point of $\operatorname{Spec} R$. Hence its image in $\operatorname{Spec} R$ is constructible, and contains a dense set of closed points. So its image in $\operatorname{Spec} R$ is all of $\operatorname{Spec} R$. In particular, $\operatorname{Mor}_{\widetilde{x}}(\mathbb{P}^1, \widetilde{X})_k^{\le n+1} = \operatorname{Mor}_x(\mathbb{P}^1, X)^{\le n+1}$ is non-empty. ☐

Theorem 7.22. *Let X be smooth projective, H ample and $f \colon C \to X$ such that $f_* C \cdot K_X < 0$. Then for any $x \in f(C)$, there exists a rational curve Γ passing through x such that*

$$H \cdot \Gamma \le 2 \dim X \frac{H \cdot f_* C}{-K_X \cdot f_* C}.$$

Proposition 7.23 (Bend-and-break 3)**.** *Let X be smooth projective, H ample, $f \colon C \to X$ be a map and $B \subset C$ be a reduced divisor. If $\dim_{[f]} \operatorname{Mor}_B(C, X) \ge 1$, then there exists a rational Γ meeting $f(B)$ such that*

$$H \cdot \Gamma \le 2 \frac{H \cdot f_* C}{\operatorname{length} B}.$$

Proof of theorem. In characteristic p, do the following. Let $b_m := \lfloor -p^m (K_X \cdot f_* C)/\dim X - g(C) \rfloor$, which is positive for $m \gg 0$. Choose any b_m points $B_m \subset C$. Then $\dim_{[f \circ \mathrm{Frob}^m]} \operatorname{Mor}_B(C, X) > 1$ for $m \gg 0$. By bend-and-break 3, there exists Γ_m meeting $f(B_m)$ such that

$$H \cdot \Gamma_m \le 2(H \cdot f_* C)(2p^m / b_m) \to \dim X / (-K_X \cdot C).$$

Hence for $m \gg 0$, we get $H \cdot \Gamma_m \le 2 \dim X (H \cdot f_* C)/\operatorname{length}(B)$. Now we can repeat the previous argument for characteristic 0. ☐

8 Isomonodromic deformations

Definition 8.1. Let X/S be smooth with a vector bundle E on X. A **connection** is a map of sheaves $\nabla \colon E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ satisfying a Leibniz rule $\nabla(fs) = df \otimes s + f \nabla s$. It is **integrable** if $\nabla^2 = 0$. Call the category of vector bundles with integrable connection $\mathsf{MIC}(X/S)$.

Assume we are over a field k of characteristic 0, and that $\pi \colon X \to S$ is smooth and S itself is also smooth. Pick a base point $0 \in S$. Suppose we are given an integrable (E, ∇) on the central fiber X_0. By Ehresmann's theorem, we expect to be able to deform the integrable bundle into ones that have the same monodromy representation.

Definition 8.2. $(E, \nabla \colon E \to E \otimes \Omega^1_{X/S})$ is an isomonodromic deformation of (E_0, ∇_0) if there exists $\widetilde{\nabla} \colon E \otimes \Omega^1_X$ integrable such that $\nabla = \widetilde{\nabla} \bmod \pi^* \Omega^S_1$.

Remark. Assume S is a disk. Given $s \in S$, we have maps $X_s \to X \leftarrow X_0$. The connection $\widetilde{\nabla}$ gives a representation $\pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$. After a choice of path between $x_1 \in X_s$ and $x_0 \in X_0$, we get an isomorphism $\pi_1(X_0) \cong \pi_1(X) \to \pi_1(X_s)$.

Example 8.3. Let $Y \xrightarrow{f} X \xrightarrow{\pi} S$ with f smooth proper. (Think: family of families.) Let $E = R^i f_* \Omega^\bullet_{Y/X}$ and $\nabla \colon E \to E \otimes \Omega^1_{X/S}$ be the Gauss–Manin connection mod $\pi^* \Omega^1_S$.

Theorem 8.4. *Let $R \coloneqq k[[x_1, \dots x_n]]$ and $X/\operatorname{Spf} R$ be smooth. Suppose $(E_0, \nabla_0) \in \mathsf{MIC}(X_0/k)$. Then there exists a unique isomonodromic deformation of (E_0, ∇_0) to $X/\operatorname{Spf} R$.*

Proof. In the special case $X = X_0 \times_k \operatorname{Spf} R \xrightarrow{\pi} X_0$, just set $(E, \widetilde{\nabla}) = \pi^*_{X_0}(E_0, \nabla_0)$. In general, take an affine cover $\{U_i\}$ of X. On each U_i, we get $U_i \xrightarrow{\pi_i} U_i \cap X_0$. We know

$$\operatorname{Hom}_{U_i \cap U_j}((E^i, \widetilde{\nabla}^i), (E^j, \widetilde{\nabla}^j)) = \operatorname{Hom}_{U_i \cap U_j}(E^i, E^j)^{\nabla=0} = \operatorname{Hom}_{U_i \cap U_j \cap X_0}(E_0^i, E_0^j)^{\nabla=0}$$

by analytic continuation. But $E_0^i = E_0^j$ and the identity is the canonical section. Hence we get a cocycle. Using this gluing data, we get $(E, \widetilde{\nabla})$. Now suppose $(E_1, \widetilde{\nabla}_1)$ and $(E_2, \widetilde{\nabla}_2)$ give isomonodromic deformations of (E_0, ∇_0) when reduced mod $\pi^*\Omega^1_S$. Then

$$\operatorname{Hom}_X((E_1, \widetilde{\nabla}_1), (E_2, \widetilde{\nabla}_2)) = \operatorname{Hom}_{X_0}((E_0, \nabla_0), (E_0, \nabla_0)),$$

which contains the identity. □

Theorem 8.5. *Let $X/\operatorname{Spec} R$ be smooth proper. Then in characteristic 0, completion gives an equivalence of categories*

$$\mathsf{MIC}(X/R) \xrightarrow{\sim} \mathsf{MIC}(\widehat{X}/\operatorname{Spf} R).$$

Proof. Suppose $(E, \nabla) \in \mathsf{MIC}(\widehat{X}/\operatorname{Spf} R)$. By formal GAGA, there exists $F \in \mathsf{Vect}(X)$ such that $E = \widehat{F}$. Now we need to algebraize ∇. This does not immediately follow from formal GAGA because ∇ is not $\mathcal{O}_X$-linear.

Claim: the map $\mathsf{Conn}(G) \to \mathsf{Conn}(\widehat{G})$ from connections on G and connections on $\widehat{G}$ is an isomorphism for any $G \in \mathsf{Vect}(X)$. This is because $\mathsf{Conn}(G)$ is a torsor for $\operatorname{End}(G) \otimes \Omega^1_{X/R}$, and similarly $\mathsf{Conn}(\widehat{G})$ is a torsor for $\operatorname{End}(\widehat{G}) \otimes \Omega^1_{\widehat{X}/R}$, and by formal GAGA these two have the isomorphic H^0. So it suffices to show $\mathsf{Conn}(G) \neq 0$ iff $\mathsf{Conn}(\widehat{G}) \neq 0$. The Atiyah class $[\mathsf{Conn}(G)]$ lives in $H^1(X, \operatorname{End}(G) \otimes \Omega^1_{X/S}) = H^1(X, \operatorname{End}(\widehat{G}) \otimes \Omega^1_{X/S})$ (by formal GAGA), which is where the Atiyah class $[\mathsf{Conn}(\widehat{G})]$ lives. A computation shows they are the same class.

Now we need to show $\nabla \in \mathsf{Conn}(G)$ integrable iff $\widetilde{\nabla} \in \mathsf{Conn}(\widehat{G})$ integrable. The curvature ∇^2 lives in $H^0(\operatorname{End}(G) \otimes \Omega^2) = H^0(\operatorname{End}(\widehat{G}) \otimes \Omega^2)$ (by formal GAGA), which is where $\widetilde{\nabla}^2$ lives. Another computation shows they are the same class.

We have proved essential surjectivity. We skip fully faithfulness. □

Remark. Here every time we take Taylor series, we are using characteristic 0.

Example 8.6 (Schlessinger)**.** Take $X_0 = \mathbb{P}^1 - \{\lambda_1, \dots, \lambda_n\}$ and $E_0 = \mathcal{O}^m$ with $\nabla_0 = \sum_{i=0}^n a_{ij}/(x - \lambda_i)dx + d$ **Fuchsian**. (Here (a_{ij}) is a matrix of constants.) The deformation will be $\mathbb{P}^1 - \{\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_n\} \to k[x_1, \dots, x_n]$ where $\widetilde{\lambda}_i \coloneqq \lambda_i + x_i$. Set $E = \mathcal{O}^m$ and $\widetilde{\nabla} = d + \sum_{i=0}^n \widetilde{A}_i/(x_i - \widetilde{\lambda}_i) d(x - \widetilde{\lambda}_i)$. We want to compute these $\widetilde{A}_i$ so that the result is flat. If Y is a flat section, then compute

$$-\frac{\partial}{\partial x} Y = \sum_{j=0}^n \sum \frac{\widetilde{A}_j}{x - \widetilde{\lambda}_j} Y, \quad -\frac{\partial}{\partial \widetilde{\lambda}_i} Y = -\frac{\widetilde{A}_i}{x - \widetilde{\lambda}_i} Y$$

up to sign. We want to pick the $\widetilde{A}_i$ such that

$$\pm \frac{\partial}{\partial x} \frac{\partial}{\partial \widetilde{\lambda}_i} Y = \frac{\widetilde{A}_i}{(x - \widetilde{\lambda}_i)^2} Y + \frac{\widetilde{A}_i}{x - \widetilde{\lambda}_i} \sum_{j=0}^n \frac{A_j}{x - \widetilde{\lambda}_j} Y$$

$$\pm \frac{\partial}{\partial \widetilde{\lambda}_i} \frac{\partial}{\partial x} Y = \sum_{j=0}^n \frac{\partial_{\lambda_i} \widetilde{A}_i}{x - \lambda_j} Y + \frac{\widetilde{A}_i}{(x - \widetilde{\lambda}_i)^2} Y + \sum \frac{\widetilde{A}_i}{x - \lambda_j} \frac{-\widetilde{A}_i}{x - \widetilde{\lambda}_i} Y$$

are equal. Equating, we get

$$\sum_j \frac{\partial_{\lambda_i} A_j}{x - \widetilde{\lambda}_j} = \sum_j \frac{[A_j, A_i]}{(x - \widetilde{\lambda}_i)(x - \widetilde{\lambda}_j)}.$$

Taking residues around each $\widetilde{\lambda}_j$, we get

$$\frac{\partial \widetilde{A}_j}{\partial \lambda_i} = \begin{cases} [A_i, A_j]/(\lambda_i - \lambda_j) & i \neq j \\ -\sum_{j \neq i} [A_i, A_j]/(\lambda_i - \lambda_j) & i = j. \end{cases}$$

These are the Schlessinger equations.

In characteristic p, MIC does not have canonical deformations, but $\mathcal{D}$-modules do. Instead of calling them $\mathcal{D}$-modules we will call them stratified sheaves.

Definition 8.7. Let X/k be a scheme, with k of characteristic p. A **stratified sheaf** on X is a sequence (E_i, σ_i) with $\sigma_i \coloneqq E_i \xrightarrow{\sim} \operatorname{Frob}^* E_{i+1}$.

Theorem 8.8 (Gieseker)**.** *If X is smooth, there is an equivalence of categories*

$$\mathcal{D}\text{–}\mathsf{mod} \xrightarrow{\sim} \mathsf{Strat}(X/k).$$

Example 8.9. Suppose we have $\operatorname{Frob}^* E = \mathcal{O}_X \otimes \operatorname{Frob}^{-1}(E)$ and D is an order-1 differential operator. Then the action is defined as

$$D(f \otimes e) \coloneqq Df \otimes e.$$

Remark. Suppose ∂ is a derivation. Then $\partial|_{\mathcal{E}_0}$ is defined by $\partial(f \otimes e') = \partial f \otimes e'$ where $f \in \mathcal{O}_X$ and $e \in \Gamma(\mathcal{E}_0)$.

Example 8.10 (More differential operators in characteristic p)**.** Let $X = \mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$ be affine. There is a differential operator $(\partial/\partial x_1)^p/p!$ which sends $x_1^a \mapsto 0$ for $a < p$ and $x_1^p \mapsto 1$.

Definition 8.11. Let $X \xrightarrow{\pi} S$. Take the Cartesian square

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "\mathrm{Frob}"] & S. \end{tikzcd}$$

Write the map $X \xrightarrow{\mathrm{Frob}} X$ making the outer square commute. The **relative Frobenius** is the resulting map $\operatorname{Frob}_{X/S} \colon X \to X^{(p)}$.

Example 8.12. If $S = \operatorname{Spec} k$ and $X = \mathbb{A}^n$, then the absolute Frobenius sends everything in $k[x_1, \dots, x_n]$ to the p-th power, including things in k. The relative Frobenius fixes k and sends only the x_i to the p-th power.

Definition 8.13. Let $X \to S$ and S be over k with $0 \in S$ a k-point. A **stratified sheaf** $(\mathcal{E}_i, \sigma_i)$ on X/S is an isomonodromic deformation of $(\mathcal{E}_i^0, \sigma_i^0)$ on X_0/S if it is restricted from a stratified sheaf on X/k.

Theorem 8.14. *Let $S \coloneqq \operatorname{Spf} k[[x_1, \dots, x_r]]$ and X/S is schematic, i.e. base change to a closed Artin subscheme produces a scheme. Then any stratified sheaf $(\mathcal{E}_i, \sigma_i)$ on X_1/k admits a unique isomonodromic deformation.*

Proof. It suffices to show the theorem for $S_n \coloneqq k[[x_1, \dots, x_n]]/\mathfrak{m}^N$. The map $S_N \xrightarrow{\mathrm{Frob}^N} S_N$ factors through $\operatorname{Spec} k$. The map $X_N \xrightarrow{\mathrm{Frob}_{X/k}} X_N$ factors through X_1. Set $\widetilde{\mathcal{E}}_0 = \widetilde{\operatorname{Frob}}_{X/k}^{N*} \mathcal{E}_N$. In general, set $\widetilde{\mathcal{E}}_i = \widetilde{\operatorname{Frob}}_{X/k}^{N*} \mathcal{E}_{i+N}$. This is obviously a stratified sheaf. Uniqueness is an exercise. ☐

Theorem 8.15. *Stratified sheaves algebraize.*

9 Cotangent complex

We give two motivations for the cotangent complex.

1. Recall if $X \to S$ is smooth and S' is a square-zero thickening of S with ideal sheaf $\mathcal{I}$, then there exists a canonical class $\mathrm{ob} \in \mathrm{Ext}^2(\Omega^1_{X/S}, \mathcal{I})$ such that $\mathrm{ob} = 0$ iff there is a flat deformation, and if $\mathrm{ob} = 0$ then deformations are a torsor for $\mathrm{Ext}^1(\Omega^1_{X/S}, \mathcal{I})$. Furthermore, the automorphism group of any deformation is $\mathrm{Hom}(\Omega^1_{X/S}, \mathcal{I})$. What happens if $X \to S$ is not smooth?

2. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, there are all kinds of interesting exact sequences. For example,
$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0,$$
and it is left-exact if $X \to Y$ is smooth. If in addition f is a closed embedding with ideal sheaf $\mathcal{I}$, then
$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0,$$
and it is left-exact if $X \to Z$ is smooth. If in addition $X \to Z$ is a closed embedding, then
$$\mathcal{I}_{X/Z}/\mathcal{I}^2_{X/Z} \to \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \to f^*\Omega_{Y/Z} \to 0,$$
and it is left-exact if $Y \to Z$ is smooth. If in addition $Y \to Z$ is a closed embedding, then
$$f^*\mathcal{I}_{Y/Z}/\mathcal{I}^2_{Y/Z} \to \mathcal{I}_{X/Z}/\mathcal{I}^2_{X/Z} \to \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \to 0$$
and it is left-exact if $X \to Y$ is lci. The guess is that there is a distinguished triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ in $D^-_{\mathrm{coh}}(X)$ with $H^0(L_{X/Z}) = \Omega^1_{X/Z}$, and sometimes we should be able to guess $H^1 = \mathcal{I}/\mathcal{I}^2$. This will be the cotangent complex.

9.1 Naive cotangent complex

The naive cotangent complex $\mathrm{NL}_{X/S}$ will be the first approximation $\tau_{\geq -1}L_{X/S}$ to $L_{X/S}$. Let $R \to S$ be a ring map. Suppose we can put $\operatorname{Spec} S \xrightarrow{f} \mathbb{A}^N \xrightarrow{g} \operatorname{Spec} R$ and f is a closed embedding. Then we get
$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{\mathbb{A}^N/\operatorname{Spec} R} \to \Omega^1_{S/R} \to 0,$$
and we should define $\mathrm{NL}_{S/R} = [\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{\mathbb{A}^N/\operatorname{Spec} R}]$. This is not obviously functorial and we want a more canonical way to do this.

Definition 9.1. View S as a set and let $R[S]$ be the polynomial R-algebra on the elements of S. There is an R-algebra map $R[S] \to S$ sending $[s] \mapsto s$ with kernel $\mathcal{I}$ generated by $[s] + [s'] - [s + s']$, $[s][s'][ss']$, and $[r] - r$ for $r \in R$. The **naive cotangent complex** $\mathrm{NL}_{S/R}$ is
$$\mathrm{NL}_{S/R} \coloneqq [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_{R[S]/R} \otimes_{R[S]} S]$$
in (cohomological) degrees -1 and 0.

Remark. This definition is evidently functorial, but is not computable.

Proposition 9.2. $H^0(\mathrm{NL}_{S/R}) = \Omega^1_{S/R}$.

Definition 9.3. A **presentation** of $R \to S$ is a polynomial R-algebra P and a surjection $P \to S$ of R-algebras. Given a presentation $\alpha\colon P \to S$ with kernel $\mathcal{I}$, define
$$\mathrm{NL}(\alpha) \coloneqq \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{P/R} \otimes_P S.$$

Suppose we have a morphism of presentations α and α'. Then there is a morphism $\mathrm{NL}(\alpha) \to \mathrm{NL}(\alpha')$

Lemma 9.4. *Suppose we have*

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

and α is a presentation of S/R and α' is a presentation of S'/R'. Then:

1. *there exists a morphism of presentations;*
2. *any two morphisms of presentations induce homotopic maps $\mathrm{NL}(\alpha) \to \mathrm{NL}(\alpha')$;*
3. *these homotopies are compatible with composition;*
4. *if $R \to R'$ and $S \to S'$ are isomorphisms, any map of presentations induces a homotopy equivalence.*

Proof. (1) is trivial. Suppose φ, φ' are morphisms $P \to P'$. Then we have two morphisms of complexes

$$\begin{tikzcd} \mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] \arrow[d, "\varphi_{-1}"', "\varphi'_{-1}"] & \Omega^1_{P/R} \otimes_P S \arrow[d, "\varphi_0"', "\varphi'_0"] \\ \mathcal{I}'/\mathcal{I}'^2 \arrow[r, "d"] & \Omega^1_{P'/R'} \otimes_{P'} S' \end{tikzcd}$$

and we want a map $h: \Omega^1_{P/R} \otimes_P S \to \mathcal{I}'/\mathcal{I}'^2$ such that $h \circ d = \varphi_{-1} - \varphi'_{-1}$ and $d \circ h = \varphi_0 - \varphi'_0$. Take $\varphi - \varphi' : P \to P'$, which factors through $\mathcal{I}'$. Compose with $\mathcal{I}' \to \mathcal{I}'/\mathcal{I}'^2$. The composition $D: P \to \mathcal{I}'/\mathcal{I}'^2$ is a derivation (exercise). Then h comes from the universal property of Ω^1. $\square$

Corollary 9.5. *For any presentation α, there is a canonical homotopy equivalence $\mathrm{NL}(\alpha) \xrightarrow{\sim} \mathrm{NL}_{S/R}$.*

Example 9.6. Suppose $R \to S$ is a polynomial algebra. Then $\mathrm{NL}_{S/R} \cong [0 \to \Omega^1_{S/R}]$. (Just write S as a presentation of itself.)

Theorem 9.7. *Let $0 \to I \to A' \to A \to 0$ be a square-zero extension, with two square-zero extensions $0 \to N_1 \to B'_1 \to B_1 \to 0$ and $0 \to N_2 \to B'_2 \to B_2 \to 0$ over it. Suppose we have maps $B_1 \to B_2$ and $N_1 \to N_2$. Then there exists a map $B'_1 \to B'_2$ extending $B_1 \to B_2$ iff the canonical class $\mathrm{ob} \in \mathrm{Ext}^1_{B_1}(\mathrm{NL}_{B_1/A}, N_2)$ vanishes. If $\mathrm{ob} = 0$, then solutions are a torsor for $\mathrm{Hom}_{B_1}(\Omega^1_{B_1/A}, N_2)$.*

Proposition 9.8. *Let A be a ring and I be an A-module.*

1. *The set of square-zero extensions $0 \to I \to A' \to A \to 0$ is canonically in bijection with $\mathrm{Ext}^1(\mathrm{NL}_{A/\mathbb{Z}}, I)$.*
2. *Given a diagram*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & A' \arrow[r] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

   of square-zero extensions $\alpha \in \mathrm{Ext}^1_A(\mathrm{NL}_{A/\mathbb{Z}}, I)$ and $\beta \in \mathrm{Ext}^1_B(\mathrm{NL}_{B/\mathbb{Z}}, J)$ respectively, a middle arrow exists iff α and β map to the same thing in $\mathrm{Ext}^1_A(\mathrm{NL}_{A/\mathbb{Z}}, J)$.

9.2 Using the cotangent complex

Theorem 9.9. *Suppose $X \xrightarrow{\pi} S$ is a geometrically reduced lci curve. Then Def_X is formally smooth.*

Proof. Obstructions live in $\mathrm{Ext}^2(L_{X/S}, \pi^*\mathcal{I})$. Use that $L_{X/S}$ is supported in degrees -1 and 0 because X is lci. There is a spectral sequence

$$\mathrm{Ext}^i(H^{-j}(L_{X/S}), \pi^*\mathcal{I}) \Rightarrow \mathrm{Ext}^{i-j}(L_{X/S}, \pi^*\mathcal{I}).$$

It suffices to check $\mathrm{Ext}^1(H^{-1}(L_{X/S}), \pi^*\mathcal{I}) = 0$. Observe that $H^{-1}(L_{X/S})$ is torsion; here we used that $L_{X/S}$ "commutes" with localization. Since $L_{X/S}$ is perfect of tor amplitude $(-1, 0)$, this implies actually that $H^{-1} = 0$. □

Theorem 9.10. *Suppose $X/\mathbb{F}_p$ is such that $\mathrm{Frob}_{abs}: X \to X$ is an isomorphism, i.e. X is* **perfect**. *Then for any local Artin $\mathbb{Z}_p$-algebra R with residue field $\mathbb{F}_p$, there exists a unique deformation of X to R. Equivalently, Def_X is the constant functor $\{pt\}$.*

Proof. We want $\mathrm{Ext}^2(L_{X/S}, \pi^* I) = \mathrm{Ext}^1(L_{X/S}, \pi^* I) = 0$. It suffices to show $L_{X/\mathbb{F}_p} = 0$. There is a triangle

$$\mathrm{Frob}^* L_{X/\mathbb{F}_p} \to L_{X/\mathbb{F}_p} \to L_{\mathrm{Frob}}$$

which arises from the diagram $X \to X \to \mathrm{Spec}\,\mathbb{F}_p$. Since Frob is an isomorphism, $\mathrm{Frob}^* L_{X/\mathbb{F}_p} \to L_{X/\mathbb{F}_p}$ is an isomorphism. This map is also zero, because $d(x^p) = 0$. (We will see this later in the construction of L.) Hence $L_{X/\mathbb{F}_p} = 0$ as well. □

Corollary 9.11. *Witt vectors exist for perfect fields.*

Example 9.12. Take an abelian variety in characteristic p. Consider the inverse limit of the system of multiplication by p maps $\varprojlim(\cdots \to A \xrightarrow{[p]} A)$. This inverse limit is perfect.

9.3 Construction and properties

Idea: for the naive cotangent complex we had the map $R[S] \to S$. Now we just continue this resolution.

Definition 9.13. The **simplex category** Δ has objects non-empty totally ordered finite sets $[n]$ and morphisms order-preserving maps. Let $d_i: [i] \to [i+1]$ be the map which "skips" i.

Definition 9.14. If $\mathcal{C}$ is a category, a **simplicial object** in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The category of simplicial objects in $\mathcal{C}$ is called $s\mathcal{C}$.

Example 9.15. The Dold–Kan equivalence says $s\mathsf{Ab}$ is equivalent to cochain complexes of abelian groups supported in non-positive degree. A wrong functor sends $A^\bullet$ to $\cdots A^\bullet[2] \xrightarrow{d} A^\bullet[1] \xrightarrow{d} A^\bullet[0]$ where $d = \sum(-1)^i d_i$. This is not an equivalence, but the resulting functor is homotopy equivalent to the functor that actually induces the equivalence.

Definition 9.16. Let $R \to S$ be a ring map. Consider

$$\cdots \to R[R[S]] \rightrightarrows R[S] \to S$$

The two maps $R[R[S]] \to R[S]$ are $[\sum a_i[s_i]] \mapsto \sum a_i[s_i]$ and $[\sum a_i s_i]$. The map back is $[s] \mapsto [[s]]$. We continue the left analogously. Write $\cdots \to P_1 \to P_0 \coloneqq R[S]$ for this complex which resolves S. The **cotangent complex** is

$$L_{S/R} \coloneqq (\Omega^1_{P_i/R} \otimes_{P_i} S).$$

Remark. There is a functor $\mathsf{Set} \to R\text{–}\mathsf{alg}$ given by the free algebra functor $R[-]$, left adjoint to the forgetful functor F. We get a map $R[-]F \to \mathrm{id}$. In this setting where we have an adjunction satisfying some mild conditions, the construction $[n] \mapsto (R[-]F)^n(S)$ gives a simplicial set, which gives a resolution of $S[0]$.

Proposition 9.17. *Properties of the cotangent complex:*

1. $\tau_{\geq 1} L_{S/R} = \mathrm{NL}_{S/R}$;
2. *(functoriality) there is a natural map* $L_{S/R} \otimes_S S' \to L_{S'/R'}$.

Remark. In characteristic 0 we can do all this with dg-algebras, and then we get a much smaller resolution of S.

Definition 9.18. Let $\Sigma \coloneqq \Sigma_2 \to \Sigma_1 \to \Sigma_0 \xrightarrow{\varepsilon} B$ be an **augmented free simplicial A-algebra**, i.e.

1. $\Sigma_i = A[X_i]$ for some set X_i,
2. $s_r(Y) = X_{ir}$ for $Y \in X_i$ where s_r is the r-th degeneracy map,

and the complex $\cdots \to \Sigma_2 \xrightarrow{d} \Sigma_1 \xrightarrow{d} \Sigma_0 \xrightarrow{\varepsilon} B \to 0$ is exact.

Lemma 9.19. $L_{B/A}$ *is canonically isomorphic in* $D^-(B)$ *to the complex*

$$\cdots \to \Omega^1_{\Sigma_i/A} \otimes_{\Sigma_i} B \to \Omega^1_{\Sigma_{i-1}/A} \otimes_{\Sigma_{i-1}} B \to \cdots.$$

Remark. In the non-affine case, a finite type $X \to Y$ gives $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, and then find a resolution of $\mathcal{O}_X$, e.g. use the canonical simplicial resolution. Then $L_{X/Y} \in D^-\mathsf{Coh}(X)$.

Theorem 9.20. *Let* $f\colon X \to Y_0$ *with* $Y_0 \hookrightarrow Y_1$ *a closed embedding defined by a square-zero ideal sheaf* $\mathcal{I}$. *There exists a functorial class* $\mathrm{ob}(f) \in \mathrm{Ext}^2_X(L_{X/Y_0}, f^*\mathcal{I})$ *such that:*

1. *a lift* X_1 *making the square Cartesian exists iff* $\mathrm{ob}(f) = 0$;
2. *if* $\mathrm{ob}(f) = 0$, *solutions are a torsor for* $\mathrm{Ext}^1_X(L_{X/Y_0}, f^*\mathcal{I})$;
3. *automorphisms of the solutions are* $\mathrm{Hom}(L_{X/Y_0}, f^*\mathcal{I})$.

Lemma 9.21. *Some properties of the cotangent complex for computation:*

1. *if* $f\colon X \to Y$ *is smooth, then* $\Omega^1_f = \Omega^1_{X/Y}[0]$;
2. *if* $g\colon Z \hookrightarrow X$ *is lci, then* $L_{Z/X} = \mathcal{I}_Z/\mathcal{I}_Z^2[1]$;
3. *if* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *there is a distinguished triangle in* $D^-(X)$

$$Lf^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \xrightarrow{[1]};$$

4. *given a Cartesian diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z, \end{tikzcd}$$

there is a natural map $L\widetilde{g}^*L_{X/Z} \to L_{X\times_Z Y/Y}$ *which is an isomorphism if* g *is flat.*

Remark. This lemma encapsulates all the exact sequences we saw when motivating the existence of the cotangent complex, at the beginning of this section.

9.4 More applications

Theorem 9.22. *Let k be a field, and X/k be a geometrically generically reduced lci curve. Then X is unobstructed.*

Proof. We want $\mathrm{Ext}^2(L_{X/k}, f^*\mathcal{I}) = 0$. Use the local-to-global spectral sequence

$$H^i(X, \mathcal{E}xt^j(L_{X/k}, f^*\mathcal{I})) \Rightarrow \mathrm{Ext}^{i+j}(L_{X/S}, f^*\mathcal{I}).$$

All terms on the E_2 page which go to Ext^2 are zero, as follows.

1. $(i = 2, j = 0)$ $H^2(X, -) = 0$ because $\dim X = 1$.
2. $(i = 0, j = 2)$ $\mathcal{E}xt^2(L_{X/k}, f^*\mathcal{I}) = 0$ because $L_{X/k}$ is perfect, so in particular H^0 of it vanishes.
3. $(i = 1, j = 1)$ Since X is generically smooth, $\mathcal{E}xt^1(L_{X/S}, f^*\mathcal{I})$ has zero-dimensional support. Hence H^1 of it vanishes. ☐

Example 9.23. Suppose X is smooth over $\mathbb{F}_p$, and let $F\colon X \to X$ be the absolute Frobenius. Write the exact triangle corresponding to $X \xrightarrow{F} X \to \mathbb{F}_p$:

$$F^* L_{X/\mathbb{F}_p} \to L_{X/\mathbb{F}_p} \to L_F \xrightarrow{[1]}.$$

This is just the triangle

$$F^*\Omega^1_X \xrightarrow{0} \Omega^1_X \to L_F \xrightarrow{[1]}$$

because d on p-th powers vanishes. Hence $L_F = [F^*\Omega^1_X \xrightarrow{0} \Omega^1_X]$.

Example 9.24. Consider $y^2 = x(x-1)(x-p)$. In characteristic 0 this is a donut; in characteristic p it is a nodal cubic, with universal cover an infinite chain of $\mathbb{P}^1$'s. The covering map has zero cotangent complex. Hence there exists a canonical lift over $\mathrm{Spf}\,\mathbb{Z}_p$.

Example 9.25. Let $A/\mathbb{F}_p$ be an abelian scheme and consider $\widetilde{A}\colon \varprojlim_{[p]} A$. Then $F\colon \widetilde{A} \to \widetilde{A}$ is an isomorphism. This is because the multiplication by $[p]$ operator factors as $A \xrightarrow{V} A \xrightarrow{F} A$, so on $\widetilde{A}$ the inverse to F is therefore V. Hence the cotangent complex $L_{\widetilde{A}} = 0$ is zero.

In fact, if we do $\widetilde{\widetilde{A}} \coloneqq \varprojlim_{[n]} A$, again $L = 0$. Let $A/\mathbb{Z}_p$ be an abelian variety and let $\overline{A} \coloneqq (\varprojlim_{[n]} A)^\wedge$ be the completion at p. This is the unique lift of $\widetilde{\widetilde{A}}/\mathbb{F}_p$. This only depends on the isogeny class of the special fiber.

Example 9.26. Let $X/\mathcal{O}_{\mathbb{C}_p}$ be a curve of genus ≥ 1. Take $\overline{X} \coloneqq (\varprojlim_{X' \to X} X')^\wedge$ where X' is finite and étale over $\mathbb{C}_p$, and the completion is at the maximal ideal $\mathfrak{m}$. Then this also only depends on X mod $\mathfrak{m}$, by the same proof as the above example. Question: if $g \geq 2$, does $\overline{X}$ depend on X at all? Related question: in particular, is it true that for X and Y smooth proper curves of genus ≥ 2 over $\overline{\mathbb{F}}_p$, do they have a finite étale cover in common?

10 Gauss–Manin connection

Let $X \to Y$ be a smooth proper map with Y smooth and everything is over $\mathbb{C}$. Consider $R^i f^{\mathrm{an}}_* \underline{\mathbb{C}} \otimes \mathcal{O} = (Rf_*\Omega^\bullet_{\mathrm{dR},X/Y})^{\mathrm{an}}$. Clearly the lhs has a connection, so the rhs has a corresponding connection called the **Gauss–Manin connection**. The rhs terms have a Hodge filtration given by

$$F^i Rf_*\Omega^\bullet_{\mathrm{dR},X/Y} \coloneqq Rf_*\Omega^{\geq i}_{\mathrm{dR},X/Y}$$

and the Gauss–Manin connection ∇ satisfies **Griffiths transversality**

$$\nabla(F^i) \cong F^{i-1} \otimes \Omega^1_Y.$$

Since $\nabla(fs) = f\nabla(s) + df \otimes s$ and $df \otimes s$ is clearly in $F^i \otimes \Omega^1$, it follows that the map

$$\nabla\colon F^i \to F^{i-1} \otimes^1_Y \to F^{i-1}/F^i \otimes \Omega^1_Y$$

is $\mathcal{O}_Y$-linear. Let $\mathrm{gr}_i \nabla\colon F^i/F^{i+1} \to F^{i-1}/F^i \otimes \Omega^1_Y$ be the associated graded of the ∇. By definition, this is the same as

$$\mathrm{gr}_i \nabla\colon R^{j-i} f_* \Omega^i_{X/Y} \to R^{j-i+1} f_* \Omega^{i-1}_{X/Y} \otimes \Omega^1_Y.$$

Pick a base point, so this becomes $H^{j-i}(X_{Y,y}, \Omega^i) \to H^{j-i+1}(X_{Y,y}, \Omega^{i-1})$. We get a canonical class in $H^1(X_y, T_{X_y})$ called the **Kodaira–Spencer class** such that this map is cupping with it.