Notes for student algebraic geometry seminar (Fall 2018): Cubic fourfolds, Rationality, etc.

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Abstract

These are my live-texed notes for (some subset of) the Fall 2018 student algebraic geometry seminar on cubic fourfolds. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 Sep 10 (Raymond)

Sorry, no notes!

2 Sep 17 (Carl): Hodge theory of cubic four-folds

Suppose $X_{/\mathbb{C}}$ is smooth projective of dimension n. Recall that there is a decomposition

$$H^k(X,\mathbb{Z})\otimes\mathbb{C}\cong\bigoplus_i H^{i,k-i}(X),\quad H^{i,k-i}(X)\coloneqq H^{k-i}(X,\Omega_X^i).$$

There are two structures on this decomposition:

1. a conjugation action on the lhs where $H^{i,k-i} = \overline{H^{k-i,i}}$;
2. a non-degenerate integral quadratic form on $H^k(X,\mathbb{Z})$ given by

$$(\alpha,\beta) \coloneqq \int \alpha \wedge \beta \wedge \omega^{n-k}.$$

(Maybe there are some signs in front, but it doesn't matter.) The Hodge–Riemann bilinear relations tell us the signature of the form $(-,-)$ and the Hodge decomposition is orthogonal with respect to it.

Now specialize to cubic four-folds. There are standard computations which give the Hodge diamond for cubic hypersurfaces in $\mathbb{P}^5$:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 1 & & 21 & & 1 & & 1 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}.$$

The only interesting part is in H^4. Make the following observations about it.

1. By universal coefficient, there is no torsion in H^4, so it is an honest lattice and the pairing really is just intersection.
2. By Poincaré duality, it is unimodular. Hodge–Riemann bilinear relations tells us it has signature $(21,2)$.
3. If $h \in H^2(X,\mathbb{Z})$ is the hyperplane, $(h^2)^2 = 3$ so that H^4 is an **odd** lattice.

From this information, the classification of lattices tells us that

$$H^4(X,\mathbb{Z}) \cong (+1)^{21} \oplus (-1)^2.$$

What about primitive cohomology, i.e. $H^4(X,\mathbb{Z})_0 \coloneqq \{x \in H^4 : x \cdot h = 0\}$? Recall that given any cubic fourfold X, we can associate to it its Fano variety of lines $F(X)$, which is smooth and projective of dimension 4 (via the hyperplane class g from the Plücker embedding). When X is a Pfaffian cubic, $F(X) \cong \mathrm{Hilb}^2(K3)$. Here $F(X)$ is an example of a holomorphic symplectic variety, but this property is deformation invariant, and all cubic fourfolds are deformation equivalent. So in fact $F(X)$ is *always* holomorphic symplectic for cubic fourfolds.

Theorem 2.1 (Beauville–Donagi)**.** *There is an Abel–Jacobi map*

$$\alpha \colon H^4(X,\mathbb{Z})_0 \xrightarrow{\sim} H^2(F,\mathbb{Z})_0(-1)$$

which is an isomorphism of polarized Hodge structures. It is given by the incidence correspondence

$$P \coloneqq \{(L,p) \in F \times X : p \in L\}.$$

Remark. Here $H^2(F,\mathbb{Z})_0$ carries the canonical Bogomolov–Beauville form, since F is holomorphic symplectic. In this case, the form is something like

$$(u,v) \coloneqq -\frac{1}{6} g^2 uv.$$

Proof sketch. First check that α respects the pairings; this is just some computation. Then it suffices to show that the two lattices are abstractly isomorphic, i.e. they have the same rank and discriminant (up to sign).

1. We know $H^4(X,\mathbb{Z})_0$ is rank 22. The full H^4 is unimodular, and the hyperplane class has norm 3, so the discriminant of its orthogonal complement is ± 3.

2. To show $H^2(F,\mathbb{Z})_0$ has the same property, it suffices to show it for one F. Assume $F = \mathrm{Hilb}^2(K3)$ arises from Pfaffian X. Then we can just do an explicit calculation using

$$H^2(\mathrm{Hilb}^2(K3)) = H^2(K3) \oplus \frac{1}{2}\mathbb{Z}(\delta)$$

   where δ is the exceptional divisor. This gives rank 22 and discriminant 3.

Then one shows $\alpha(h^2) = g$ geometrically via the correspondence, from which we get compatibility of Hodge structures. □

By explicitly computing $H^2(F)_0$, the conclusion is that

$$H^4(X)_0 = H^2(F)_0 = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2} \oplus (-E_8)^{\oplus 2}.$$

Theorem 2.2 (Torelli for cubic four-folds (Voisin, Looijenga, Charles))**.** *Let X, X' be cubic four-folds. Suppose there is an isomorphism of polarized Hodge structures*

$$\varphi\colon H^4(X,\mathbb{Z}) \xrightarrow{\sim} H^4(X',\mathbb{Z}).$$

Then φ comes from a unique projective isomorphism $X \xrightarrow{\sim} X'$, i.e. it preserves the hyperplane class.

Definition 2.3. The period map takes X to the embedding of a line $H^{3,1} \subset H^4(X) \otimes \mathbb{C}$. More formally, fix a marked lattice (L,u) abstractly isomorphic to (H^4, h^2) and let $L^0 \coloneqq u^{\perp}$. Let $\mathcal{M}$ be the moduli of **marked cubic four-folds**

$$\mathcal{M} \coloneqq \{(X, \phi\colon (H^4(X,\mathbb{Z}), h^2) \xrightarrow{\sim} (L,u))\}.$$

The **period domain** is

$$\mathcal{D} \coloneqq \{x \in \mathbb{P}(L_0 \otimes \mathbb{C}) : (x,x) = 0,\ (x,\bar{x}) > 0\}.$$

The **period map** is

$$p\colon \mathcal{M} \to \mathcal{D}, \quad X \mapsto \phi(H^{3,1}).$$

Remark. Fact: $\mathcal{M}$ is an algebraic variety (since all X live in the same projective space; contrast this with K3s). A quick dimension count:

1. $\dim \mathcal{M} = (\binom{8}{3} - 1) - (6^2 - 1) = 20$ (PGL action on equation in $\mathbb{P}^5$);

2. $\dim \mathcal{D} = (22 - 1) - 1 = 20$.

There is a standard computation (Griffiths residue formula) one can do to show that p is a local isomorphism (see Voisin's book).

Theorem 2.4 (Torelli)**.** *p is injective.*

Analogue for holomorphic symplectic varieties:

1. $\mathcal{M}'$ the connected component of **marked** holomorphic symplectic varieties, i.e. with $\varphi\colon (L', u') \to (H^2(F,\mathbb{Z}), g)$;

2. period map $p'\colon \mathcal{M}' \to \mathcal{D}'$ given by $(F,\varphi) \mapsto \varphi(H^{2,0}(F))$.

Theorem 2.5 (Torelli for hyperkählers (Verbitsky, Huybrechts, Markman))**.** *p' is generically injective and a local isomorphism.*

Remark. Note that p' can only fail to be an isomorphism at non-separated points, and non-separated points do exist on $\mathcal{M}'$. (Think moduli of K3s, where birational K3s have the same period, but can be related only by Mukai flops.)

Fix a copy of $L_0 \subset L'$ where $L_0 = u^\perp$. Let $\mathcal{N}$ be the incidence correspondence

$$\mathcal{N} := \{(X, F(X), \phi, \psi) : \psi \circ \alpha = \varphi\}$$

i.e. require compatibility with Abel–Jacobi. So we now have

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\pi'"] \arrow[d, "\pi"'] & \mathcal{M}' \arrow[d, "p'"'] \\ \mathcal{M} \arrow[r, "p"] & \mathcal{D} \end{tikzcd}$$

We know π is injective, p' is generically injective (by Torelli for hyperkählers).

Lemma 2.6. *π, π' are injective, and π, π', p, p' are local isomorphisms.*

Proof. Injectivity of π: above (X, ϕ) sits $(F(X), \psi)$ and we need

$$\psi\colon (H^2(F(X)), g) \to (L', u')$$

to be uniquely specified. This is true up to sign; $\mathcal{M}$ apparently has two connected components and $\mathcal{N}$ has one.

The content of injectivity of π' is that given a polarized F, it can only come from one (polarized) X; this is something classical. If we have $X, X' \subset \mathbb{P}(V)$ with $F, F' \subset \mathrm{Gr} \subset \mathbb{P}(\wedge^2 V)$ with projective isomorphism $F \to F'$, the intermediate step is to show the isomorphism must preserve the Grassmannian Gr. □

Proof of theorem. Let $O(L, u)$ be the orthogonal group of L fixing u. Let $O^+(L, u)$ be the index 2 part with positive determinant. These both act on $\mathcal{M}$ and $\mathcal{D}$ and p is equivariant. We know $\mathcal{M}$ and $\mathcal{D}$ each have two components fixed by O^+.

It suffices to show p is injective on one connected component of $\mathcal{M}$. Let $\mathcal{N}_0$ be a connected component of $\mathcal{N}$, and let $\mathcal{M}_0$ and $\mathcal{M}'_0$ be the closures of the images of $\mathcal{N}_0$. In

$$\begin{tikzcd} \mathcal{N}_0 \arrow[r, "\pi'"] \arrow[d] & \mathcal{M}'_0 \arrow[d] \\ \mathcal{M}_0 \arrow[r] & \mathcal{D} \end{tikzcd}$$

we know π, π' are injective and dominant, and $\mathcal{M}'_0 \to \mathcal{D}$ is generically injective. Hence $p' \circ \pi'$ is generically injective, and with π dominant this implies p generically injective. Since $\mathcal{M}_0$ is separated and p is a local isomorphism, it follows that p is injective. □

3 Oct 01 (Raymond): Special cubic fourfolds

In general, the Picard rank of a cubic hypersurface is 1. So we want to look for cubic fourfolds with extra algebraic classes.

Definition 3.1. A cubic fourfold X is **special** if it has an algebraic surface $S \hookrightarrow X$ not homologous to a complete intersection.

Remark. This condition is equivalent to asking $\operatorname{rank} H^{2,2}(X, \mathbb{Z}) \geq 2$, i.e. it has to contain more than just the hyperplane class. For this equivalence, we need to know that the (integral) Hodge conjecture is true for cubic fourfolds. Also equivalently,

$$H^{2,2}(X, \mathbb{Z}) \cap H^4(X)_0 \neq \varnothing.$$

Definition 3.2. We relate this to period domains. A **labeling** of a special cubic fourfold X is a rank 2 saturated, positive definite lattice $K \subset H^{2,2}(X,\mathbb{Z})$ containing h^2. Quick recap: the period domain is

$$\mathcal{D} \coloneqq \{x \in \mathbb{P}(L_0 \otimes_{\mathbb{Z}} \mathbb{C}) : \langle x, x\rangle = 0,\ \langle x, \bar{x}\rangle > 0\},$$

where $L \coloneqq H^4(X,\mathbb{Z})$ contains $L_0 \coloneqq (h^2)^{\perp}$. Hence $K^{\perp} \subset L_0$. To each K, associate

$$\mathcal{D}_K \coloneqq \{x \in \mathbb{P}(K^{\perp} \otimes \mathbb{C}) : \cdots\},$$

which is a linear section of the period domain. Hence we get a square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] & \mathcal{D} \\ \mathcal{C}_K \arrow[u] \arrow[r] & \mathcal{D}_K \arrow[u] \end{tikzcd}$$

where $\mathcal{C}_K$ are divisors in $\mathcal{C}$, possibly empty. Here $\mathcal{C}$ is the moduli of *marked* cubic fourfolds. Act on this whole picture by $\mathrm{Aut}(L, h^2)$, since everything has a marking, to get rid of the markings. Descend to the quotient and abuse notation by writing

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] & \mathcal{D} \\ \mathcal{C}_{[K]} \arrow[u] \arrow[r] & \mathcal{D}_{[K]} \arrow[u] \end{tikzcd}$$

so now $[K]$ is the $\mathrm{Aut}(L, h^2)$-orbit of the lattice K.

Lemma 3.3. *Given two saturated non-degenerate sublattices $K, K' \subset L$ containing h^2, then*

$$[K] = [K'] \iff \mathrm{disc}(K) = \mathrm{disc}(K').$$

(Here $\mathrm{disc}(K) \coloneqq \det(\langle -,-\rangle_K)$ *is the determinant of the intersection form.)*

Definition 3.4. This lets us write $\mathcal{C}_d \coloneqq \mathcal{C}_{[K]}$ where $d \coloneqq \mathrm{disc}(K)$. These are **Hassett divisors**. Now the natural question is: which d occur?

Lemma 3.5. *Let (X, K_d) be a labeled cubic fourfold, with lattice K_d of discriminant d. Then $d > 0$, is even, and $d \equiv 0, -1 \bmod 3$, i.e. $d \equiv 0, 2 \bmod 6$.*

Proof. We know $K_d = \mathrm{span}\{h^2, T\}$ for some T, and

$$d = \det\begin{pmatrix} \langle h^2, h^2\rangle & \langle h^2, T\rangle \\ \langle h^2, T\rangle & \langle T, T\rangle \end{pmatrix} = 3\langle T, T\rangle - \langle h^2, T\rangle^2$$

using $\langle h^2, h^2\rangle = 3$. Finally, we know d is even because L_0 is an even lattice, and $K^{\perp}$ is a sublattice of L_0. □

Theorem 3.6. *Let $d > 6$ and $d \equiv 0, 2 \bmod 6$. Then $\mathcal{C}_d \neq \varnothing$ is an irreducible divisor.*

Proof sketch. Idea: produce a singular cubic fourfold and smooth it out in various ways. Precisely, we will construct

$$K_d = \begin{pmatrix} 3 & 0 \\ 0 & d/3 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 1 \\ 1 & (d-2)/3 \end{pmatrix}$$

corresponding to $d \equiv 0 \bmod 6$ and $d \equiv 2 \bmod 6$ respectively. The strategy is to make a K3 surface S with Picard lattice

$$\mathrm{Pic}(S) = \begin{pmatrix} 6 & 0 \\ 0 & 2d/3 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 2 \\ 2 & (2d-4)/3 \end{pmatrix},$$

and then use S to construct a *singular* cubic fourfold X_0. With the X_0, let $x_0 \coloneqq \tau(X_0) \in \mathcal{D}_d$ and take a small disk $\Delta \subset \mathcal{D}_d$ where x_0 is the origin $0 \in \Delta$. Let $\tilde{\mathcal{C}}$ denote the moduli of possibly singular cubic fourfolds with an ordinary double point. Using $\tilde{\mathcal{C}} \hookrightarrow \mathcal{D}$, form the map

$$\Delta \to \tilde{\mathcal{C}}$$

where $0 \in \Delta$ is the only point mapping to the boundary $\partial\tilde{\mathcal{C}}$. In other words, take a family whose special fiber is the singular cubic fourfold X_0. If we look at the projectivized tangent cone of X_0 at p, we get a smooth quadric. Hence if we choose coordinates where $p = [1:0:0:0:0:0]$, then the equation defining X_0 (in an affine chart) splits as

$$f(x_1,\ldots,x_5) \coloneqq f_2(x_1,\ldots,x_5) + f_3(x_1,\ldots,x_5)$$

where f_2 is degree 2 and f_3 is degree 3. The condition that p is an ordinary double point says $\{f_2 = 0\} \subset \mathbb{P}^4$ is a smooth quadric.

The observation now is that on X_0, the linear projection from the special point p gives a birational map $X_0 \dashrightarrow \mathbb{P}^4$. Such X_0 are therefore rational. We can resolve this map by blowing up at p, to get $\mathrm{Bl}_p(X_0) \to \mathbb{P}^4$. All lines through p are contracted, and therefore the space of such lines is a type $(2,3)$ complete intersection in $\mathbb{P}^4$ (by plugging a line $p + \lambda y$ into $f = f_2 + f_3$). This is a K3 surface! It follows that $\mathrm{Bl}_p(X_0) = \mathrm{Bl}_S(\mathbb{P}^4)$, giving a bijection

$$\{\text{cubic fourfold w/ o.d.p}\} \leftrightarrow \{\text{smooth c.i. of smooth quadric and cubic in } \mathbb{P}^4\}.$$

So it suffices to construct K3s on the rhs with desired Picard lattice. □

Definition 3.7. Recall that rank $H^4(X) = 23$. Call the rank-21 sublattice

$$W_{X,K_d} \coloneqq K_d^\perp \subset H^4(X)$$

the **non-special cohomology**. The special cubic fourfold (X, K) has **associated K3** (S, f) if

$$[K^\perp \subset H^4(X)] \cong [f^\perp \subset H^2(S)(-1)]$$

(where $f \in H^{1,1}(S)$ is the polarization) are isomorphic as Hodge structures.

Theorem 3.8. *(X, K_d) has an associated K3 iff*

1. *$4 \nmid d$ and $9 \nmid d$;*
2. *$p \nmid d$ for all odd primes $p \equiv -1 \bmod 3$.*

Example 3.9. If we list $d \in \mathbb{Z}_{>0}$ with $d \equiv 0, 2 \bmod 6$ and also satisfy the conditions of the theorem, $d = 14$ is the smallest. In fact, $\mathcal{C}_{14}$ are exactly the Pfaffians.

Remark. Globally, this means there should be a map $\mathcal{C}_d \to \mathcal{N}_d$ where $\mathcal{N}_d$ denotes the moduli of polarized degree-d K3s. It turns out that for this map to be well-defined, we must keep track of the labeling K_d, and also remember the embedding $K_d \subset H^4(X)$. It turns out that

$$[\mathcal{C}_d^{\text{marked}} \coloneqq \{(X, K_d \hookrightarrow H^4(X))\}] \to [\mathcal{C}_d^{\text{labeled}} \coloneqq \{(X, K_d)\}]$$

is an isomorphism when $d \equiv 0 \bmod 6$ and is 2-to-1 when $d \equiv 2 \bmod 6$. Hence up to some 2-to-1 cover, there is an actual map

$$\mathcal{C}_d^{\text{marked}} \to \mathcal{N}_d.$$

Remark. If we knew our cubic fourfold X were rational, choose a map $X \dashrightarrow \mathbb{P}^4$. Using weak factorization, if everything passes through a common blowup $X \leftarrow Y \to \mathbb{P}^4$, we get that there exist smooth surfaces $S_1,\ldots,S_m,T_1,\ldots,T_n$ such that

$$H^4(X) \oplus \bigoplus_{i=1}^n H^2(T_i)(-1) \cong \bigoplus_{i=1}^m H^2(S_i)(-1)$$

as Hodge structures. This equality is why we might expect K3 surfaces to be involved in the rationality of X. Note that we know there is a part of $H^4(X)$ that looks like $H^2(S)$ for some K3 S. Question: X rational iff it has an associated K3? This holds in the Pfaffian locus $\mathcal{C}_{14}$, but it is unclear in other cases.

4 Oct 08 (Noah): Kuznetsov components

Goal: explain the statement that given a smooth cubic fourfold $W \subset \mathbb{P}^5$, its Kuznetsov component

$$\mathrm{Ku}(W) \coloneqq \langle \mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2)\rangle^\perp$$

is a non-commutative K3 surface.

Conjecture 4.1 (Kuznetsov)**.** *W is rational iff $\mathrm{Ku}(W) \cong D^b\mathsf{Coh}(X)$ for X some K3.*

4.1 Serre functors and FM transforms

Let k be a field and $\mathcal{C}$ be a k-linear category with finite-dimensional Homs.

Definition 4.2. A **Serre functor** is an autoequivalence $S: \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms

$$\mathrm{Hom}(A, S(B)) \cong \mathrm{Hom}(B, A)^\vee.$$

Example 4.3. Let $\mathcal{C} = D(X)$ for X a smooth projective variety over k. Then a Serre functor is

$$- \otimes \omega_X[\dim X].$$

Proposition 4.4. *A Serre functor on $\mathcal{C}$ is unique if it exists.*

Corollary 4.5. *Given two smooth projective varieties X, Y with $\Phi: D(X) \cong D(Y)$, then $\dim X = \dim Y$.*

Proof. By the proposition, $S_Y \circ \Phi = \Phi \circ S_X$. Apply this to the object $\kappa(x) \in D(X)$ for some point $x \in X$:

$$\begin{aligned} S_Y \circ \Phi(\kappa(x)) &= \Phi(\kappa(x)) \otimes \omega_Y[\dim Y] \\ \Phi(\kappa(x) \otimes \omega_X[\dim X]) &= \Phi(\kappa(x))[\dim X]. \end{aligned}$$

Comparing both sides, $\Phi(\kappa(x)) = \Phi(\kappa(x)) \otimes \omega_Y[\dim Y - \dim X]$. By looking at highest non-vanishing cohomology, we get $\dim Y - \dim X = 0$. ☐

Definition 4.6. Let $Y \xleftarrow{q} X \times Y \xrightarrow{p} X$. A **Fourier–Mukai (FM) transform** with kernel $K \in D(X \times Y)$ is

$$\Phi_K: D(X) \to D(Y), \quad E \mapsto q_*(p^*E \otimes K).$$

Example 4.7. Take $f: X \to Y$ and let $K \coloneqq \mathcal{O}_{\Gamma_f}$ be the structure sheaf of the graph of f. Compute

$$\Phi_K(E) = q_*(p^*E \otimes \mathcal{O}_{\Gamma_f}) = q_*\Gamma_{f*}E = f_*E.$$

Example 4.8. Take the Serre functor $- \otimes \omega_X[\dim X]$. This is a FM transform with $K \coloneqq \Delta_*\omega_X[\dim X]$.

Example 4.9. Φ_K has left and right adjoints which are also FM transforms with kernels

$$\begin{aligned} K_L &\coloneqq K^\vee \otimes q^*\omega_Y[\dim Y] \\ K_R &\coloneqq K^\vee \otimes p^*\omega_X[\dim X]. \end{aligned}$$

We can also compose two FM transforms to get another FM transform.

Theorem 4.10 (Orlov)**.** *If $\Phi: D(X) \to D(Y)$ is a k-linear exact functor which is fully faithful, then there exists $K \in D(X \times Y)$ unique up to isomorphism such that $\Phi = \Phi_K$.*

Corollary 4.11. *If $D(X) \cong D(Y)$, then their (anti)canonical rings are isomorphic.*

Remark. When $k = \mathbb{C}$ and $F\colon D(X) \to D(Y)$ is exact, we get a functor $K^0(X) \to K^0(Y)$. If $F = \Phi_K$, then there exists a map $H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ which makes the following commute:

$$\begin{tikzcd} K^0(X) \arrow[r] \arrow[d, "\mathrm{ch}"'] & K^0(Y) \arrow[d, "\mathrm{ch}"'] \\ H^*(X,\mathbb{Z}) \arrow[r] & H^*(Y,\mathbb{Z}) \end{tikzcd} .$$

This map is the "cohomological FM transform" associated to $\mathrm{ch}(K)\sqrt{\mathrm{td}(X \times Y)} \in H^*(X \times Y)$. Caution:

1. this map does not respect cup products (but does respect the Mukai pairing);
2. this map does not respect the grading on cohomologies. However it does send

$$\bigoplus_{p-q=i} H^{p,q}(X,\mathbb{C}) \to \bigoplus_{p-q=i} H^{p,q}(Y,\mathbb{C}).$$

So for example if X is K3 and $D(Y) \cong D(X)$ then Y is K3.

4.2 Non-commutative varieties

Definition 4.12. A **non-commutative** smooth projective variety over k is an admissible subcategory $\mathcal{A} \subset D(X)$ for some smooth projective variety X. Here admissible means a full triangulated subcategory which is k-linear and the inclusion has left and right adjoints.

Definition 4.13. Let $\mathcal{D}$ be a triangulated category. A **semi-orthogonal decomposition** $\langle \mathcal{D}_1, \ldots, \mathcal{D}_m \rangle$ of $\mathcal{D}$ is a sequence of full triangulated subcategories satisfying:

1. $\mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_j) = 0$ for $i > j$;
2. given $F \in \mathcal{D}$ there is a filtration $0 = F_m \to \cdots \to F_1 \to F_0 = F$ such that $\mathrm{cone}(F_i \to F_{i-1}) \in \mathcal{D}_i$ for all i.

Lemma 4.14. *The filtration and its factors are unique and functorial, and*

$$\delta_i(F) \coloneqq \mathrm{cone}(F_i \to F_{i-1})$$

is a functor $\mathcal{D} \to \mathcal{D}_i$. If $m = 2$, then:

1. *δ_1 is left adjoint to the inclusion $\mathcal{D}_1 \to \mathcal{D}$;*
2. *δ_2 is right adjoint to the inclusion $\mathcal{D}_2 \to \mathcal{D}$.*

Proof. Let $m = 2$, so that $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Take $F, K \in \mathcal{D}$ with a morphism $F \to K$. Then there exist distinguished triangles

$$\begin{tikzcd} F_1 \arrow[r] & F \arrow[r] \arrow[d] & \delta_1(F) \arrow[r, "{[1]}"] & {} \\ K_1 \arrow[r] & K \arrow[r] & \delta_1(K) \arrow[r, "{[1]}"] & {} \end{tikzcd} ,$$

and we claim there exists a unique morphism $F_1 \to K_1$ that makes the square commute. So we are interested in $\mathrm{Hom}(F_1, K_1) \to \mathrm{Hom}(F_1, K)$. This fits into

$$\mathrm{Ext}^{-1}(F_1, \delta_1(K)) \to \mathrm{Hom}(F_1, K_1) \to \mathrm{Hom}(F_1, K) \to \mathrm{Hom}(F_1, \delta_1(K)) = 0$$

by semi-orthogonality of the decomposition. We also get a unique $\delta_1(F) \to \delta_1(K)$, by the same method. □

Lemma 4.15. *Let $\mathcal{C} \subset \mathcal{D}$ be an admissible subcategory. Then we get two semi-orthogonal decompositions*

$$\mathcal{D} = \langle \mathcal{C}, {}^{\perp}\mathcal{C} \rangle = \langle \mathcal{C}^{\perp}, \mathcal{C} \rangle$$

where $\mathcal{C}^{\perp} \coloneqq \{K \in \mathcal{D} : \mathrm{Hom}(F, K) = 0 \ \forall F \in \mathcal{C}\}$, and ${}^{\perp}\mathcal{C}$ is for $\mathrm{Hom}(K, F)$.

Proof. Let $\mathcal{C} \to \mathcal{D}$ be the inclusion with right adjoint $R{:}\,\mathcal{D} \to \mathcal{C}$. For $F \in \mathcal{D}$, form the triangle

$$RF \to F \to \mathrm{cone} \xrightarrow{[1]},$$

and note that $RF \in \mathcal{C}$ and $\mathrm{cone} \in {}^{\perp}\mathcal{C}$. □

Definition 4.16. We say $E \in \mathcal{D}$ is **exceptional** if

$$\mathrm{Ext}^i(E, E) = \begin{cases} k & i = 0 \\ 0 \text{otherwise.} \end{cases}$$

A collection $E_1, \dots, E_m$ is an **exceptional collection** is all E_i are exceptional and $\mathrm{Ext}^*(E_i, E_j) = 0$ for $i > j$.

Proposition 4.17. *Let E be exceptional, and $\mathcal{C} \coloneqq \langle E \rangle$ be the full triangulated subcategory in $\mathcal{D}$ generated by E. Suppose $\mathcal{D}$ is proper, i.e. $\dim_k \bigoplus \mathrm{Ext}^i(F, G) < \infty$ for all F, G. Then $\mathcal{C}$ is admissible.*

Proof. The map $D(k) \to \mathcal{D}$ given by $V^\bullet \mapsto V^\bullet \otimes E$ is actually an equivalence of categories $D(k) \xrightarrow{\sim} \mathcal{C}$. Now just write down the adjoints. Example: the right adjoint to $\mathcal{C} \hookrightarrow \mathcal{D}$ is

$$K \mapsto \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i(E, K) \otimes E[-n].$$ □

Remark. Thus given $E_1, \dots, E_m$ an exceptional collection, we get semi-orthogonal decompositions

$$\mathcal{D} = \langle \mathcal{C}^{\perp}, E_1, \dots, E_m \rangle = \langle E_1, \dots, E_m, {}^{\perp}\mathcal{C} \rangle$$

where here E_i stands for $\langle E_i \rangle$ and $\mathcal{C} \coloneqq \langle E_1, \dots, E_m \rangle$.

Example 4.18. If W is a smooth cubic fourfold, then $\mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2)$ form an exceptional collection. To show this we have to check

$$H^*(W, \mathcal{O}_W) = k[0], \quad H^*(W, \mathcal{O}_W(-i)) = 0 \ \forall i = 1, 2.$$

Hence $D(W) = \langle \mathrm{Ku}(W), \mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2) \rangle$, where $\mathrm{Ku}(W)$ is the right orthogonal. To figure out the other adjoint, we figure out the Serre functor on $\mathrm{Ku}(W)$.

Lemma 4.19. *If $\mathcal{A} \subset D(X)$ is a non-commutative variety, then $\mathcal{A}$ has a Serre functor given by*

$$S = R \circ S_X, \quad S^{-1} = L \circ S_X^{-1}$$

where L and R are the adjoints.

Proof. This comes from verifying $\mathrm{Hom}(A, RS_X B) = \mathrm{Hom}(i_* A, S_X B) = \mathrm{Hom}(B, A)^\vee$. □

Definition 4.20. A non-commutative $\mathcal{D}$ is called a non-commutative **Calabi–Yau** of dimension n if its Serre functor is just $[n]$.

5 Oct 15 (Dmitrii)

Sorry, no notes!

6 Oct 22 (Dmitrii): Addington–Thomas

Conjecture 6.1 (Hassett, 2000)**.** *A cubic fourfold Y is rational iff there is $T \in H^{2,2}_{prim}(Y,\mathbb{Z})$ such that*

$$\langle h^2, T\rangle \subset H^{2,2}(Y,\mathbb{Z})$$

has discriminant d satisfying

$$d \equiv 0, 2 \bmod 6, \quad d > 6, \quad d \textit{not divisible by } 4, 9, \textit{ and any odd prime } \equiv 2 \bmod 3. \tag{*}$$

Equivalently, d is even and there exists a primitive vector $v \in A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ with norm d.

Remark. If d satisfies (*), then $C_d \subset \mathcal{M}$ is a non-empty irreducible divisor.

Conjecture 6.2 (Kuznetsov)**.** *A cubic fourfold Y is rational iff in its derived category*

$$D(Y) = \langle \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle,$$

the Kuznetsov component $\mathcal{A}_Y$ is geometric, i.e. there exists a K3 surface S and an equivalence $D(S) \cong \mathcal{A}_Y$.

Theorem 6.3 (Addington–Thomas)**.**

1. *If $\mathcal{A}_Y$ is geometric, then $Y \in C_d$ for some C_d satisfying (*).*
2. *For d satisfying (*), there exists a Zariski-open non-empty $U_d \subset C_d$ of cubics with geometric $\mathcal{A}_Y$.*

First we talk about some generalities. If we have such a $T \in H^{2,2}(Y,\mathbb{Z})$, what can we say about $\mathcal{A}_Y$?

Definition 6.4. The **Mukai lattice** for a K3 surface S is $\tilde{H}(S,\mathbb{Z})$. As an abelian group, it is $H^*(S,\mathbb{Z})$, with modified Hodge structure

$$\tilde{H}^{1,1} \coloneqq H^0(S) \oplus H^{1,1}(S) \oplus H^4(S), \quad \tilde{H}^{0,2} = H^{0,2}(S)$$

and modified intersection pairing.

Theorem 6.5 (Mukai–Orlov)**.** *Two K3 surfaces are derived equivalent iff their Mukai lattices are isometric.*

So there is a unique $\tilde{H}(S,\mathbb{Z})$ coming from $D(S)$, but how do we construct it? The solution by Addington–Thomas is to use $K_{\text{top}}(S)$.

Definition 6.6. The **topological K-theory** $K_{\text{top}}(S)$ is an abelian group generated by topological vector bundles.

1. There is a map
$$K_{\text{top}}(S) \xrightarrow{E \mapsto \text{ch}(E)\sqrt{\text{td}\, S}} H^*(S,\mathbb{Q})$$
which is injective and the image is a full rank lattice.
2. There is a pairing $\chi(-,-)$ coming from pushforward $\pi\colon K_{\text{top}}(S) \to K_{\text{top}}(\text{pt}) = \mathbb{Z}$:
$$\chi(E,F) \coloneqq p_*(E^\vee \otimes F).$$
3. Define a Hodge structure by pulling back $H^0 \oplus H^{1,1} \oplus H^4$.

Fact: if S is K3, this pairing and Hodge structure defines an isomorphism between the Mukai lattice and $K_{\text{top}}(S)$.

Definition 6.7. Let Y be a cubic fourfold. Define

$$K_{\mathrm{top}}(\mathcal{A}_Y) = \{E \in K_{\mathrm{top}}(Y) : \chi(E, [\mathcal{O}(i)]) = 0\ \forall i = 0, 1, 2\}.$$

Set

$$\begin{aligned} K_{\mathrm{top}}(\mathcal{A}_Y)^{1,1} &\coloneqq \nu^{-1}(H^0(Y) \oplus H^1(Y) \oplus H^{2,2}(Y) \oplus H^3(Y) \oplus H^4(Y)) \\ K_{\mathrm{top}}(\mathcal{A}_Y)^{2,0} &\coloneqq \nu^{-1}(H^{3,1}(Y)). \end{aligned}$$

Proposition 6.8. *If $D(S) \cong \mathcal{A}_Y \subset D(Y)$, then the embedding comes from a Fourier–Mukai kernel $P \in D(S \times Y)$, and there exists an induced map*

$$\Phi_p^{1,1} : H^*(S, \mathbb{Q}) \to H^*(Y, \mathbb{Q}).$$

Its associated $\Phi_p^K : K_{top}(S) \to K_{top}(Y)$ identifies the Mukai lattice structure on $K_{top}(S)$ with our structure on $K_{top}(\mathcal{A}_Y)$.

Corollary 6.9. *$\chi(-,-)$ on $K_{top}(\mathcal{A}_Y)$ is symmetric for every Y.*

Proof sketch. This is true when $\mathcal{A}_Y$ is geometric, and is preserved under deformations. ☐

Proposition 6.10. *Let λ_1, λ_2 be the classes of projections to $\mathcal{A}_Y$ of $\mathcal{O}_{line}(1), \mathcal{O}_{line}(2) \in D(Y)$. Then:*

1. *the lattice is*
$$\langle \lambda_1, \lambda_2 \rangle = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix};$$
2. *the Mukai vector gives an isomorphism*
$$K_{top}(\mathcal{A}_Y)/\langle \lambda_1, \lambda_2 \rangle \xrightarrow{\sim} H^4(X, \mathbb{Z})/\langle h^2 \rangle;$$
3. *the pre-image $H^{2,2}(Y, \mathbb{Z})$ is exactly the image of $K_{alg}(\mathcal{A}_Y) \to K_{top}(\mathcal{A}_Y)$.*

Proof. Proof omitted. Check that the two lattices have the same signature and discriminant and then use lattice theory. Use the integral Hodge conjecture proved by Voisin. ☐

Theorem 6.11. *Let Y be a cubic fourfold. TFAE:*

1. *$Y \in C_d$ for some d satisfying (*);*
2. *the image of $K_{alg}(\mathcal{A}_Y) \to K_{top}(\mathcal{A}_Y)$ contains a sublattice $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Proof sketch. (1) implies there exists $T \in H^{2,2}(Y, \mathbb{Z})$ such that $\langle h^2, T \rangle \subset H^4(X, \mathbb{Z})$ has discriminant d. Hence there exists $K \in K_{\mathrm{top}}(\mathcal{A}_Y)$ (some arbitrary pre-image of T) such that $\langle \lambda_1, \lambda_2, K \rangle \subset K_{\mathrm{top}}(\mathcal{A}_Y)$ has discriminant d. One of the equivalent conditions for (*) is the following:

there exists an embedding $(-d) \hookrightarrow U^3 \oplus E_8^2$ such that $(-d)^\perp \cong \langle \lambda_1, \lambda_2, K \rangle^\perp$ in $K_{\mathrm{top}}(\mathcal{A}_Y)$.

Then by some lattice theory, $(-d) \oplus U \cong \langle \lambda_1, \lambda_2, K \rangle$.

Conversely, (2) implies there exists $K_1, K_2 \in K_{\mathrm{top}}$ forming U. Then we look at $\langle \lambda_1, \lambda_2, K_1, K_2 \rangle$. By some reasons involving discriminants, this can only be either rank 3 or rank 4.

1. If it is rank 3, then we will get a factor of U which splits off, to get d.
2. If it is rank 4, then look at $\langle \lambda_1, \lambda_2, xK_1 + yK_2 \rangle$. Some of these lattices will have discriminant satisfying (*). (There is a surprising amount of number theory hidden here; we have to use Chebotarev density.)

☐

Corollary 6.12. *If $\mathcal{A}_Y$ is geometric, then $Y \in C_d$ for some d.*

Proof. If $\mathcal{A}_Y \cong D(S)$, then the classes $[\mathcal{O}_{\mathrm{pt}}]$ and $[\mathcal{I}_{\mathrm{pt}}]$ generate a copy of U. □

Proposition 6.13. *For every d satisfying (*), there exists $Y \in C_d \cap C_8$ such that $\mathcal{A}_Y$ is geometric.*

Proof. Note (Voisin) that $C_8 = \{\text{cubics containing a plane}\}$. If $\mathbb{P}^2 \subset Y$, then look at the linear projection

$$\mathrm{Bl}_{\mathbb{P}^2}(Y) \to \mathbb{P}^2.$$

1. This is a quadric fibration.
2. If there exists $T \in H^{2,2}(Y, \mathbb{Z})$ satisfying $T \cdot (h^2 - P) = 1$, then Y is rational. (This is the intersection with a fiber of the map.) Here $P = [\mathbb{P}^2]$.

By lattice theory, we can find:

1. $h^2, P, T \subset H^4(Y, \mathbb{Z})$ with expected pairings;
2. $\sigma \subset H^4(Y, \mathbb{C})$ such that $\sigma^\perp \cap H^4(Y, \mathbb{Z}) = \langle h^2, P, T\rangle$.

The period map is not surjective; Laza–Looijenga has a description of its image that shows there exists some Y with $\langle \sigma \rangle = H^{3,1}$. Hence Y is in $C_8 \cap C_d$ and $\mathcal{A}_Y$ is geometric. □

Fact (Hassett): there exists a smooth quasi-projective variety C_d^v and families $\mathcal{Y}$ and $\mathcal{S}$ of cubic fourfolds with $T \in H^{2,2}$ and K3 surfaces such that

$$\mathcal{Y} \to C_d^v \to C_d$$

is a surjective finite morphism. There is also a morphism $H^*(\mathcal{S}_t) \to H^*(\mathcal{Y}_t)$ comparing their Hodge structures.

Idea: over $z \in C_d^v$, we have a FM kernel P_z on $\mathcal{S}_z \times \mathcal{Y}_z$ defining a fully faithful embedding with image $\mathcal{A}_Y = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle^\perp$. Its image lying in $\mathcal{A}_Y$ is an open condition. So we can try to deform P_0. There is an explicit obstruction living in $\mathrm{Ext}^2_{S_z \times Y_z}(P_0, P_0)$ (Huybrechts–Thomas). It is a sum of two contributions: one from S_z, and the other from Y_z. By some calculation with Atiyah classes and the definition of the obstruction, the two contributions basically compute the same thing and cancel each other. This gives a first-order deformation.

Theorem 6.14 (Lieblich)**.** *Let $\mathcal{X} \to B$ be a proper flat family of finite presentation. Then there exists a stack $\mathcal{M}$ of perfect complexes E on fibers such that*

1. $\mathrm{Ext}^{<0}(E, E) = 0$*, and*
2. *$\mathcal{M}$ is an algebraic stack locally of finite presentation.*

Remark. This is the necessary algebraization procedure for our liftings (to all orders) of P_0. Now we get a deformation in some Zariski opens.

7 Oct 29 (Carl): Galkin–Shinder

Theorem 7.1 (Galkin–Shinder, 2014)**.** *Assume cancellation holds in $K_0(\mathsf{Var}_{/\mathbb{C}})$. Then a very general cubic fourfold is not rational.*

Idea: do a computation in $K_0(\mathsf{Var}_{/\mathbb{C}})$ relating $Y \subset \mathbb{P}^5$ with $F(Y)$. The conclusion will be that

$$F(Y) \dashrightarrow \mathrm{Hilb}^2(K3)$$

is birational. A theorem of Addington gives that Y is associated to this K3, i.e. $Y \in \mathcal{C}_d$ for some conditions on d, and these conditions make Hassett's conjecture false as stated. In fact, they are strictly stronger.

Theorem 7.2 (Borisov)**.** *Cancellation is false.*

Definition 7.3. Quickly recall that $K_0(\mathsf{Var}_{/\mathbb{C}})$ is the $\mathbb{Z}$-algebra generated by isomorphism classes of $\mathsf{Var}_{/\mathbb{C}}$ with scissor relations

$$[X] = [U] + [Z], \quad Z \hookrightarrow X \text{ closed }, U = X - Z.$$

The product structure is $[X] \times [Y] \coloneqq [X \times Y]$. For example,

$$[\mathrm{pt}] = 1, \quad [\mathbb{A}^1] =: \mathbb{L}, \quad [\mathbb{P}^n] = \sum_{i=1}^{n} [\mathbb{L}^i].$$

Remark (Homomorphisms)*.* There are all kinds of interesting invariants which arise from homomorphisms out of the Grothendieck ring.

1. There exists a homomorphism $K_0(\mathsf{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ counting points, which can be souped up to some kind of zeta function.
2. Over $\mathbb{C}$, we can take $K_0(\mathsf{Var}_{/\mathbb{C}})$ returning Hodge structures.

Work over $\mathbb{C}$ from now on. Here are some facts.

1. The association $X \mapsto \operatorname{Sym}^n X$ makes sense on $K_0(\mathsf{Var})$. This works by

$$\operatorname{Sym}^n(X+Y) = \sum_{i+j=n} \operatorname{Sym}^i X \cdot \operatorname{Sym}^j Y$$
$$\operatorname{Sym}^n(X \times Y) = \operatorname{Sym}^n X \times \operatorname{Sym}^n Y.$$

2. Suppose $X \to S$ is a Zariski-locally trivial fibration with fiber F. Then

$$[X] = [F][S].$$

   This is by cutting S into pieces where $X \to S$ really is a trivial fibration. One important case is a blow-up of a smooth locus $Z \subset X$ inside a smooth X. Then the exceptional divisor $E \to Z$ is a bundle, so

$$[\mathrm{Bl}_Z X] - [E] = [X] - [Z],$$

   and $[E] = [Z][\mathbb{P}^{c-1}]$ where c is the codimension.

Theorem 7.4 (Bittner)**.** *$K_0(\mathsf{Var}_{/\mathbb{C}})$ is generated by classes of smooth projective varieties, with all relations coming from blow-up relations as above.*

Corollary 7.5. *Suppose X, X' are birational smooth projective varieties of dimension d. Then*

$$[X] - [X'] = \mathbb{L} \cdot M$$

where M is a linear combination of smooth projective varieties of dimension $d-2$.

Proof. By weak factorization, it suffices to prove this in the case $X' \to X$ is a blow-up. Then

$$[X'] = [X] + [Z]([\mathbb{P}^{c-1}] - 1) = [X] + \mathbb{L}([Z] \times [\mathbb{P}^{c-2}]).$$ □

Definition 7.6. Given Y smooth projective, the **rational defect** of Y is

$$M_Y \coloneqq \frac{[Y] - [\mathbb{P}^n]}{\mathbb{L}} \in K_0(\mathsf{Var}_{/\mathbb{C}})_{\mathbb{L}}.$$

The above corollary shows that if Y is in fact rational, then this class is integral.

Theorem 7.7 (Larson–Luntz)**.** *$K_0(\mathsf{Var})/\mathbb{L}$ is precisely the free abelian group on stable birational equivalence classes of smooth projective varieties, i.e. there exists a (clearly surjective) map*

$$K_0(\mathsf{Var}) \to \mathbb{Z}[\textit{stable birational classes}]$$

with kernel $\mathbb{L}$.

Remark. Recall that X, Y are **stably birational** if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for some (possibly different) m, n.

Proof sketch. For smooth varieties the construction of this map is clear. For singular things, cut it up. To make sure it is well-defined, relate two different resolutions by weak factorization. Both these things require us to be in characteristic 0. ☐

Corollary 7.8. *If*

$$[X] \equiv \sum [Y_i] - \sum [Z_j] \bmod \mathbb{L}$$

where X, Y_i, Z_j are smooth projective, then X is stably birational to one of the terms on the rhs.

Conjecture 7.9 (Cancellation (known to be false))**.** *$\mathbb{L}$ is not a zero divisor in $K_0(\mathsf{Var}_{/\mathbb{C}})$.*

Theorem 7.10. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold and let*

$$Y^{[2]} \coloneqq \mathrm{Hilb}^2 Y, \quad Y^{(2)} \coloneqq \mathrm{Sym}^2 Y,$$

with $F(Y)$ its Fano variety of lines. Then:

1. $[Y^{[2]}] = [\mathbb{P}^4][Y] + \mathbb{L}^2[F(Y)]$;
2. $[Y^{(2)}] = (1 + \mathbb{L}^4)[Y] + \mathbb{L}^2[F(Y)]$.

Proof. (2) follows easily from (1), so we prove (1). Form an incidence correspondence

$$W \coloneqq \{(x, L) : x \in Y,\ L \subset \mathbb{P}^5 \text{ a line, } x \in L\},$$

with projections $W \to \mathrm{Gr}(1, \mathbb{P}^5)$ and $W \to Y$. The former is generically finite of degree 3, and the latter is a $\mathbb{P}^4$-bundle. So W has dimension 8 and is in fact smooth. Make a birational map

$$\phi \colon Y^{[2]} \dashrightarrow W$$

by forming the line through the two points (or the point and tangent vector) to get the third point of intersection. As long as this line is not completely contained in Y, we are OK, i.e. the base locus is

$$Z \coloneqq \{(x, y) \in Y \times Y : \text{corresponding line is in } Y\}.$$

Let $U \coloneqq Y^{[2]} - Z$, with image $U' \subset W$. Then we get

$$Z' \coloneqq \{(x, L) : x \in L \subset Y\}.$$

On one hand, $Z \to F(Y)$ is a $\mathrm{Sym}^2 \mathbb{P}^1 = \mathbb{P}^2$-bundle. On the other, $Z' \to F(Y)$ is a $\mathbb{P}^1$-bundle. Hence

$$\begin{aligned}
[Y^{[2]}] &= [Z] + [U] = [Z] + [U'] \\
&= [\mathbb{P}^2][F(Y)] + [W] - [Z'] \\
&= [\mathbb{P}^2][F(Y)] + [\mathbb{P}^4][Y] - [\mathbb{P}^1][F(Y)] = \mathbb{L}^2[F(Y)] + [\mathbb{P}^4][Y].
\end{aligned}$$

To get (2) from (1), use that $Y^{[2]} = \mathrm{Bl}_\Delta Y^{(2)}$, whose blow-up relation is

$$[Y^{(2)}] = [Y^{[2]}] - [\mathbb{P}^3][Y] + [Y] = ([\mathbb{P}^4] - [\mathbb{P}^3] + 1)[Y] + \mathbb{L}^2[F(Y)],$$

as desired. ☐

Theorem 7.11. *In $K_0(\mathsf{Var})_{\mathbb{L}}$, we have*

$$[F(Y)] = \mathrm{Sym}^2(M_Y + [\mathbb{P}^2]) - \mathbb{L}^2.$$

Proof. Just compute. □

Example 7.12. Assume the cancellation conjecture. Then if Y is rational, this formula holds in $K_0(\mathsf{Var})$. In particular, it holds in $K_0(\mathsf{Var})/\mathbb{L}$, to give something like

$$[F(Y)] = \mathrm{Sym}^2(\sum[V_i] - [W_j]) = \sum \mathrm{Sym}^2 V_i + \sum \mathrm{Sym}^2 W_j + \sum V_i \times W_j + V_i \times V_j + W_i \times W_j.$$

But mod $\mathbb{L}$, we know $\mathrm{Sym}^2 \equiv \mathrm{Hilb}^2$. Hence by Larsen–Luntz, $F(Y)$ is stably birational to something of the form $\mathrm{Hilb}^2 V$, or $V \times W$. We will show it cannot be the latter, and that if it is the former, $V = K3$.

By explicit geometry, prove that $F(Y)$ is not uniruled by showing its canonical class is trivial and therefore not negative. By the existence of an MRC (maximally rationally connected) fibration, we can remove the word "stable", i.e. $F(Y)$ is birational to $V \times W$ or $\mathrm{Hilb}^2 S$ for a surface S.

Remark. This argument fails in dimension > 4, because $F(Y)$ has negative canonical class.

Lemma 7.13. *$F(Y)$ is not birational to $V \times W$.*

Proof. Look at p-forms $\psi_Z(t) = \sum_{p\geq 0} h^{p,0}(Z)t^p$, which is a birational invariant coming from $K_0(\mathsf{Var})$. But $\psi_{F(Y)}(t) = 1 + t^2 + t^4$, which is irreducible in $\mathbb{Z}[t]$. □

Lemma 7.14. *$F(Y)$ birational to $\mathrm{Hilb}^2 S$ implies S is K3.*

Proof. First show that $\chi(S) = 0$. From ψ, conclude that $h^{1,0} = 0$ and $h^{2,0} = 1$. Then apply classification of surfaces. □

Theorem 7.15 (Addington)**.** *In this case, $Y \in \mathcal{C}_d$ where $d = (2n^2 + 2n + 2)/a^2$ for $a, n \in \mathbb{Z}$.*

Remark. There are values of d that satisfy this condition that do not satisfy the conditions on d in Hassett's conjecture. So if we believe this conjecture, Hassett's conjecture must be false.

Proof sketch. Here are the ingredients.

1. (Markman) Given a variety of K3 type, we can build lattices $\tilde{\Lambda}_{F(Y)} \supset H^2(F(Y),\mathbb{Z})$ and $\tilde{\Lambda}_{\mathrm{Hilb}^2(S)} \supset H^2(\mathrm{Hilb}^2(S),\mathbb{Z})$. Then $F(Y)$ birational to $\mathrm{Hilb}^2(S)$ implies these two lattices are isomorphic, and the isomorphism preserves the H^2.

2. More generally, if M is a moduli space of stable sheaves on S with Mukai vector v, then $H^2(M,\mathbb{Z})$ is identified with $v^\perp \subset \tilde{\Lambda}_M$. In particular, if $M = \mathrm{Hilb}^2 S$, then $v = (1, 0, 1 - n)$.

3. On the cubic fourfold side, $H^2(F(Y),\mathbb{Z}) \subset \tilde{\Lambda}_{F(Y)}$ is identified with the embedding $H^2(F(Y)) \subset K_{\mathrm{top}}(\mathcal{A}_Y)$ into the topological K-theory of the Kuznetsov component.

4. Use the corresponding $w \in \tilde{\Lambda}_{F(Y)}$ to produce some rank-2 sublattice of $H^2(F(Y))$. □

8 Nov 26 (Dmitrii):

The motivation for today's talk is as follows. Let X be a K3 or abelian surface. Then the moduli of stable vector bundles (with any fixed Chern character) is smooth and has a 2-form which is non-degenerate everywhere ('84) and closed ('88). Beauville–Donagi proved in '85 that for Y a cubic fourfold, its Fano variety of lines $F(Y)$ has a symplectic form. Today we will discuss a paper by Kuznetsov–Markushevich ('09) which gives:

1. a general way to construct closed forms on moduli spaces of sheaves (which can be used to do both of these constructions);
2. for $F(Y)$, and other spaces related to cubic fourfolds, a way to check non-degeneracy.

There is also work of Bottacin ('08, '09) which did the first part independently.

Definition 8.1 (Naive version of Atiyah class)**.** Let X be a smooth variety over $\mathbb{C}$. Let v be a vector field on X, and $a\colon \mathbb{A}^1 \times X \to X$ be its flow. Let

$$\operatorname{Spec} k[\epsilon] \times X \xrightarrow{j} \mathbb{A}^1 \times X \xrightarrow{a} X.$$

Given any $E \in D(X)$, take $j^* a^* E$, which is a flat family over dual numbers. This is an infinitesimal deformation of E, and therefore gives a class

$$\operatorname{at}_v(E) \in \operatorname{Ext}^1(E, E) = \operatorname{Hom}(E, E[1]).$$

Proposition 8.2. *Some properties of this Atiyah class* $\operatorname{at}_v(-)$*:*

1. *it is functorial;*
2. *it commutes with triangles in* $D(X)$*;*
3. *it commutes with restrictions to subvarieties invariant under the flow.*

Example 8.3. Let $X = \mathbb{A}^n = \mathbb{A}(V)$. Then $v \in V$ produces a constant vector field, and the map

$$v \mapsto \operatorname{at}_v(\mathcal{O}_0)$$

is an isomorphism $V \cong \operatorname{Ext}^1(\mathcal{O}_0, \mathcal{O}_0)$. The middle term for $\operatorname{at}_v(\mathcal{O}_0)$ is

$$\{f \in \operatorname{Sym} V^\vee : f(0) = 0,\, \partial_v f(0) = 0\}.$$

Example 8.4. Exercise: $\operatorname{at}_v(E \otimes F)$ is a sum

$$E \otimes (\operatorname{at}_v(F)\colon F \to F[1]) + (\operatorname{at}_v(E)\colon E \to E[1]) \otimes F.$$

Definition 8.5 (Actual Atiyah class)**.** Let X be smooth, $\Delta \subset X \times X$ be the diagonal, and I its ideal sheaf. Then there is a SES

$$0 \to I/I^2 \to \mathcal{O}_{X\times X}/I^2 \to \mathcal{O}_{X\times X}/I \to 0$$

of sheaves on $X \times X$. Note that $\mathcal{O}/I = \Delta_* \mathcal{O}_X$, and $I/I^2 = \Delta_* \Omega^1_X$. So we get a morphism

$$\Delta_* \mathcal{O}_X \to \Delta_* \Omega^1_X[1]$$

in $D(X \times X)$. This we can consider as a map of Fourier–Mukai kernels, which is a natural transformation of the corresponding functors:

$$\operatorname{at}(E) \coloneqq E \mapsto E \otimes \Omega^1_X[1].$$

This is the **Atiyah class**.

Proposition 8.6. *Some properties of this Atiyah class* $\operatorname{at}_v(-)$*:*

1. *it is functorial;*
2. *it commutes with triangles in* $D(X)$*;*
3. *it commutes with restrictions to subvarieties* $j\colon Z \hookrightarrow X$ *in the obvious way;;*

4. it satisfies a Leibniz rule

$$\mathrm{at}(E \otimes F) = \mathrm{at}(E) \otimes F + E \otimes \mathrm{at}(F).$$

Remark. There is a projection

$$\mathrm{Ext}^1(E, E \otimes \Omega^1) \to H^0(X, \mathcal{E}xt^1(E, E \otimes \Omega^1))$$

and locally the image of $\mathrm{at}(E)$ is given by the previous differential-geometric construction. So the actual construction contains more data.

Example 8.7. Exercise: $\mathrm{at}_v(\mathcal{O}_X) = 0$.

Definition 8.8. Define the **second Atiyah class**

$$\mathrm{at}_2(E): E \to E \otimes \Omega^1[1] \to (E \otimes \Omega^1[1]) \otimes \Omega^1[1] \twoheadrightarrow E \otimes \Omega^2[2].$$

Similarly, define $\mathrm{at}_3(E), \ldots$ by iterating the same construction. (There is also a Fourier–Mukai construction.)

Definition 8.9 (Traces in derived category)**.** View $E \to E \otimes F$ as an element of $E^\vee \otimes E \otimes F$. (Here we only need to consider E which are perfect.) The **trace** map is $E^\vee \otimes E \otimes F \to F$, and in general is

$$\mathrm{tr}: \mathrm{Hom}(E, E \otimes F) \to \mathrm{Hom}(\mathcal{O}, F).$$

Proposition 8.10. *Let X be smooth and quasiprojective, and $E \in D_{perf}(X)$. Then*

$$\mathrm{tr}(\mathrm{at}_i(E)) \in H^i(X, \Omega^i)$$

is d-closed (under de Rham differential).

Proof sketch. The additivity of trace and the splitting principle imply it is enough to prove this for a line bundle. By the Leibniz rule, it is enough to prove this for a very ample line bundle. Hence by functoriality it is enough to check for $\mathcal{O}_{\mathbb{P}^n}(1)$. But on $\mathbb{P}^n$ we can apply Hodge theory to get that they are all closed. □

Remark. There is another definition of Atiyah class by differential geometry, as an obstruction to the existence of a holomorphic connection on E, in the case where E is a vector bundle.

Theorem 8.11 (Chern character via Atiyah class)**.** *Let X be smooth proper. Define*

$$\exp(\mathrm{at}(E)) \coloneqq \left(\mathrm{id}_E + \mathrm{at}(E) + \frac{\mathrm{at}_2(E)}{2!} + \cdots \right) \in \bigoplus \mathrm{Hom}(E, E \otimes \Omega^i[i]).$$

Then $\mathrm{tr}(\exp(\mathrm{at}(E))) \in \bigoplus H^i(X, \Omega^i) = \bigoplus H^{i,i}(X)$ *is the Chern character.*

Definition 8.12. Let S be affine and smooth, and $F \to S \times Y$ be a flat family. Let

$$\mathcal{E}xt^i_{\mathrm{pr}}(F, G)$$

be the sheaf on S whose fiber at $s \in S$ is $\mathrm{Ext}^i(F_s, G_s)$. Since S is affine, $H^0(\mathcal{E}xt^1_{\mathrm{pr}}(F, G)) = \mathrm{Ext}^1(F, G)$. In particular, $\mathrm{at}(F)$ gives a global section of $\mathcal{E}xt^1_{\mathrm{pr}}(F, F \otimes G)$.

Proposition 8.13 (Kodaira–Spencer map via Atiyah class)**.** *In this situation, the Kodaira–Spencer map*

$$\mathrm{KS}: T_S \to \mathcal{E}xt^1_{pr}(F, F)$$

is equal to

$$T_S \xrightarrow{-\otimes \mathrm{at}(F)} T_S \otimes \mathcal{E}xt^1_{pr}(F, F \otimes \Omega^1_{S \times Y}) \xrightarrow{pairing} \mathcal{E}xt^1_{pr}(F, F).$$

8.1 Closed forms on moduli of sheaves

Let $\mathcal{M}$ be some moduli space of sheaves on Y. Recall from deformation theory that the tangent space at $[F] \in \mathcal{M}$ is $\mathrm{Ext}^1(F,F)$. The obstruction map is squaring to $\mathrm{Ext}^2(F,F)$. In particular, at a smooth point $[F] \in \mathcal{M}$, the product

$$\mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^2(F,F)$$

is skew-symmetric. Recall that having a 2-form on the smooth locus on $\mathcal{M}$ means that for any smooth affine base S, a family F on $S \times Y$ has a 2-form.

Definition 8.14 (Construction)**.** Let Y be smooth projective. Fix an element $\omega \in H^r(Y, \Omega^{r+2})$ and set $q \coloneqq n - r - 2$. Given a family (S, F), make a 2-form in the following way.

1. Use the Kodaira–Spencer map to get

$$T_{S,s} \times T_{S,s} \xrightarrow{\mathrm{KS} \times \mathrm{KS}} \mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F).$$

2. Multiply by $\mathrm{at}_q(F)$ to get

$$\mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) \times \mathrm{Ext}^q(F, F \otimes \Omega^q).$$

3. Take their product followed by the trace to get an element in $H^{q+2}(Y, \Omega^q)$. Multiplying by ω gives an element in $Y^n(Y, \Omega^n) \cong \mathbb{C}$.

Theorem 8.15. *This 2-form is closed, but may be degenerate.*

Proof. Since we can write the Kodaira–Spencer map in terms of Atiyah classes, the 2-form may be described as

$$T_S \times T_S \to T_S \times T_S \times \mathcal{E}xt^1_{\mathrm{pr}}(F, F \otimes \Omega^1)^{\times 2} \times \mathcal{E}xt^q_{\mathrm{pr}}(F, F \otimes \Omega^q).$$

Taking traces and plugging in T_S commute. So the whole 2-form is

$$T_S \times T_S \to T_S \times T_S \times \mathrm{Ext}^{q+2}(F, F \otimes \Omega^q) \to T_S \times T_S \times \mathrm{Ext}^{q+2} \times H^r(\Omega^{r+2})$$

followed by plugging in T_S and taking a trace. So basically this operation is to look at $\mathrm{at}_{q+2}(F)$, take the trace, and plug in two tangent vectors. We know

$$\mathrm{tr}(\mathrm{at}_{q+2}(F)) \in H^{q+2}(S \times Y, \Omega^{q+2}_{S \times Y})$$

is closed. By Künneth formula,

$$H^{q+2}(S \times Y, \Omega^{q+2}_{S \times Y}) = \bigoplus H^0(S, \Omega^j_S) \otimes H^{q+2}(Y, \Omega^{q-j}_Y)$$

because S is affine. All second components are d-closed from Hodge theory. Hence all first components are d-closed as well. Multiplication by ω and integrating just picks one of the elements in $H^0(S, \Omega^2_S)$ from $\mathrm{at}_{q+2}(F)$. It remains to check commutativity of some diagrams, which we omit. □

Recall that if Y is a cubic fourfold, then $D(Y) = \langle \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$. Pick $\omega \in H^{1,3}(Y)$. Then we have the following theorem.

Theorem 8.16. *If $\mathcal{M}$ parameterizes sheaves only from $\mathcal{A}_Y$, then the 2-form associated to ω is non-degenerate on $\mathcal{M}$ everywhere.*

Proof. Compare the recipe that we had with the recipe that uses Serre duality on $\mathcal{A}_Y$. □