Notes for Enumerative geometry seminar (Fall 2018): GW/DT correspondence

Henry Liu

December 5, 2018

Webpage: `http://math.columbia.edu/~hliu/seminars/f18-enumerative-geometry.html`

Abstract

These are my live-texed notes for the Fall 2018 student enumerative geometry seminar on the GW/DT correspondence. These notes have known omissions in the earlier talks. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 GW theory

1.1 Sep 11 (Melissa): Hodge/Hurwitz numbers and ELSV

Today we prove the ELSV formula, relating Hodge integrals with Hurwitz numbers.

Definition 1.1. A **Hodge integral** is an integral over $\bar{\mathcal{M}}_{g,n}$ of the form

$$\int_{[\bar{\mathcal{M}}_{g,n}]} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

Remark. Using Mumford's GRR, Hodge integrals can be expressed in terms of ψ integrals (called **descendant integrals**). Then Witten's conjecture that descendants satisfy the KdV hierarchy (and the string equation) determines all descendants from the initial value $\int_{\bar{\mathcal{M}}_{0,3}} 1 = 1$.

Witten's conjecture was first proved by Kontsevich. Another proof is in Okounkov–Pandharipande's GW/Hurwitz papers, in part 2, which derives Witten's conjecture from the ELSV formula by localization on $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$. The proof of ELSV is due to Graber–Vakil in "Hodge integrals and Hurwitz numbers via virtual localization".

Definition 1.2. Let $f \colon C \to \mathbb{P}^1$ be a ramified *connected* cover of genus g and degree d, with branch points $b_1, \ldots, b_r \in \mathbb{C} \subset \mathbb{P}^1$ and ∞. Assume that $b_1, \ldots, b_r$ are **simple** branch points, i.e. $f^{-1}(b_i)$ contains exactly $d-1$ points. Write

$$f^{-1}(\infty) =: \mu_1 x_1 + \cdots \mu_n x_n$$

as a divisor. **(Simple) Hurwitz numbers** $H_{g,\mu}$ count the number of such ramified covers for given μ. We know $|\mu| := \sum \mu_i = d$, i.e. μ is a partition of d, and we write $\ell(\mu) := n$ for the **length**.

Remark. By Riemann–Hurwitz,

$$2 - 2g = \chi(C) = d\chi(\mathbb{P}^1) + R = 2d - r - (d+n),$$

so that there are a total of $r = 2g - 2 + d + n = 2g - 2 + |\mu| + \ell(\mu)$ ramification points.

Remark. The monodromy around a simple branch point gives a transposition of two out of d sheets. Hence the simple Hurwitz number is

$$H_{g,\mu} = \frac{1}{Z_\mu} \# \left\{ \sigma_1, \ldots, \sigma_r, \sigma_\infty \in S_d : \begin{array}{c} \sigma_1, \ldots, \sigma_r \text{transpositions, } \sigma_\infty \in C_\mu \\ \sigma_1 \cdots \sigma_r \sigma_\infty = 1 \\ \text{(connectedness) } \langle \sigma_1, \ldots, \sigma_r \rangle \text{ acts transitively on } \{1, \ldots, d\} \end{array} \right\}$$

Here C_μ is the conjugacy class of the cycle class of μ, and $Z_\mu := \mu_1 \cdots \mu_n |\operatorname{Aut}(\mu)|$, where $\operatorname{Aut}(\mu)$ is non-trivial if there are identical numbers $\mu_i = \mu_{i+1} = \cdots$ in the partition.

Definition 1.3. The **disconnected Hurwitz numbers** $H^\bullet_{g,\mu}$ are exactly the same as above, but without the connectedness requirement. Both can be put into generating functions

$$\sum_g H_{g,\mu} \lambda^{2g-2+\ell(\mu)}, \quad \sum_g H^\bullet_{g,\mu} \lambda^{2g-2+\ell(\mu)}.$$

These can be explicitly evaluated via Burnside's formula.

Theorem 1.4 (ELSV formula)**.**

$$H_{g,\mu} = \frac{2g-2+|\mu+\ell(\mu)}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{[\bar{\mathcal{M}}_{g,n}]} \frac{1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{\ell(\mu)} (1 - \mu_i \psi_i)}.$$

Remark. The idea is to set up the moduli of relative stable maps, and then do localization. We will define the moduli only for $(\mathbb{P}^1, \infty)$.

Definition 1.5. The **moduli of relative stable maps to** $(\mathbb{P}^1, \infty)$ is given as follows.

1. Define the moduli space $\mathcal{M}_g(\mathbb{P}^1, \mu)$ of

$$f\colon (C, x_1, \dots, x_n) \xrightarrow{\deg d} \mathbb{P}^1$$

where $f^{-1}(\infty) = \mu_1 x_1 + \cdots + \mu_n x_n$ and C is smooth of genus g.

2. Define the compactification $\bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)$ of

$$f\colon (C, x_1, \dots, x_n) \to \mathbb{P}^1[m]$$

by allowing C to become nodal, and $\mathbb{P}^1[m]$ denotes attaching m extra $\mathbb{P}^1$ to the original $\mathbb{P}^1$. Let q_i' denote nodes in the $\mathbb{P}^1[m]$. Require compatibility conditions:

   (a) $f_i\colon C_i \to \mathbb{P}^1$ is degree $d = |\mu|$ where C_i is the preimage of the i-th $\mathbb{P}^1$;
   (b) $f^{-1}(q_m') = \mu_1 x_1 + \cdots + \mu_n x_n$;
   (c) (predeformable) $f^{-1}(q_i')$ is a union of nodes in C with the same contact order (so that we can simultaneously smooth nodes in the target and the source);
   (d) (stability) $\mathrm{Aut}(f)$ is finite.

   A homomorphism of two such relative stable maps is a commuting square

$$\begin{array}{ccc} (C, x_1, \dots, x_n) & \xrightarrow{f} & \mathbb{P}^1[m] \\ \phi\downarrow & & \downarrow \psi\in(\mathbb{C}^*)^m \\ (C', x_1', \dots, x_n') & \xrightarrow{f'} & \mathbb{P}^1[m] \end{array}$$

   where ψ can re-parameterize the extra $\mathbb{P}^1$. (The number m is bounded by stability once we fix g and μ.)

The compactification is the moduli of relative stable maps. It is a proper DM stack with perfect obstruction theory.

Definition 1.6. There is a **branch morphism** extending the usual one for *smooth* projective varieties

$$\mathrm{Br}\colon \bar{\mathcal{M}}_g(\mathbb{P}^1, \mu) \to \mathrm{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$$
$$[f\colon (C, x_1, \dots, x_n) \to \mathbb{P}^1[m]] \mapsto \sum \mathrm{Br}(f_i) + \sum (2g(B_i) - 2)[f(B_i)] + f_* N,$$

where in the normalization $\tilde{C} \to \mathbb{P}^1$ of $C \to \mathbb{P}^1$:

1. $f_i\colon A_i \to D$ are maps of uncontracted components;
2. $B_i \subset \tilde{C}$ are contracted components;
3. N is the divisor consisting of the nodes in C.

Hence the Hurwitz number $H_{g,\mu}$ is the degree of Br:

$$\begin{aligned} H_{g,\mu} &= \frac{1}{|\mathrm{Aut}(\mu)|} \deg(\mathrm{Br}\colon \bar{\mathcal{M}}_g(\mathbb{P}^1, \mu) \to \mathbb{P}^r) \\ &= \frac{1}{|\mathrm{Aut}(\mu)|} \int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} \mathrm{Br}^*(H^r) \end{aligned}$$

since $H^r = \mathrm{PD}(\mathrm{pt})$.

1.2 Sep 18 (Melissa): ELSV formula

Last time we identified Hurwitz numbers $H_{g,\mu}$ with the degree of a branch morphism. Today we will compute this via localization. First, a brief review of localization.

Definition 1.7. Let G be a Lie group and EG be a contractible topological space with free G-action. The **classifying space** of G is $BG \coloneqq EG/G$, defined up to homotopy equivalence. If X is a topological space with continuous G-action, then we can form the associated X-bundle

$$X_G \coloneqq EG \times_G X,$$

called the **homotopy orbit space**, with projection $\pi\colon X_G \to BG$. The G**-equivariant cohomology** $H^*_G(X, R) \coloneqq H^*(X_G, R)$ for any coefficient ring R.

Example 1.8. If G acts on X freely, then

$$H^*_G(X, R) = H^*(X_G, R) = H^*(X/G, R)$$

since X_G is homotopic to X/G by contractibility of EG. If G acts on X trivially, then

$$H^*_G(X, R) = H^*(X \times BG, R) = H^*(X, R) \otimes_R H^*(BG, R).$$

Remark. In general, because of

$$\pi^*\colon H^*(BG) \to H^*(X_G),$$

the G-equivariant cohomology of any space is always a $H^*_G(\mathrm{pt})$-module. There is also an inclusion $i\colon X \to X_G$ which induces

$$i^*\colon H^*(X_G) \to H^*(X),$$

the specialization to non-equivariant cohomology; this is well-defined because all fibers are homotopic.

Example 1.9. For $G = \mathbb{C}^*$, we see that it acts freely on $\mathbb{C}^\infty - \{0\}$. Hence

$$B\mathbb{C}^* = (\mathbb{C}^\infty - \{0\})/\mathbb{C}^* = \mathbb{C}^\infty \coloneqq \lim_{N\to\infty} \mathbb{CP}^n.$$

The equivariant cohomology is

$$H^*(B\mathbb{C}^*, \mathbb{Z}) = \lim_{N\to\infty} H^*(\mathbb{CP}^N, \mathbb{Z}) = \lim_{N\to\infty} \mathbb{Z}[u]/u^{N+1} = \mathbb{Z}[u].$$

Here $u \coloneqq c_1(\mathcal{O}_{\mathbb{CP}^N}(-1))$.

Definition 1.10. Let $V \to X$ be a G-equivariant complex vector bundle of rank r. Then $V_G \to X_G$ is still a vector bundle of rank r. The G**-equivariant Chern class** of V is

$$c_k^G(V) \coloneqq c_k(V_G) \in H^{2k}(X_G, \mathbb{Z}) = H^{2k}_G(X, \mathbb{Z}).$$

Example 1.11. Let $\mathbb{C}^*$ act on $\mathbb{C}$ via the standard weight, i.e.

$$\mathbb{C}^* \to \mathrm{GL}(1, \mathbb{C}), \quad t \mapsto t.$$

This gives a $\mathbb{C}^*$-equivariant line bundle $\mathbb{C}_t$. On $B\mathbb{C}^*$, this is

$$\mathbb{C}_{\mathbb{C}^*} = \mathcal{O}_{\mathbb{CP}^\infty}(-1) \to \mathbb{CP}^\infty = B\mathbb{C}^*.$$

We see that $c_1^{\mathbb{C}^*}(\mathbb{C}_t) = u$.

Example 1.12. The standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ induces an action on $\mathbb{P}^r = \operatorname{Sym}^r \mathbb{P}^1$ given by

$$t \cdot [a_0 : \cdots : a_r] \coloneqq [a_0 : t^{-1}a_1 : t^{-2}a_2 : \cdots : t^{-r}a_r].$$

The two fixed points $q_0, q_1 \in \mathbb{P}^1$ induce fixed points

$$\mathbb{P}^r \ni p_i \coloneqq \{a_j = 0 \; \forall j \neq i\} \leftrightarrow iq_0 + (r-i)q_1 \in \operatorname{Sym}^r \mathbb{P}^1.$$

Remark. If X is a compact complex manifold of dimension n, then there is a pushforward induced by $\pi\colon X_G \to BG$:

$$\pi_*\colon H^*_G(X) \to H^*_G(\mathrm{pt}), \quad \alpha \mapsto \int_{[X]} \alpha.$$

Suppose $G = \mathbb{C}^*$ for simplicity. Then this pushforward commutes with specialization to non-equivariant cohomology:

$$\begin{array}{ccc} H^*_{\mathbb{C}^*}(X) & \longrightarrow & H^{*-2n}_{\mathbb{C}^*}(\mathrm{pt}) \\ {\scriptstyle u\mapsto 0}\downarrow & & {\scriptstyle u\mapsto 0}\downarrow \\ H^*(X) & \longrightarrow & H^{*-2n}(\mathrm{pt}) \end{array} .$$

Example 1.13. Let $D_i \coloneqq \{a_i = 0\} \subset \mathbb{P}^r$ be a T-invariant divisor, and there are $r+1$ of them. We know $\mathrm{PD}(D_i) = c_1(\mathcal{O}_{\mathbb{P}^r}(D_i)) = H \in H^2(\mathbb{P}^r, \mathbb{Z})$. The cohomology $H^*(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}H/\langle H^{r+1}\rangle$ is saying $D_1 \cdot \cdots \cdot D_r = 0$. But equivariantly,

$$H_i \coloneqq \mathrm{PD}_{\mathbb{C}^*}(D_i) = c_1^{\mathbb{C}^*}(\mathcal{O}_{\mathbb{P}^r}(D_i)) \in H^2_{\mathbb{C}^*}(\mathbb{P}^r, \mathbb{Z}),$$

and $H_0|_{p_0} = 0 \in H^2_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{Z}u$ and

$$H_i|_{p_0} = -iu.$$

Hence $H_i = H_0 - iu$, and

$$H^*_{\mathbb{C}^*}(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}[H, u] / \prod_{i=0}^{r} (H_0 - iu).$$

Back to our situation: we want to compute

$$|\operatorname{Aut}(\mu)| H_{g,\mu} = \int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)]^{\mathrm{vir}}} \mathrm{Br}^*(H^r).$$

We can do this equivariantly, by the equivariant lift of H^r coming from the example above:

$$\int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)]^{\mathrm{vir}}} \mathrm{Br}^*(H^r) = \int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)]^{\mathrm{vir}}} \mathrm{Br}^* \prod_{i=0}^{r-1} (H_0 - iu).$$

We need to identify fixed points in order to apply equivariant localization. The branch morphism Br is $\mathbb{C}^*$-equivariant, so that

$$\mathrm{Br}\colon \bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)^{\mathbb{C}^*} \mapsto (\mathbb{P}^r)^{\mathbb{C}^*} = \{p_0, p_1, \ldots, p_r\}$$

where as we identified earlier, $p_i = iq_0 + (r-i)q_1$. Define $F_i \coloneqq \mathrm{Br}^{-1}(p_i)$. By localization,

$$|\operatorname{Aut}(\mu)| H_{g,\mu} = \sum_{j=0}^{r} \int_{[F_j]^{\mathrm{vir}}} \frac{i_j^* \mathrm{Br}^* \prod_{i=0}^{r-1} (H_0 - iu)}{e^{\mathbb{C}^*}(N_{F_j}^{\mathrm{vir}})}.$$

But over p_j, we have $\prod_{i=0}^{r-1} (H_0 - iu)|_{p_j} = 0$ unless $j = r$. Hence

$$|\operatorname{Aut}(\mu)| H_{g,\mu} = \int_{[F_j]^{\mathrm{vir}}} \frac{r! u^r}{e^{\mathbb{C}^*}(N_{F_r}^{\mathrm{vir}})}.$$

It suffices now to identify the fixed locus $\xi \in F_r \subset \bar{\mathcal{M}}_g(\mathbb{P}^1, \mu)$ such that $\mathrm{Br}(\xi) = rq_1$.

1.3 Sep 25 (Melissa): Resolved conifold

Let X be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, called the **resolved conifold**. It is a non-compact toric CY3. Let

$$i_0 \colon \mathbb{P}^1 \to X$$

denote the inclusion of the zero section. Let $\bar{\mathcal{M}}_g(X, d)$ denote the moduli of genus g degree d stable maps $f \colon C \to X$. The inclusion i_0 induces two things.

1. Because X is homotopy equivalent to $\mathbb{P}^1$, the induced $(i_0)_* \colon H_2(\mathbb{P}^1, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is an isomorphism. Hence $f_*[C] = d[\mathbb{P}^1]$ for $d \in \mathbb{Z}_{\geq 0}$.
2. There is an induced map $i_0 \colon \bar{\mathcal{M}}_g(\mathbb{P}^1, d) \to \bar{\mathcal{M}}_g(X, d)$.
   (a) When $d = 0$ we get $\bar{\mathcal{M}}_g(\mathbb{P}^1, 0) = \bar{\mathcal{M}}_g \times \mathbb{P}^1$; note that $\bar{\mathcal{M}}_g(X, 0) = \bar{\mathcal{M}}_g \times X$ is not compact.
   (b) When $d > 0$, compose $f \colon C \to X$ with the blow-down map $\pi \colon X \to X_0 \coloneqq \{ad - bc = 0\} \subset \mathbb{C}^4$. Since X_0 is affine, the induced $\tilde{f} \colon C \to X_0$ is constant, and hence the image of f must lie in $\pi^{-1}(0) = \mathbb{P}^1 \subset X$. Hence
   $$i_0 \colon \bar{\mathcal{M}}_g(\mathbb{P}^1, d) \hookrightarrow \bar{\mathcal{M}}_g(X, d)$$
   is actually an isomorphism of DM stacks, and since the former is proper, so is the latter. It follows that
   $$[\bar{\mathcal{M}}_g(X, d)]^{\mathrm{vir}} = e(V_{g,d}) \cap [\bar{\mathcal{M}}_g(\mathbb{P}^1, d)]^{\mathrm{vir}}$$
   where $V_{g,d}$ is a complex vector bundle of rank $2(d + g - 1)$, to be defined.

For any $g \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, define the genus g degree d GW invariant of the resolved conifold is defined by

$$N_{g,d} \coloneqq \deg[\bar{\mathcal{M}}_g(X, d)]^{\mathrm{vir}} = \int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1, d)]^{\mathrm{vir}}} e(V_{g,d}) \in \mathbb{Q}.$$

What is $V_{g,d}$? It should measure the difference of the deformations of a map to $\mathbb{P}^1$ vs a map to X. Given $[f \colon C \to X] \in \bar{\mathcal{M}}_g(X, d)$, the tangent space T^1_ξ and the obstruction space T^2_ξ fit into the following exact sequence of $\mathbb{C}$-vector spaces:

$$0 \to \mathrm{Aut}(C) \to \mathrm{Def}(f) \to T^1_\xi \to \mathrm{Def}(C) \to \mathrm{Obs}(f) \to T^2_\xi \to 0$$

where:

1. $\mathrm{Aut}(C) \coloneqq \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C)$ is infinitesimal automorphisms of the domain C (when C is smooth, this is just the space $H^0(C, T_C)$ of vector fields);
2. $\mathrm{Def}(C) \coloneqq \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ is infinitesimal deformations of the domain C (when C is smooth, this is just $H^1(C, T_C)$, which is first-order deformations of complex structures);
3. $\mathrm{Def}(f) \coloneqq H^0(C, f^*T_X)$ is infinitesimal deformations of the map f with fixed domain curve C;
4. $\mathrm{Obs}(f) \coloneqq H^1(C, f^*T_X)$ is infinitesimal obstructions to such deformations.

Imagine now we do the same thing for $\mathbb{P}^1$. Then the only difference in the tangent-obstruction theory is the difference between f^*T_X and $f^*T_{\mathbb{P}^1}$:

$$f^*T_X = f^*T_{\mathbb{P}^1} \oplus f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)).$$

Hence we have a splitting $H^i(C, f^*T_X) = H^i(C, f^*T_{\mathbb{P}^1}) \oplus H^i(C, f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))$. Since H^0 vanishes, the excess $H^1(C, f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))$ over each point ξ glue to form a vector bundle over $\bar{\mathcal{M}}_g(\mathbb{P}^1, d)$, and this vector bundle is precisely $V_{g,d}$.

Definition 1.14. Define the generating series

$$F(u,v) \coloneqq \sum_{d>0}\sum_{g\geq 0} N_{g,d} v^d u^{2g-2}.$$

We will compute $F(u,v)$ by virtual localization. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $t\cdot[x:y] \coloneqq [tx:y]$. Call $0=[0:1]$ and $\infty=[1:0]$. Then $T_0\mathbb{P}^1=\mathbb{C}_u$ and $T_\infty\mathbb{P}^1=\mathbb{C}_{-u}$. Lift the $\mathbb{C}^*$-action to $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. There are many possible linearizations; for every $a,b\in\mathbb{Z}$ we can choose weights

$$\begin{array}{ccc} & 0=[0:1] & \infty=[1:0] \\ \mathcal{O}(-1) & \mathbb{C}_{au} & \mathbb{C}_{(a+1)u} \\ \mathcal{O}(-1) & \mathbb{C}_{bu} & \mathbb{C}_{(b+1)u} \end{array}.$$

But it turns out the most convenient one is $a=-1$ and $b=0$. So we get a full $\mathbb{C}^*$-action on $\bar{\mathcal{M}}_g(\mathbb{P}^1,d)$ and $V_{g,d}$ is equivariant with respect to this action. Hence we can apply equivariant localization. We need to identify fixed components, i.e. components in $\bar{\mathcal{M}}_g(\mathbb{P}^1,d)^{\mathbb{C}^*}$, and the virtual normal bundle.

Definition 1.15. Given $\xi=[f\colon C\to\mathbb{P}^1]\in\bar{\mathcal{M}}_g(\mathbb{P}^1,d)^{\mathbb{C}^*}$, we can associate to it a decorated graph as follows:

1. (vertices) for each connected component C_v of $f^{-1}(\{0,\infty\})$, associate a vertex $v\in V(\Gamma)$;
2. (edges) for each connected component $O_e\cong\mathbb{C}^*$ of $f^{-1}(\mathbb{P}^1-\{0,\infty\})$, associate an edge $e\in E(\Gamma)$, and let $C_e \coloneqq \bar{O}_e\cong\mathbb{P}^1$;
3. (**flags**) $F(\Gamma)\coloneqq\{(e,v)\in E(\Gamma)\times V(\Gamma)\}: v\in e\}$;
4. (genus) label each vertex with its arithmetic genus $\vec{g}\colon V(\Gamma)\to\mathbb{Z}_{\geq 0}$, mapping C_v to its arithmetic genus;
5. (degree) label each edge with its degree $\vec{d}\colon E(\Gamma)\to\mathbb{Z}_{>0}$, so that $f|_{C_e}\colon\mathbb{P}^1\to\mathbb{P}^1$ is a degree d cover;
6. (marking) label each vertex with its marked points $\vec{f}\colon V(\Gamma)\to\{\text{subset of }\{1,\dots,n\}\}$.

Note that the total degree is $d=\sum_{e\in E(\Gamma)} d_e$ and the total genus is $g=\sum_{v\in V(\Gamma)} g_v + b_1(\Gamma)$ where b_1 is the first Betti number of the graph. Define the following subsets.

1. (stable vertices) $V^s(\Gamma)\coloneqq\{v\in V(\Gamma): C_v \text{ is stable}\}$.

Let $G_g(\mathbb{P}^1,d)$ be the set of such decorated graphs. (For each g,d clearly there are finitely many such graphs.)

These decorated graphs index fixed components. The structure of the fixed component associated to $(\Gamma,\vec{f},\vec{g},\vec{d})$ is

$$F_{\vec{\Gamma}} \coloneqq [(\prod_{v\in V_s(\Gamma)} \bar{\mathcal{M}}_{g_v,n_v})/A_{\vec{\Gamma}}]$$

where $A_{\vec{\Gamma}}$ is the stabilizer of the whole fixed component. It fits into a SES

$$1\to\prod_{e\in E(\Gamma)}\mathbb{Z}/d_e\mathbb{Z}\to A_{\vec{\Gamma}}\to\mathrm{Aut}(\vec{\Gamma})\to 1.$$

Lemma 1.16. *If Γ contains a vertex with valency $n_v>1$, then the restriction of $e_{\mathbb{C}^*}(V_{g,d})$ to the locus $F_{\vec{\Gamma}}$ is zero.*

Proof sketch. This arises from our convenient choice of linearization as follows. By normalization exact sequence, check that if there are any nodes of valency greater than 1, we will get zero weights and $e_{\mathbb{C}^*}(V_{g,d})=0$. □

Hence the only remaining graph with (possibly) non-zero contribution is from $\vec{\Gamma}$ with a single edge of degree d, from a vertex of genus g_1 to a vertex of genus g_2. Its normal bundle $N_\xi^{\mathrm{vir}} = T_\xi^{1,m} - T_\xi^{2,m}$ is the moving part in the tangent-obstruction sequence

$$0 \to (B_1 \coloneqq \mathrm{Aut}(C)) \to (B_2 \coloneqq \mathrm{Def}(f)) \to T_\xi^1 \to (B_4 \coloneqq \mathrm{Def}(C)) \to (B_5 \coloneqq \mathrm{Obs}(f)) \to T_\xi^2 \to 0.$$

It remains to evaluate the weights of each term:

1. $B_1 = \mathrm{Aut}(C_0, y_1, y_2) = H^0(C_0, TC_0(-y_1 - y_2)) = B_1^f$ because the only vector field fixing 0 and ∞ is $z\partial_z$, with trivial weight;
2. $B_4^m = T_{y_1}C_0 \otimes T_{y_1}C_1 \oplus T_{y_2}C_0 \otimes T_{y_2}C_2$, where note that C_1 is a d-fold cover of $\mathbb{P}^1$ so that $(T_yC_1)^{\otimes d} = T_y\mathbb{P}^1$;
3. We will continue next time!

1.4 Oct 02 (Melissa): Resolved conifold II

Theorem 1.17. *We have*

$$\sum_{g\geq 0} N_{g,d} u^{2g-2} = \frac{1}{d(2\sin(du/2))^2}.$$

Remark. On the GW side, we expand near $u = 0$. Hence we have

$$Z'_{GW} = \exp \sum_{d=1}^{\infty} \frac{v^d}{d(2\sin du/2)^2}.$$

We use $u^{-\chi}$ so that the series behaves nicely under degeneration.

Definition 1.18. Recall that fixed components $\xi = [f\colon C \to \mathbb{P}^1] \in \bar{\mathcal{M}}_g(\mathbb{P}^1, d)^{\mathbb{C}^*}$ correspond to decorated graphs $(\vec{\Gamma} \coloneqq (\Gamma, \vec{f}, \vec{g}, \vec{d})$ as follows.

1. (Vertices) For each connected component $C_v \in f^{-1}(\{0, \infty\})$, we associate a vertex $v \in V(\Gamma)$. Define two labels on vertices:
   (a) $\vec{f}(v) \coloneqq f(C_v)$, i.e. the label is either 0 or ∞;
   (b) $\vec{g}(v)$ is the genus of C_v (or 0 if C_v is a point).

   Let $V_0(\Gamma), V_\infty(\Gamma)$ be all vertices sitting over 0 and ∞ respectively.
2. (Edges) For each connected component $O_e \cong \mathbb{C}^*$ of $f^{-1}(\mathbb{P}^1 - \{0, \infty\})$, associate an edge $e \in E(\Gamma)$. Define the label on edges $\vec{d}\colon E(\Gamma) \to \mathbb{Z}_{>0}$ giving the degree of $f|_{O_e}$. Write $C_e \coloneqq \bar{O}_e \cong \mathbb{P}^1$.
3. (Flags) Define $F(\Gamma) \coloneqq \{(e, v) : v \in e\}$. Also, define $E_v \coloneqq \{e : v \in e\} \subset E(\Gamma)$ to be all edges incident to v, and let $n_v \coloneqq |E_v|$ be the valency of v. Hence

$$2g_v - 2 + n_v > 0.$$

Write $V(\Gamma) = V^I(\Gamma) \sqcup V^{II}(\Gamma) \sqcup V^s(\Gamma)$ where

$$V^I(\Gamma) = \{v : (g_v, n_v) = (0, 1)\}$$
$$V^{II}(\Gamma) = \{v : (g_v, n_v) = (0, 2)\}$$
$$V^s(\Gamma) = \{v : C_v \text{ is a curve}\}.$$

Similarly, write F^s to mean flags which involve stable vertices.

Do a partial normalization, at the nodes in $F^s(\Gamma) \sqcup V^{II}(\Gamma)$. Given a labeled graph $\vec{\Gamma}$, write

$$\mathcal{M}_{\vec{\Gamma}} := (\prod_{v \in V^s(\Gamma)} \bar{\mathcal{M}}_{g_v,n_v}), \quad F_{\vec{\Gamma}} := [\mathcal{M}_{\vec{\Gamma}}/A_{\vec{\Gamma}}].$$

Lemma 1.19. *We have*

$$[F_{\vec{\Gamma}}]^{vir} = \frac{1}{|A_{\vec{\Gamma}}|}(i_{\vec{\Gamma}})_*[\mathcal{M}_{\vec{\Gamma}}], \quad [\mathcal{M}_{\vec{\Gamma}}] = \prod_{v \in V^s(\Gamma)} [\bar{\mathcal{M}}_{g_v,n_v}].$$

Remark. By virtual localization, it follows that

$$N_{g,d} = \sum_{\vec{\Gamma} \in G_g(\Gamma,d)} I_{\vec{\Gamma}}, \quad I_{\vec{\Gamma}} := \frac{1}{|A_{\vec{\Gamma}}|} \int_{[\mathcal{M}_{\vec{\Gamma}}]^{\mathrm{vir}}} i^*_{\vec{\Gamma}} \frac{e_{\mathbb{C}^*}(V_{g,d})|_{F_{\vec{\Gamma}}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\vec{\Gamma}})}.$$

We will do the virtual normal bundle now.

Take a point $\xi = [f\colon C \to \mathbb{P}^1] \in F_{\vec{\Gamma}}$. We get an exact sequence

$$0 \to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, f^*T\mathbb{P}^1) \to T^1_\xi \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^1(C, f^*T\mathbb{P}^1) \to T^2_\xi \to 0.$$

We call the terms B_1, B_2, B_4, B_5 for short. Hence the virtual normal bundle is the difference of the *moving* parts, i.e. parts with non-trivial weight:

$$(N^{\mathrm{vir}}_{\vec{\Gamma}})_\xi = T^{1,m}_\xi - T^{2,m}_\xi.$$

Hence

$$\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\vec{\Gamma}})} = \frac{e_{\mathbb{C}^*}(B_1^m)e_{\mathbb{C}^*}(B_5^m)}{e_{\mathbb{C}^*}(B_2^m)e_{\mathbb{C}^*}(B_4^m)}.$$

So we just have to identify the weights of each piece B_i^m. These details are in Melissa's " Equivariant Gromov-Witten Invariants of Algebraic GKM Manifolds" paper, applied to $X = \mathbb{P}^1$.

We go into some detail about the vanishing for the obstruction bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, by request. The most general linearization is

$$\begin{array}{ccc} & 0 = [0:1] & \infty = [1:0] \\ \mathcal{O}(-1) & \mathbb{C}_{au} & \mathbb{C}_{(a+1)u} \\ \mathcal{O}(-1) & \mathbb{C}_{bu} & \mathbb{C}_{(b+1)u} \end{array}.$$

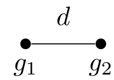
By normalization exact sequence, we get

$$\begin{aligned} 0 &\to H^0(C, f^*\mathcal{O}(-1)) \to \bigoplus_{v \in V^s} H^0(C_v) \oplus \bigoplus_{e \in E} H^0(C_e) \to (\mathbb{C}_{au})^{|F_0^s|+|V_0^2|} \oplus (\mathbb{C}_{(a+1)u})^{|F_\infty^s|+|V_\infty^2|} \\ &\to H^1(C, f^*\mathcal{O}(-1)) \to \bigoplus_{v \in V^s} H^1(C_v) \oplus \bigoplus_{e \in E} H^1(C_e) \to 0. \end{aligned}$$

This will give an Euler class

$$\begin{aligned} e_{\mathbb{C}^*}(V_{g,d}) = &\prod_{v \in V_0^s(\Gamma)} \Lambda^\vee_{g_v}(au)\Lambda^\vee_{g_v}(bu)((au)(bu))^{n_v-1} \\ &\prod_{v \in V_\infty^s(\Gamma)} \Lambda^\vee_{g_v}((a+1)u)\Lambda^\vee_{g_v}((b+1)u)(((a+1)u)((b+1)u))^{n_v-1} \\ &\prod_{e \in E} \prod_{j=1}^{d_e-1} (a + \frac{1}{d_e})(b + \frac{1}{d_e})u^{2d_e-1}. \end{aligned}$$

This vanishes when $a = 0$ and $b = -1$ for any vertex, stable or unstable, whenever there are vertices with $n_v > 1$. This is why only the graph



contributes, where $g_1 + g_2 = g$. The contribution of the g_1 vertex (over 0) is of the form

$$\frac{\Lambda^\vee_{g_1}(u)\Lambda^\vee_{g_1}(0)\Lambda^\vee_{g_1}(-1)}{u/d - \psi_1}$$

and similarly for the g_2 vertex (over ∞). Let

$$b_g := \begin{cases} 1 & g = 0 \\ \int_{\bar{\mathcal{M}}_{g,1}} \frac{\lambda_g}{1-\psi_1} & g > 0 \end{cases}$$

so that

$$N_{g,d} = \sum_{g_1+g_2=g} \frac{1}{d} \int_{\bar{\mathcal{M}}_{g,1}} \frac{\lambda_{g_1} u^{2g_1}}{u/d - \psi_1} \int_{\bar{\mathcal{M}}_{g,1}} \frac{\lambda_{g_2} (-u)^{2g_2}}{-u/d - \psi_1} \frac{(\frac{(d-1)!}{d^{d-1}})^2 u^{2d-2} (-1)^{d-1}}{(\frac{d!}{d^d})^2 u^{2d} (-1)^d}.$$

Putting everything into a generating function, we get

$$\sum_{g \geq 0} N_{g,d} u^{2g-2} = \frac{1}{u^2 d^3} \left(\sum_{g \geq 0} b_g (du)^{2g} \right)^2.$$

From the Faber–Pandharipande evaluation of Hodge integrals, we know

$$\sum_{g \geq 0} b_g t^{2g} = \frac{t/2}{\sin(t/2)}.$$

Simplifying gives the desired theorem from the beginning of today's lecture.

1.5 Oct 09 (Melissa): Relative GW theory

Let's quickly review the tangent-obstruction theory for (absolute) GW theory. This is to prepare for the tangent-obstruction theory in relative GW theory. Fix X a non-singular projective variety over $\mathbb{C}$ and $\beta \in H_2(X, \mathbb{Z})$ an effective curve class. Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle \pi}\downarrow & & \\ \bar{\mathcal{M}}_{g,n}(X, \beta) & & \end{array}$$

where $\pi \colon \mathcal{C} \to \bar{\mathcal{M}}_{g,n}$ is the universal domain and $\tilde{f} \colon \mathcal{C} \to X$ is the universal map. We can also consider the forgetful map

$$q \colon \bar{\mathcal{M}}_{g,n}(X, \beta) \to \mathfrak{M}^{\mathrm{pre}}_{g,n}$$

to the Artin stack of prestable curves of genus g with n marked points. At the point $\xi = [(C, x_1, \ldots, x_n)] \in \mathfrak{M}^{\mathrm{pre}}_{g,n}$, we have

$$\mathrm{Lie}(\mathrm{Aut}(\xi)) = \mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C), \quad \mathrm{Def}(\xi) = \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C)$$

where $D := x_1 + \cdots + x_n$. Inside $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ sits the proper smooth DM stack $\bar{\mathcal{M}}_{g,n}$ of stable curves, which inherits this tangent-obstruction theory. Hence to compute the virtual dimension of $\bar{\mathcal{M}}_{g,n}(X, \beta)$, we can first compute $\mathrm{vdim}\, \bar{\mathcal{M}}_{g,n} = 3g - 3 + n$, and then say

$$\mathrm{vdim}\, \bar{\mathcal{M}}_{g,n}(X, \beta) = 3g - 3 + n + (\text{relative dimension of } q).$$

But the relative tangent-obstruction theory for q is $\operatorname{Def}(f) - \operatorname{Obs}(f)$, i.e. the deformation theory of the map f, which we know is

$$\operatorname{Def}(f) - \operatorname{Obs}(f) = H^0(C, f^*T_X) - H^1(C, f^*T_X) = \chi(C, f^*T_X).$$

Putting this all together, we get

$$\operatorname{vdim} \bar{\mathcal{M}}_{g,n}(X, \beta) = \int_\beta c_1(T_X) + (\dim X - 3)(1 - g) + n.$$

Now we do this in the relative case.

Definition 1.20. Let X be a non-singular projective variety over $\mathbb{C}$, with a smooth divisor $D \subset X$. Fix an effective curve class $\beta \in H_2(X, \mathbb{Z})$ such that $\beta \cdot D \coloneqq \int_\beta c_1(\mathcal{O}(D)) \geq 0$. Let $\mu = \mu_1 \geq \cdots \geq \mu_\ell > 0$ be a partition of $\beta \cdot D$. Define the **moduli space of relative stable maps** $\bar{\mathcal{M}}_{g,n}(X/D, \beta, \mu)$ to parametrize objects

$$f\colon (C, x_1, \ldots, x_n, y_1, \ldots, y_\ell) \to X[k] \coloneqq X \cup_{D_0} \Delta_1 \cup_{D_1} \cdots \cup_{D_k} \Delta_k = D_k$$

where $\Delta_i \coloneqq \mathbb{P}(N_{D/X} \oplus \mathcal{O})$ and $D_i \cong D$ for $i = 0, 1, \ldots, k$, such that $f^{-1}(D_k) = \sum_{i=1}^\ell \mu_i y_i$. Again we have a universal domain and universal target

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\tilde{f}} & \mathcal{T} \\
\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \tilde{\pi}} & \\
\bar{\mathcal{M}}_{g,n}(X/D, \beta, \mu) & \xrightarrow{\text{target}} & \mathcal{B} \\
\downarrow{\scriptstyle q} & & \\
\mathfrak{M}^{\text{pre}}_{g,n+\ell} & &
\end{array}$$

To understand what the universal target $\mathcal{B}$ is, look at $\mathfrak{X} \coloneqq \lim[\mathcal{X}[k]/\mathbb{G}_m^k]$ mapping to $\mathcal{B} \coloneqq \lim[\mathbb{A}^k/\mathbb{G}_m^k]$, where $\mathcal{X}[k]$ is constructed as follows.

1. Set $\mathcal{X}[1] \coloneqq \operatorname{Bl}_{D \times 0}(X \times \mathbb{A}^1)$. When $t = 0 \in \mathbb{A}^1$, we get $X \sqcup_{D_0} \Delta$, and otherwise we just get (X, D). There is a $\mathbb{G}_m$-action acting on the $\mathbb{A}^1$, giving

   $$[\mathcal{X}[1]/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$$

   and we are supposed to view $[\mathbb{A}^1/\mathbb{G}_m]$ as the moduli corresponding to the total space $[\mathcal{X}[1]/\mathbb{G}_m]$.

2. Set $\mathcal{X}[2] \coloneqq \operatorname{Bl}_{D[1] \times \mathbb{A}^1}(\mathcal{X}[1] \times \mathbb{A}^1)$. Now there are two parameters $t = (t_1, t_2)$, and when $t_1 = t_2 = 0$ we get $X[2]$. There is now a $\mathbb{G}_m^2$-action, and we get

   $$[\mathcal{X}[2]/\mathbb{G}_m^2] \to [\mathbb{A}^2/\mathbb{G}_m^2].$$

3. Continue in a similar fashion.

The universal target $\mathcal{T}$ is formed by pullback of $\mathcal{X} \to \mathcal{B}$ to $\bar{\mathcal{M}}$, i.e.

$$\begin{array}{ccc}
\mathcal{T} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\bar{\mathcal{M}}_{g,n}(X/D, \beta, \mu) & \longrightarrow & \mathcal{B}
\end{array}$$

is Cartesian.

Now let's describe the tangent-obstruction theory for $\bar{\mathcal{M}}_{g,n}(X/D,\beta,\mu)$. Fix $\xi \coloneqq [f\colon (C, D_x, D_y) \to X]$ where D_x is the marked points on the domain and D_y is marked points intersecting with D. Then we get

$$\begin{aligned} 0 &\to \operatorname{Ext}^0(\Omega_C(D_x+D_y), \mathcal{O}_C) \to H^0(C, f^*\Omega_X(\log D)^\vee) \to T^1_\xi \\ &\to \operatorname{Ext}^1(\Omega_C(D_x+D_y), \mathcal{O}_C) \to H^1(C, f^*\Omega_X(\log D)^\vee) \to T^2_\xi \to 0. \end{aligned}$$

Here the two terms $H^i(C, f^*\Omega_X(\log D)^\vee)$ are the relative tangent-obstruction theory for the map q at the point ξ. If there were no $\log D$, then this would just be f^*T_X. But the $\log D$ requires the section to vanish to some degree: if $z_1, \ldots, z_N$ are local coordinates on X with $D = \{z_N = 0\}$, then locally

$$\Omega_X(\log D) = \langle dz_1, \ldots, dz_{N-1}, \frac{dz_N}{z_N} = d\log(z_N)\rangle.$$

It follows that

$$\operatorname{vdim} \bar{\mathcal{M}}_{g,n}(X/D,\beta,\mu) = 3g-3+n+\ell+\chi(C, f^*\Omega_X(\log D)^\vee).$$

But $\Omega_X(\log D)$ is a vector bundle over C of degree $\int_\beta c_1(T_X) - \beta\cdot D$ and rank $\dim X$. Putting this all together, we get the following.

Proposition 1.21. *The virtual dimension of the moduli of relative stable maps is*

$$\operatorname{vdim} \bar{\mathcal{M}}_{g,n}(X/D,\beta,\mu) = (\int_\beta c_1(T_X) + (\dim X - 3)(1-g) + n) + (\ell - \beta\cdot D).$$

We see that the second term is new, and can also be written as $\ell - \sum_{i=1}^\ell \mu_i$. View this as the codimension arising from the relative condition. In the generic case $\mu = (1,\ldots,1)$, there is no codimension.

Now in the general case of $\xi \coloneqq [f\colon (C, D_x, D_y) \to X[k]]$, we need the more general exact sequence

$$\begin{aligned} 0 &\to H^0(C, f^*\Omega_{X[k]}(\log D_k)^\vee) \to \bigoplus_{m=0}^{r-1} H^0_{\mathrm{et}}(R_m) \to H^0(D) \\ &\to H^1(C, f^*\Omega_{X[k]}(\log D_k)^\vee) \to \bigoplus_{m=0}^{r-1} H^1_{\mathrm{et}}(R_m) \to H^1(D) \to 0. \end{aligned}$$

What is $H^i_{\mathrm{et}}(R_m)$? Think: it is supposed to be the deformation theory of q which is "compatible" with the smoothing of nodes in the domain at D_i. Over each D_i there are line bundles $L_i \coloneqq N_{D_i/\Delta_i} \otimes N_{D_i/\Delta_{i+1}}$. Define

$$\begin{aligned} H^0_{\mathrm{et}}(R_i) &\coloneqq \bigoplus_{q\in f^{-1}(D_i)} \mathcal{O}_{D_\ell} \\ H^1_{\mathrm{et}}(R_i) &\coloneqq H^0(D_i, L_i)^{\oplus n_i}/\Delta, \quad n_i \coloneqq \# f^{-1}(D_i). \end{aligned}$$

Here Δ is the diagonal. We can view R_i as the ramification divisor at D_i.

Now let's look at gluing formulas. Take a simple degeneration $\mathcal{Y} \to \mathbb{A}^1$, with:

$$Y_t = Y, \quad Y_0 = X_1 \sqcup_D X_2.$$

When we look at $\bar{\mathcal{M}}_{g,n}(Y_t,\beta)$, think of β as an element of $\operatorname{Hom}(\operatorname{Pic}(\mathcal{Y}), \mathbb{Z})$, because now in general we can have monodromy when we go around 0. This is in general coarser than H_2. There is a cobordism argument that says that in Chow,

$$[\bar{\mathcal{M}}_{g,n}(Y_0,\beta)]^{\mathrm{vir}} = [\bar{\mathcal{M}}_{g,n}(Y_t,\beta)]^{\mathrm{vir}}, \quad \forall t \neq 0.$$

The rhs is GW invariants on $Y = Y_t$. The lhs can be expressed using the relative moduli $\bar{\mathcal{M}}_{g,n}(X_i/D,\beta,\mu)$ as follows. To write the formula, it is more convenient to do the disconnected invariants $\bar{\mathcal{M}}^\bullet_{g,n}(\ldots)$, because if we break a curve it may become disconnected. Let $\beta_1 + \beta_2 = \beta$ and $\mu_1 \geq \cdots \geq \mu_\ell > 0$ be a partition of $\beta \cdot D$. There are evaluation maps which fit into a square

$$\begin{array}{ccc} \mathcal{M}_1 \times_{D^l} \mathcal{M}_2 & \longrightarrow & \bar{\mathcal{M}}^\bullet_{g,n}(X/D, \beta_2, \mu) =: \mathcal{M}_2 \\ \big\downarrow & & \big\downarrow \scriptstyle{\bar{\mathrm{ev}}} \\ \mathcal{M}_1 := \bar{\mathcal{M}}^\bullet_{g,n}(X/D, \beta_1, \mu) & \xrightarrow{\bar{\mathrm{ev}}} & D^\ell \end{array}$$

Let $\Delta \colon D^\ell \to D^{2\ell}$ be the diagonal. Then we have another diagram

$$\begin{array}{ccc} \mathcal{M}_1 \times_{D^\ell} \mathcal{M}_2 & \longrightarrow & D^\ell \\ \big\downarrow & & \big\downarrow \scriptstyle{\Delta} \\ \mathcal{M}_1 \times \mathcal{M}_2 & \xrightarrow{\bar{\mathrm{ev}} \times \bar{\mathrm{ev}}} & D^\ell \times D^\ell \end{array}$$

with $[\mathcal{M}_1 \times_{D^\ell} \mathcal{M}_2]^{\mathrm{vir}} = \Delta^!([\mathcal{M}_1]^{\mathrm{vir}} \times [\mathcal{M}_2]^{\mathrm{vir}})$. Hence we have an equality

$$[\bar{\mathcal{M}}^\bullet_{g,n}(Y_0, \beta)]^{\mathrm{vir}} = \sum_{\substack{\mu \vdash \beta_1 \cdot D = \beta_2 \cdot D \\ \mathrm{length}(\mu) = \ell}} \frac{\mu_1 \cdots \mu_\ell}{|\operatorname{Aut}(\mu)|} [\bar{\mathcal{M}}_{g_1, n_1}(X_1/D, \beta_1, \mu) \times_{D^\ell} \bar{\mathcal{M}}_{g_2, n_2}(X_2/D, \beta_2, \mu)]^{\mathrm{vir}}$$

for $g_1 + g_2 = g$ and $\beta_1 + \beta_2 = \beta$.

1.6 Oct 16 (Henry): The GW local curves TQFT

All manifolds are oriented, and we work over $\mathbb{C}$. Given a manifold Y, denote by $-Y$ the same manifold with opposite orientation. (We also assume our QFTs are anomaly-free.)

Definition 1.22. A $(n+1)$**-dimensional TQFT** is a symmetric monoidal functor

$$Z \colon (n+1)\mathsf{Cob} \to \mathsf{Vect}_{\mathbb{C}}$$

from the category of cobordisms to the category of vector spaces. Concretely, this means the following data.

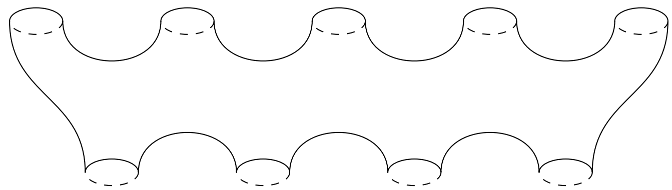
1. Associated to each closed n-dimensional manifold Y is a vector space $\mathcal{H}_Y$ called the **(quantum) state space** satisfying:
   - (gluing) $\mathcal{H}_\emptyset = \mathbb{C}$ and $\mathcal{H}_{Y_1 \sqcup Y_2} = \mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$;
   - (orientation) $\mathcal{H}_{-Y} = \mathcal{H}_Y^*$.
   - (functoriality) if $f \colon Y \to Y'$ is a diffeomorphism, then there is an induced isomorphism $f_* \colon \mathcal{H}_Y \to \mathcal{H}_{Y'}$.
2. Associated to each compact $(d+1)$-dimensional manifold X is an element $Z_X \in \mathcal{H}_{\partial X}$ called the **partition function**. To work with Z_X it helps to imagine $\mathcal{H}_{\partial X}$ as the collection of functions on "boundary conditions" on X, and Z_X as a function that takes a boundary condition and spits out the number of states satisfying that boundary condition on ∂X. This assignment must satisfy the following.
   - (Functoriality) if $f \colon X \to X'$ is a diffeomorphism with $\partial f \colon \partial X \to \partial X'$, then $(\partial f)_* Z_X = Z_{X'}$. (This is why we say the theory is "topological".)
   - (Gluing) Suppose $X = X_1 \sqcup_Y X_2$, i.e. X is obtained by gluing $(d+1)$-folds X_1 and X_2 along a common boundary Y.
   $$Z_X = \mathrm{tr}_{\mathcal{H}_Y}(Z_{X_1} \otimes Z_{X_2}).$$

Think: Z_{X_i} counts how many states on X_i satisfy a given boundary condition $Q \in \mathcal{H}_Y$ on Y, so if $\{Q_i\}$ is a basis for $\mathcal{H}_Y$, then

$$\#(\text{states in } X) = \sum_i \#(\text{states in } X_1 \text{ satisfying } Q_i) \cdot \#(\text{states in } X_2 \text{ satisfying } Q_i),$$

which is exactly the formula above.

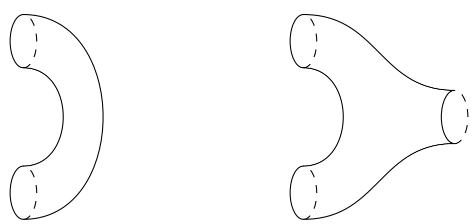
In $(1+1)$ dimensions, TQFTs have a structure that we can really get our hands on. The key idea is that any compact orientable surface S with boundary and genus zero looks like this:



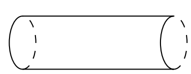
This is because the boundary ∂S is a closed 1-manifold, which is always a disjoint union of a finite number of circles S^1. So the only state space we need to consider is $\mathcal{H} = \mathcal{H}_{S^1}$, associated to "incoming" circles, and its dual $\mathcal{H}^* = \mathcal{H}_{-S^1}$, associated to "outgoing" circles, which have the opposite orientation. A surface S with m incoming circles and n outgoing circles will correspond to a map

$$\mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}.$$

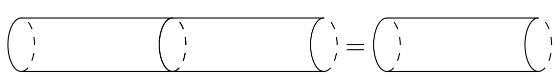
Example 1.23. The following is an inner product $\langle -, - \rangle \colon \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ and a multiplication operator $m(-,-) \colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$:



Example 1.24 (Identity map)**.** Consider the cylinder

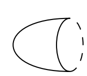
 $:= Z_C \colon \mathcal{H} \to \mathcal{H}.$

Usually we restrict our state space $\mathcal{H}$ so that Z_C is surjective. Then

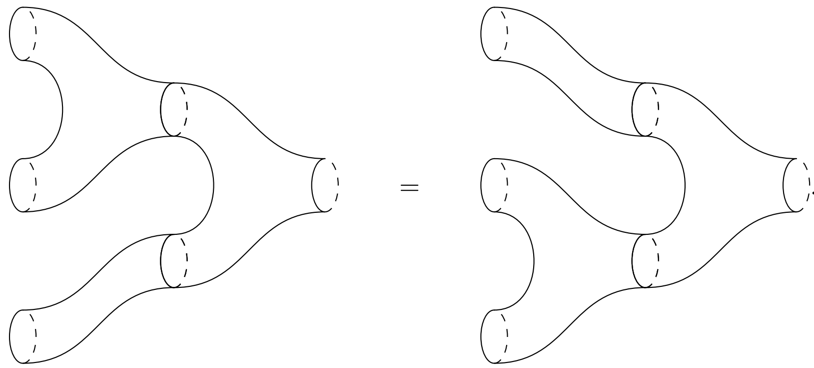


Hence $Z_C = Z_C \circ Z_C$. Idempotents are the identity on their image, so $Z_C = \mathrm{id} \colon \mathcal{H} \to \mathcal{H}$.

Proposition 1.25 (2d TQFT = Frobenius algebra)**.** *$\mathcal{H}$ with $\langle -, - \rangle$ and $m(-,-)$ has the structure of a Frobenius algebra:*

1. *it is a commutative and associative algebra, with unit $D^2 =$*  *$\in \mathcal{H}$;*
2. *multiplication is compatible with the non-degenerate inner product, i.e. $\langle ab, c \rangle = \langle a, bc \rangle$.*

Proof. The diagrammatic proof of associativity is as follows:



This uses the diffeomorphism invariance of the partition function. The others are left as an exercise. □

Remark. Clearly we don't have to map to Vect_k; we can do Mod_R for any commutative unital ring R. Later we will take $R = \mathbb{Q}(t_1, t_2)((u))$.

Remark. The following are equivalent:

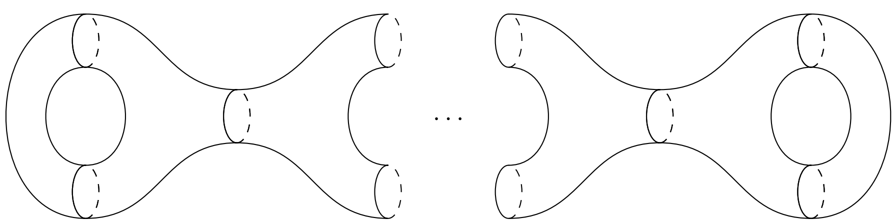
1. $\mathcal{H} = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ is a semisimple algebra;
2. $\mathcal{H}$ has an idempotent basis $\{e_i\}$ (with dual basis $\{e^i\}$ using $\langle -, - \rangle$);
3. $\langle -, - \rangle$ is a non-degenerate inner product.

Semisimplicity is a very important structural result: it means we can piece together partition functions for whole surfaces using partition functions of pieces, as follows.

Proposition 1.26. *For a semisimple 2-TQFT, let $\lambda_i := \langle e_i, e_i \rangle$ be its structure constants. Then*

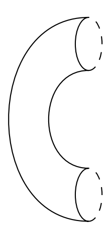
$$Z_{\Sigma_g} = \sum_i \lambda_i^{1-g}.$$

Proof. Do a pair of pants decomposition of Σ_g:



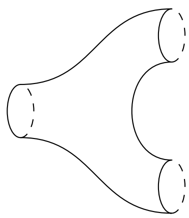
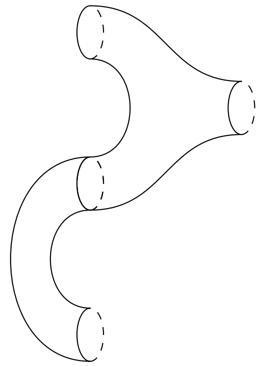
We need to compute the two pieces we don't know yet.

1. Compute the value of  $\in \mathcal{H} \otimes \mathcal{H}$ as follows. It arises from dualizing the second factor in id $= \sum_i e_i \otimes e^i \in \mathcal{H} \otimes \mathcal{H}^*$ where $\{e^i := e_i/\langle e_i, e_i \rangle\}$ is the dual basis to e_i. This means



$$= \sum_i e_i \otimes \frac{e_i}{\langle e_i, e_i \rangle} \in \mathcal{H} \otimes \mathcal{H}.$$

2. Using this, we compute

 $=$  $= \left[x \mapsto \sum_i (xe_i) \otimes \frac{e_i}{\langle e_i, e_i \rangle} \right] : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}.$

To get Z_{Σ_g}, compose all the pieces in the pairs of pants decomposition from left to right. Recalling that $e_i e_j = 0$ for $i \neq j$, we get

$$\sum_i \frac{e_i \otimes e_i}{\lambda_i} \mapsto \sum_i \frac{e_i}{\lambda_i} \mapsto \sum_i \frac{e_i \otimes e_i}{\lambda_i^2} \mapsto \sum_i \frac{e_i}{\lambda_i^2} \mapsto \cdots \mapsto \sum_i \frac{e_i \otimes e_i}{\lambda_i^g} \mapsto \sum_i \frac{\lambda_i}{\lambda_i^g}.$$

Hence the final result is $Z_{\Sigma_g} = \sum_i \lambda_i^{1-g}$, as desired. □

Definition 1.27. The **local curve** case for GW involves the following data:

1. a smooth irreducible projective curve X of genus g;
2. a rank-2 bundle $N \coloneqq L_1 \oplus L_2$ over X, of degrees or **level** (k_1, k_2);
3. a *possibly disconnected* source curve C of genus h whose image has degree β.

Let T^2 act on N with equivariant parameters t_1, t_2, so that

$$[\bar{\mathcal{M}}_h^\bullet(N, d[X])^T]^{\mathrm{vir}} = [\bar{\mathcal{M}}_h^\bullet(X, d)]^{\mathrm{vir}}.$$

Here $(-)^\bullet$ denotes disconnected invariants. By localization we can define:

1. the **reduced GW partition function**
$$Z_d'(N) \coloneqq \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\bar{\mathcal{M}}_h^\bullet(X,d)]^{\mathrm{vir}}} e(-R^\bullet \pi_* f^*(L_1 \oplus L_2)) \in \mathbb{Q}(t_1, t_2)((u));$$
2. the **GW generating function**
$$GW_d(g; k_1, k_2) \coloneqq u^{d(2-2g+k_1+k_2)} Z_d'(N) \in \mathbb{Q}(t_1, t_2)((u)).$$

   We pick the exponent of u so that gluing rules (later) are nice. In particular, $(2h-2) + d(2-2g)$ is the dimension of $\bar{\mathcal{M}}_h(X, d)$, and levels add.

Remark. Every vector bundle on a curve is deformation equivalent to a sum of line bundles; this is why it suffices to do the split case. This is because every vector bundle $\mathcal{E}$ on a curve has a filtration by line bundles: twist so that $\mathcal{E}(n)$ is globally generated, but a generic global section has zero locus of dimension $\dim X - \operatorname{rank} \mathcal{E} < 0$, so

$$0 \to \mathcal{O}_X \to \mathcal{E}(n) \to \mathcal{F} \to 0$$

and we can induct. In K-theory this means $\mathcal{E}$ is a positive linear combination of line bundles. This is not true in higher dimension: $T_{\mathbb{P}^2} = 2\mathcal{O}(1) - \mathcal{O}$ requires that negative term. Now within each extension of line bundles, we can "deform the Ext class", i.e. form a universal family over $X \times \mathrm{Ext}^1(L_2, L_1)$ to make it trivial, and we are done.

Remark. Write down the dependence of $GW_d(g;k_1k_2)$ on its variables, to get rid of the sum over h (by dimension axiom).

1. To make the dependence on t_1, t_2 clear, define

$$GW_d^{b_1,b_2}(g;k_1,k_2) \coloneqq \int_{[\bar{\mathcal{M}}_h^\bullet(X,d)]^{\mathrm{vir}}} c_{b_1}(-R^\bullet \pi_* f^* L_1) c_{b_2}(-R^\bullet \pi_* f^* L_2),$$

   so that the total $GW_d(g;k_1,k_2)$ is a sum over b_1, b_2 of these pieces. The nice thing about these pieces is t_1, t_2 pull out of them as follows.

   (a) The degree of t_1 in c_{b_1} is

$$\operatorname{rank}(-R^\bullet \pi_* f^* L_1) - b_1 = -\chi(C, f^*L_1) - b_1 = -(\deg f^* L_1 + 1 - h) - b_1 = h - 1 - dk_1 - b_1.$$

   (b) We don't want a dependence on the genus h of the source curve, because that will vary. Compute the virtual dimension

$$b_1 + b_2 = \operatorname{vdim} \bar{\mathcal{M}}_h^\bullet(X,d) = (\dim X - 3)(1-h) + \int_{d[X]} c_1(T_X) = 2h-2+d\deg T_X = 2h-2+d(2-2g).$$

   Hence $h - 1 = (1/2)(b_1+b_2) + d(g-1)$.

   It follows that the exponent of t_1 is $(1/2)(b_2 - b_1) + d(g-1-k_1)$.

2. The variable u indexes the quantity

$$2h - 2 + \int_{d[X]} c_1(T_N) = 2h - 2 + d(2 - 2g + k_1 + k_2) = b_1 + b_2 + d(k_1 + k_2).$$

In total, we have

$$GW_d(g;k_1,k_2) = u^{d(k_1+k_2)} t_1^{d(g-1-k_1)} t_2^{d(g-1-k_2)} \sum_{b_1,b_2=0}^{\infty} u^{b_1+b_2} t_1^{\frac{1}{2}(b_2-b_1)} t_2^{\frac{1}{2}(b_1-b_2)} GW_d^{b_1,b_2}(g;k_1,k_2).$$

This will be super helpful later, because it suffices to compute the *number* $GW_d^{b_1,b_2}(g;k_1,k_2)$, and insert t_1, t_2 manually.

Definition 1.28. Let $\bar{\mathcal{M}}_h(X, \lambda^1, \ldots, \lambda^r)$ be the moduli of **relative stable maps** to a curve X of genus g, with prescribed ramification profiles $\lambda^1, \ldots, \lambda^r$ (all partitions of d) at given points $x_1, \ldots, x_r \in X$. Melissa showed us that for one ramification, the codimension of $\bar{\mathcal{M}}_h(X,\lambda)$ is

$$|\lambda| - \ell(\lambda) = d - \ell(\lambda),$$

so that the codimension for multiple ramifications is

$$\operatorname{co\,vdim} \bar{\mathcal{M}}_h(X, \lambda^1, \ldots, \lambda^r) = \delta \coloneqq \sum_{i=1}^{r} (d - \ell(\lambda^i)).$$

As with the absolute case, we can shift $Z'(N)_{\lambda^1,\ldots,\lambda^r}$ by $u^{d(2-2g+k_1+k_2-r)+\sum_{i=1}^r \ell(\lambda^i)}$ so that

$$GW(g;k_1,k_2)_{\lambda^1,\ldots,\lambda^r} \coloneqq u^{d(k_1+k_2)} t_1^{d(g-1-k_1)} t_2^{d(g-1-k_2)} \sum_{b_1,b_2=0}^{\infty} u^{b_1+b_2} t_1^{\frac{1}{2}(b_2-b_1+\delta)} t_2^{\frac{1}{2}(b_1-b_2+\delta)} GW_d^{b_1,b_2}(g;k_1,k_2).$$

The idea now is to make a 2-TQFT out of the partition functions $GW(g;k_1,k_2)_{\lambda^1,\ldots,\lambda^r}$, where each incoming/outgoing state is a ramification condition λ^i. This means we need some prescription for turning incomings into outgoings, i.e. for dualizing. The factor we use is whatever makes the gluing formula work.

Definition 1.29. To raise indices, use $\mathfrak{z}(\lambda)(t_1t_2)^{\ell(\lambda)}$, i.e. define

$$GW(g;k_1,k_2)^{\nu^1\cdots\nu^t}_{\mu^1\cdots\mu^s} \coloneqq GW(g;k_1,k_2)_{\mu^1\cdots\mu^s,\nu^1\cdots\nu^t}\prod_{i=1}^t \mathfrak{z}(\nu^i)(t_1t_2)^{\ell(\nu^i)}.$$

Theorem 1.30. *For $g=g'+g''$ and $k_i=k_i'+k_i''$,*

$$GW(g;k_1,k_2)^{\nu^1\cdots\nu^t}_{\mu^1\cdots\mu^s} = \sum_{\lambda\vdash d} GW(g';k_1',k_2')^{\lambda}_{\mu^1\cdots\mu^s} GW(g'';k_1'',k_2'')^{\nu^1\cdots\nu^t}_{\lambda},$$

and

$$GW(g;k_1,k_2)_{\mu^1\cdots\mu^s} = \sum_{\lambda\vdash d} GW(g-1;k_1,k_2)^{\lambda}_{\mu^1\cdots\mu^s\lambda}.$$

Proof. We prove a simpler case:

$$GW(g;k_1,k_2) = \sum_{\lambda\vdash d} GW(g';k_1',k_2')_\lambda GW(g'';k_1'',k_2'')_\lambda \mathfrak{z}(\lambda)(t_1t_2)^{\ell(\lambda)}.$$

The general case requires a little more work (see Theorem 21, Jun Li's lecture notes on relative GW invariants). Melissa showed us last time that in a degeneration of Y to $Y_0 = X_1\cup_D X_2$,

$$\begin{aligned}[\bar{\mathcal{M}}^\bullet_{g,n}(Y,\beta)]^{\mathrm{vir}} &= [\bar{\mathcal{M}}^\bullet_{g,n}(Y_0,\beta)]^{\mathrm{vir}} \\ &= \sum_{\substack{\mu\vdash\beta_1\cdot D=\beta_2\cdot D\\ \mathrm{length}(\mu)=\ell}} \mathfrak{z}(\mu)[\bar{\mathcal{M}}_{g_1,n_1}(X_1/D,\beta_1,\mu)\times_{D^\ell}\bar{\mathcal{M}}_{g_2,n_2}(X_2/D,\beta_2,\mu)]^{\mathrm{vir}}.\end{aligned}$$

Specifically, let's focus on the component of $\bar{\mathcal{M}}^\bullet_{g,n}(Y_0,\beta)$ which corresponds to degenerations of type $\mu=\lambda$ (in the sum). We already see all the factors except $(t_1t_2)^{\ell(\lambda)}$. This factor comes from the integrand $e(-R^*\pi_*f^*(L_1\oplus L_2))$ as follows. If we degenerate the target and source

$$X = X'\cup X'',\quad C = C'\cup C'',$$

the line bundles L_1,L_2 must split with degrees $k_1=k_1'+k_1''$ and $k_2=k_2'+k_2''$, and for each line bundle L_i there is a normalization sequence

$$0\to f^*(L_i)|_C \to f^*(L_i)|_{C'}\oplus f^*(L_i)|_{C''}\to f^*(L_i)|_{C'\cap C''}\to 0.$$

But $|C'\cap C''| = \ell(\lambda)$, and this last term is trivial with weight t_i. Hence

$$-R^\bullet\pi_*f^*(L_1|_{C'}\oplus L_2|_{C''}) + (t_1t_2)^{\ell(\lambda)} = -R^\bullet\pi_*f^*(L_1\oplus L_2)|_C.$$

Level and genus add, which is why u behaves fine in gluing too. □

1.7 Oct 23 (Henry): Local curve computations

Definition 1.31. Let $2\mathsf{Cob}^{L_1,L_2}$ enrich $2\mathsf{Cob}$ by asking morphisms $Y_1\to Y_2$ to be equivalence classes of triples (W,L_1,L_2) where:

1. W is a cobordism from Y_1 to Y_2;
2. L_1,L_2 are line bundles on W trivialized on ∂W.

Topologically, vector bundles are classified by degree and rank, so it suffices to label W with the level (k_1, k_2) of (L_1, L_2).

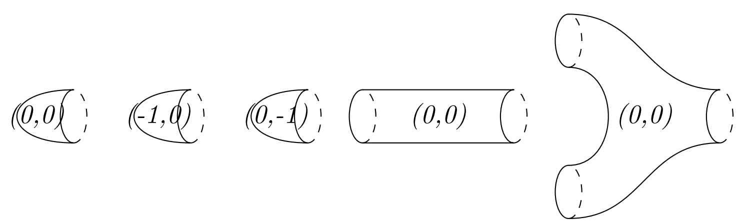
Definition 1.32. Let $R \coloneqq \mathbb{Q}(t_1, t_2)((u))$, and define the R-valued 2-TQFT

$$GW\colon \mathsf{2Cob}^{L_1,L_2} \to \mathsf{Mod}_R$$

by the data of:

1. the state space $GW(S^1) \coloneqq H \coloneqq \bigoplus_{\lambda \vdash d} Re_\lambda$;
2. the morphisms
$$e_{\eta^1} \otimes \cdots e_{\eta^s} \mapsto \sum_{\mu^1,\dots,\mu^t \vdash d} GW(g; k_1, k_2)^{\mu^1 \cdots \mu^t}_{\eta^1 \cdots \eta^s} e_{\mu^1} \otimes \cdots \otimes e_{\mu^t}$$
associated to a genus g cobordism from s inputs to t outputs of level (k_1, k_2).

Theorem 1.33. *GW is a well-defined functor, and is uniquely determined by its value on*



Proof. Everything follows from gluing laws. The only thing we really have to check is that the tube is sent to the identity morphism, i.e. that
$$GW(0; 0, 0)^\nu_\mu = \delta^\nu_\mu.$$
We will check this later. $\square$

We will show GW is a semisimple 2-TQFT by showing it in level 0, i.e. for $GW(g; 0, 0)$. This is the part of the theory which gives classical contributions. Then we lift to the whole TQFT by the following lemma.

Lemma 1.34 (TQFT Nakayama lemma)**.** *Let $(R, \mathfrak{m})$ be a complete local ring, and let A be a Frobenius algebra over R. Suppose A is a free R-module and $A/\mathfrak{m}A$ is a semisimple Frobenius algebra over $R/\mathfrak{m}$. Then A is semisimple (over R).*

Proof. Since $A/\mathfrak{m}A$ is semisimple, pick an idempotent basis represented by elements $e_1, \dots, e_n \in A$, i.e.

$$e_i^2 - e_i \in \mathfrak{m},\ \forall i, \quad e_i e_j \in \forall i \neq j,$$

and by the regular Nakayama lemma, $\{e_i\}$ is a basis for A. We modify it inductively so that it is idempotent mod $\mathfrak{m}^k$. Suppose we had the relations for $\mathfrak{m}^k$ instead of $\mathfrak{m}$. Then define

$$b_i \coloneqq e_i^2 - e_i \in \mathfrak{m}^k, \quad e_i' = e_i + b_i(1 - 2e_i)^{-1}.$$

Note that $1 - 2e_i$ is invertible only because R is complete. We picked it so that terms cancel out in

$$\begin{aligned}(e_i')^2 - e_i' &= e_i^2 - e_i + (2e_ib_i - b_i)(1 - 2e_i)^{-1} + b_i^2(1 - 2e_i)^{-2} \\ &= b_i^2(1 - 2e_i)^{-2} \in \mathfrak{m}^{2k}.\end{aligned}$$

Also, $e_i'e_j' \in \mathfrak{m}^{k+1}$ just by checking all the terms are. Hence we can inductively construct an idempotent basis $\{e_i^{(k)}\}$ for $A/\mathfrak{m}^{k+1}A$ for every k. By completeness again, there exists $\tilde{e}_i \in A$ with $\tilde{e}_i \equiv e_i^{(k)} \bmod \mathfrak{m}^{k+1}$ for all k. This is the idempotent basis we want. $\square$

Proposition 1.35. *Let $\tilde{R} := \mathbb{Q}(t_1^{1/2}, t_2^{1/2})((u))$. Then the level $(0,0)$ sector of GW in degree d is semisimple over $\tilde{R}$.*

Proof. Since $\mathfrak{m} = (u)$, the structure constants of multiplication in the Frobenius algebra are given by the pair of pants structure constants $GW(0;0,0)^{\gamma}_{\alpha\beta}|_{u=0}$. Hence we care only about $b_1 = b_2 = 0$. Recall that $b_1 + b_2 = \text{vdim}$, so here expected dimension is 0. Hence

$$\begin{aligned} GW(0;0,0)^{\gamma}_{\alpha\beta}|_{u=0} &= \mathfrak{z}(\gamma)(t_1t_2)^{\ell(\gamma)} GW(0;0,0)_{\alpha\beta\gamma}|_{u=0} \\ &= \mathfrak{z}(\gamma)(t_1t_2)^{\frac{1}{2}(d-\ell(\alpha)-\ell(\beta)+\ell(\gamma))} H_d^{\mathbb{P}^1}(\alpha,\beta,\gamma) \end{aligned}$$

where $H_d^{\mathbb{P}^1}$ is a Hurwitz number. These we know how to compute by Burnside's formula

$$H_d^{\mathbb{P}^1}(\alpha,\beta,\gamma) = \sum_{\rho\vdash d} \frac{d!}{\dim\rho} \frac{\chi^\rho_\alpha \chi^\rho_\beta \chi^\rho_\gamma}{\mathfrak{z}(\alpha)\mathfrak{z}(\beta)\mathfrak{z}(\gamma)}.$$

Hence we have an explicit formula for the structure constants. The resulting Frobenius algebra is (up to t_1t_2) Yang–Mills with finite gauge group S_d and is well-known to be semisimple. We can actually explicitly write an idempotent basis

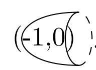
$$v_\rho := \frac{\dim\rho}{d!} \sum_\alpha (t_1^{1/2} t_2^{1/2})^{\ell(\alpha)-d} \chi^\rho_\alpha e_\alpha.$$

This requires the extension to $\tilde{R}$. ☐

Corollary 1.36. *There are universal series $\lambda_\rho, \eta_\rho \in \tilde{R}$ indexed by partitions ρ such that*

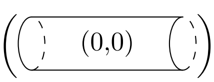
$$GW_d(g;k_1,k_2) = \sum_{\rho\vdash d} \lambda_\rho^{1-g} \eta_\rho^{-k_1} \bar{\eta}_\rho^{-k_2}$$

where bar means swapping t_1 and t_2.

Proof. Same proof as earlier, except we have series η_ρ associated to the *level adding operator* 

. ☐

Example 1.37 (Level $(0,0)$ tube)**.** This is given by the series



$$F\left(\cdot \right) := GW(0;0,0)_{\lambda\mu} = \begin{cases} \frac{1}{\mathfrak{z}(\lambda)(t_1t_2)^{\ell(\lambda)}} & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

as follows. For *connected* domains, the only contribution to $GW(0;0,0)_{\alpha\beta}$ can be from degree-d covers $\mathbb{P}^1 \xrightarrow{d} \mathbb{P}^1$, because of the following.

1. Since the L_i are trivial,

$$c(-R^\bullet \pi_* f^* L_i) = c(R^1\pi_*\mathcal{O}_{\bar{\mathcal{C}}_h} - R^0\pi_*\mathcal{O}_{\bar{\mathcal{C}}_h}) = c(\mathbb{E}^\vee)/1,$$

   and hence the terms in $GW(0;0,0)_{\alpha\beta}$ are

$$\int_{[\bar{\mathcal{M}}_h(\mathbb{P}^1,\alpha,\beta)]^{\text{vir}}} c_{b_1}(\mathbb{E}^\vee) c_{b_2}(\mathbb{E}^\vee).$$

2. Do a dimension count: $\text{vdim}\,\bar{\mathcal{M}}_h(\mathbb{P}^1,\alpha,\beta) = 2h - 2 + \ell(\alpha) + \ell(\beta)$, but $\mathbb{E}^\vee$ is rank h and hence the integrand is dimension at most $2h$. Hence $\ell(\alpha) = \ell(\beta) = 1$ and $b_1 = b_2 = h$. But Mumford's relation says

$$c_h(\mathbb{E}^\vee)^2 = 0 \quad \forall h > 0.$$

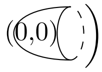
   Hence $h = 0$ as well, i.e. we have a totally ramified $\mathbb{P}^1 \xrightarrow{d} \mathbb{P}^1$.

Disconnected maps which contribute must therefore be a disjoint union of totally ramified covers. Such maps are isolated in moduli and have automorphism group of order $\mathfrak{z}(\alpha)$, i.e.

$$GW^{b_1,b_2}(0;0,0)_{\alpha\beta} = \begin{cases} 1/\mathfrak{z}(\alpha) & b_1 = b_2 = 0,\ \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

which gives the desired expression for the tube.

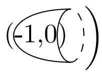
Example 1.38 (Level $(0,0)$ cap)**.** This is given by the series

$F\Big($  $\Big)$

$$:= GW(0;0,0)_\lambda = \begin{cases} \frac{1}{d!(t_1t_2)^d} & \lambda = (1^d) \\ 0 & \lambda \neq (1^d) \end{cases}$$

as follows. Now $\operatorname{vdim} \bar{\mathcal{M}}_h(\mathbb{P}^1, \lambda) = 2h-2+d+\ell(\lambda)$, and we require $d = \ell(\lambda) = 1$. Then $h = 0$ by Mumford's relation. Hence we can only have isomorphisms $\mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$. Accounting for disconnected covers, we get d copies of isomorphisms, i.e. $\lambda = (1^d)$, with S_d automorphism group.

Example 1.39 (Level $(-1,0)$ cap)**.** This is given by the series

$F\Big($  $\Big)$

$$:= GW(0;-1,0)_\lambda = (-1)^{|\lambda|}(-t_2)^{-\ell(\lambda)} \frac{1}{\mathfrak{z}(\lambda)} \prod_{i=1}^{\ell(\lambda)} \left(2\sin\frac{\lambda_i u}{2}\right)^{-1}$$

as follows. Again do the connected case. Look at the terms of the integrand:

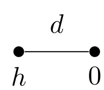
1. $-R^\bullet\pi_* f^*\mathcal{O}(-1)$ has fibers $-H^\bullet(C, \mathcal{O}_C(-d))$, which by Riemann–Roch is rank $-(d+1-h)$;
2. $-R^\bullet\pi_* f^*\mathcal{O}$ has fibers $-H^\bullet(C, \mathcal{O}_C)$, whose Chern class (up to a trivial factor) is just $c(\mathbb{E})$.

So by the usual inequalities, we require $\ell(\lambda) = 1$, i.e. $\lambda = (d)$ and $b_1 = h - 1 + d$ and $b_2 = h$. Compute

$$\int_{[\bar{\mathcal{M}}_h(\mathbb{P}^1,(d))]^{\mathrm{vir}}} e(-R^\bullet\pi_* \operatorname{ev}^* \mathcal{O}(-1)) e(-R^\bullet\pi_* \operatorname{ev}^* \mathcal{O})$$

via $\mathbb{C}_q^*$-localization for the usual action of $\mathbb{C}_q^*$ on $\mathbb{P}^1$. Pick the linearization $(-1,0)$ and $(0,0)$ on $\mathcal{O}(-1)$ and $\mathcal{O}$ respectively, so that:

1. there is a unique vertex over ∞ because of the ramification profile (d), and it has genus 0 because it carries the class $c_{g(v)}(\mathbb{E}^\vee)^2$, which vanishes unless $g(v) = 0$;
2. the vertex over ∞ cannot have valence > 1, using our choice of linearization as in the proof of Aspinwall–Morrison; (note that we can only run this argument for $\mathcal{O}(-1)$ because $\mathcal{O}$ has non-trivial H^0, in the LES induced from normalization exact sequence)
3. the vertex at 0 must be of genus h for the total genus to be h;
4. the vertex at ∞ is rigid, i.e. gives no contributions at all to the integral, because it cannot be deformed within this moduli space.

Hence the only contribution is from a graph of the form 

. We compute its contribution via $\mathbb{C}_q^*$-localization.

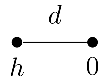
1. The vertex contribution is done in the Faber–Pandharipande linear Hodge integral calculation. From the genus-h vertex at 0 we get

   $$\int_{\bar{\mathcal{M}}_{h,1}} (-1)^h\Lambda(q)(-1)^h\Lambda(0) \cdot \frac{(-1)^h\Lambda(-q)}{q/d - \psi},$$

   where $\Lambda(q)\Lambda(0)$ comes from $e(-\mathbb{E}_{-1})e(-\mathbb{E})$. (Here $\mathbb{E}_{-1}$ is $\mathbb{E}$ with linearization -1, coming from $\mathcal{O}(-1)$ term.)
2. To be continued.

1.8 Oct 30 (Henry): Cap and pants

We continue the $(-1,0)$ cap computation from last time. The edge contribution from the single graph



is as follows. Both deformations of the map and the integrand contribution involve weights of sections $H^\bullet(\mathbb{P}^1, f_d^*\mathcal{O}(k))$, so we care only about what the linearization is at 0 and ∞. Note however that we cannot deform the map at the degenerate vertex ∞.

1. (Denominator) From the linearization $(1,0)$ of $T_{\mathbb{P}^1}(-\infty)$ (because we can't deform the degenerate vertex at ∞), we get weights kq/d for $k \in \{0,\ldots,d\} - \{0\}$. The product is $d!(q/d)^d$.
2. (Numerator) From the linearization $(-1,0)$ of $\mathcal{O}(-1)$, we get weights kq/d for $k \in \{-1,\ldots,-(d-1)\}$. The product is $(-1)^{d-1}(d-1)!(q/d)^{d-1}$.

Collecting everything together and using Mumford's relation, the total contribution is a sum over h of the terms

$$\begin{aligned}&\frac{1}{d}\int_{\bar{\mathcal{M}}_{h,1}} (-1)^h q^{2h} \frac{(-1)^h c_h(\mathbb{E})}{q/a - \psi} \cdot \frac{(-1)^{d-1}(d-1)!(q/d)^{d-1}}{d!(q/d)^d} \\ &= \frac{1}{d}\int_{\bar{\mathcal{M}}_{h,1}} q^{2h} c_h(\mathbb{E}) \frac{\psi^{2h-2}}{(q/d)^{2h-1}} \frac{(-1)^{d-1}}{q} = (-1)^{d-1} d^{2h-2} \int_{\bar{\mathcal{M}}_{h,1}} c_h(\mathbb{E})\psi^{2h-2}.\end{aligned}$$

(Note that all the q's cancel, as they should!) This is a linear Hodge integral, and can be evaluated via Faber–Pandharipande's formula

$$\sum_{h \geq 0} (du)^{2h} \int_{\bar{\mathcal{M}}_{h,1}} \psi_1^{2h-2}\lambda_h = \frac{du/2}{\sin(du/2)}.$$

Plugging this into the explicit expression for $GW(g;k_1,k_2)_\lambda$, we get the desired result. For example:

1. since $\delta = d-1$ and $b_1 = h-1+d$ and $b_2 = h$, we see that $(1/2)(b_2 - b_1 + \delta) = 0$ and $d(g-1-k_2) + (1/2)(b_1 - b_2 + \delta) = -1$;
2. disconnected invariants are products of $\ell(\lambda)$ connected invariants, so in total we have $t_2^{-\ell(\lambda)}$.

This finishes the $(-1,0)$ cap computation.

The pair of pants is hard, because now there are no dimensionality arguments:

$$\operatorname{vdim} \bar{\mathcal{M}}_h(\mathbb{P}^1, \lambda, \mu, \nu) = 2h - 2 - d + \ell(\lambda) + \ell(\mu) + \ell(\nu),$$

so now the contributions even from connected sources is complicated. However for small cases, we still have dimensionality arguments. We will compute a modified version of $GW(-)$ called $GW^*(-)$; it involves a prefactor which will make the GW/DT correspondence hold on the nose:

$$GW^*(-) = (-i)^{d(2-2g+k_1+k_2)-\delta} GW(-).$$

This requires us to make a modification to the (inverse of the) metric, which is now $\mathfrak{z}(\nu)(-t_1t_2)^{\ell(\nu)}$, i.e. there is an extra minus sign.

Example 1.40 ($GW^*(0;0,0)_{(d),(d),(2)}$)**.** Let (2) denote $(2, 1^{d-2})$. In this case, we have vdim $= 2h-1$ but the integrand is $c_{b_1}(\mathbb{E}^\vee)c_{b_2}(\mathbb{E}^\vee)$, so the only two possibilities are

$$(b_1, b_2) = (h, h-1), \quad (b_1, b_2) = (h-1, h).$$

So it suffices to compute $\int_{[\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2))]^{\mathrm{vir}}} \rho^*(-\lambda_h\lambda_{h-1})$ where $\lambda_k \in \bar{\mathcal{M}}_{h,2}$ and

$$\rho\colon \bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2)) \to \bar{\mathcal{M}}_{h,2}$$

takes a relative stable map to the domain marked with the two totally ramified points. We use this to reduce to integrals over $\bar{\mathcal{M}}_{h,2}$ by computing

$$\rho_*[\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2))]^{\mathrm{vir}} = 2h[\bar{H}_d] + B$$

where $H_d \subset \mathcal{M}_{h,2}$ is the image of ρ on the smooth locus and $\bar{H}_d \subset \bar{\mathcal{M}}_{h,2}$ is the closure, and B is some cycle on the boundary which we can neglect, as follows.

1. Note that $\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2))$ is unobstructed and hence the virtual class is the usual fundamental class. On the open locus, let $H_d \subset \mathcal{M}_{h,2}$ be the image of ρ. Then

$$\rho\colon \mathcal{M}_h(\mathbb{P}^1,(d),(d),(2)) \to H_d$$

is a proper degree $2h$ cover, because by Riemann–Hurwitz there are $R' = 2h$ other ramification points, and we can choose which one is the one we call (2). (Here $2h-2 = d(-2)+(d-1)+(d-1)$.) Hence

$$\rho_*[\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2))]^{\mathrm{vir}} = 2h[\bar{H}_d] + B$$

where B is supported on $\rho(\partial\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2)))$.

2. Let $\epsilon\colon \bar{\mathcal{M}}_{h,2} \to \bar{\mathcal{M}}_{h,1}$ be the forgetful map. Then actually

$$\rho(\partial\bar{\mathcal{M}}_h(\mathbb{P}^1,(d),(d),(2))) \subset \epsilon^{-1}(\partial\bar{\mathcal{M}}_{h,1}).$$

This is because we know the image lies in $\partial\bar{\mathcal{M}}_{h,2}$, but the only stratum there (i.e. singular curve with 2 marked points) that does not come from $\partial\mathcal{M}_{h,1}$ (i.e. singular curve with 1 marked point) must have one $\mathbb{P}^1$ component holding both marked points, which contracts onto the main component of genus h when we forget one marked point. Such a component is not a valid source curve in the compactification of relative stable maps. Now to disregard B, use that

$$\lambda_h\lambda_{h-1}|_{\partial\bar{\mathcal{M}}_{h,n}} = 0.$$

This is because there are two kinds of components in $\partial\bar{\mathcal{M}}_g$, and it suffices to verify $\lambda_g\lambda_{g-1} = 0$ on both.

   (a) ($\bar{\mathcal{M}}_{g-1,2}$) Here there is a surjection $i^*\mathbb{E}_g \twoheadrightarrow \mathcal{O}$ given by taking residue at one of the marked points. Hence $c_g(\mathbb{E}_g) = 0$.

   (b) ($\bar{\mathcal{M}}_{h,1} \times \bar{\mathcal{M}}_{g-h,1}$) Here $i^*\mathbb{E}_g$ factors as $p_1^*\mathbb{E}_h \oplus p_2^*\mathbb{E}_{g-h}$, so

$$\begin{aligned} i^*\lambda_g &= p_1^*\lambda_h p_2^*\lambda_{g-h} \\ i^*\lambda_{g-1} &= p_1^*\lambda_h p_2^*\lambda_{g-h-1} + p_1^*\lambda_{h-1}p_2^*\lambda_{g-h}. \end{aligned}$$

   Use the vanishing $\lambda_h^2 = \lambda_{g-h}^2 = 0$.

Collecting all this together, we get

$$GW^*(0;0,0)_{(d),(d),(2)} = i\frac{t_1+t_2}{t_1t_2}\sum_{h\geq 1} u^{2h-1}c_h(d), \quad c_h(d) := 2h\int_{[\bar{H}_d]} \lambda_h\lambda_{h-1}.$$

The next step is to reduce to the $d=2$ case and explicitly compute on the hyperelliptic locus.

1. H_d is the locus of curves (C, x_1, x_2) with $\mathcal{O}(x_1 - x_2) \in \mathrm{Pic}^0(C)$ being a non-trivial d-torsion point. In other words, if $\mathcal{P}ic^0 \xrightarrow{\pi} \mathcal{M}_{h,2}$ is the universal Picard bundle with section $s\colon (C, x_1, x_2) \mapsto \mathcal{O}_C(x_1 - x_2)$, then

$$[H_d] = \pi_*(s_*[\mathcal{M}_{h,2}] \cap P_d) \in A_*(M_{h,2})$$

where P_d is the locus of *non-zero* d-torsion points.

2. A result of Looijenga says given any family of abelian varieties $\mathcal{A} \to S$, the class of the locus of d-torsion points is a multiple of the zero section in Chow. Hence

$$\frac{1}{d^{2h} - 1}[P_d] = [0] = \frac{1}{2^{2h} - 1}[P_2]$$

and this descends to $c_h(d) = (d^{2h} - 1)/(2^{2h-1})c_h(2)$.

Hence we have reduced to $d = 2$. This case is easy, because $[\bar{H}_2]$ relates to the hyperelliptic locus $\bar{H} \subset \bar{\mathcal{M}}_h$ almost by definition.

1. The extra data in $\bar{H}_2$ is which two of the Weierstrass points we choose to call (2). By Riemann–Hurwitz, $2h - 2 = -2 \cdot 2 + r$, so there are $r = 2h + 2$ Weierstrass points, i.e.

$$(\bar{\mathcal{M}}_{h,2} \to \bar{\mathcal{M}}_h)_*[\bar{H}_2] = (2h + 2)(2h + 1)[\bar{H}].$$

2. Use Faber–Pandharipande's evaluation of $\mathrm{ch}(\mathbb{E})$ on the hyperelliptic locus $\bar{H} \subset \bar{\mathcal{M}}_h$ to get

$$GW^*(0; 0, 0)_{(2),(2),(2)} = \frac{i}{2}\frac{t_1 + t_2}{t_1 t_2} \tan\frac{u}{2}.$$

From this, we get $c_h(2)$, which gives $c_h(d)$, and therefore the general expression

$$GW^*(0; 0, 0)_{(d),(d),(2)} = \frac{i}{2}\frac{t_1 + t_2}{t_1 t_2}\left(d \cot\frac{du}{2} - \cot\frac{u}{2}\right).$$

Remark. It will be helpful to rewrite this with $q := -e^{iu}$ as

$$GW^*(0; 0, 0)_{(d),(d),(2)} = \frac{1}{2}\frac{t_1 + t_2}{t_1 t_2}\left(d\frac{(-q)^d + 1}{(-q)^d - 1} - \frac{(-q) + 1}{(-q) - 1}\right).$$

Theorem 1.41 (Bryan–Faber–Okounkov–Pandharipande reconstruction result)**.** *The pair of pants series $GW^*(0; 0, 0)_{\lambda\mu\nu}$ can be uniquely reconstructed from $GW^*(0; 0, 0)_{(d),(d),(2)}$, lower degree series of level $(0, 0)$, and Hurwitz numbers of $\mathbb{P}^1 \to \mathbb{P}^1$.*

Proof. We show uniqueness. Idea: write an invertible linear system of equations for $GW^*(0; 0, 0)_{\lambda\mu\nu}$ whose coefficients are matrix elements of $GW^*(0; 0, 0)_{\mu,(2),\nu}$; let

$$\langle\lambda|M_2|\mu\rangle := (-1)^{|\lambda|} GW^*(0; 0, 0)_{\lambda,(2),\mu}\delta_{|\lambda|,|\mu|}$$

be these matrix elements. In Fock space formalism, it is easy to express disconnected invariants in terms of connected ones:

$$-M_2 \propto \sum_{k>0} GW^*(0; 0, 0)_{(k),(2),(k)}\alpha_{-k}\alpha_k + \sum_{k,l>0}\left(GW^*(0; 0, 0)_{(k+l),(2),(k,l)}|_{u=0}\alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l\right).$$

This is because there are two types of contributions. The virtual dimension is $2h - 2 - d + \ell(\mu) + \ell(\nu) + d - 1 = 2h - 3 + \ell(\mu) + \ell(\nu)$.

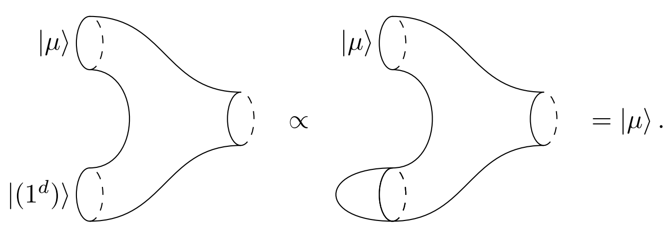
1. (Quantum contribution) If $\ell(\mu)+\ell(\nu)=2$ then $\ell(\mu)=\ell(\nu)=1$ and we are necessarily in the case of connected invariants $GW^*(0;0,0)_{(d),(2),(d)}$.
2. (Classical contribution) Otherwise $\ell(\mu)+\ell(\nu)=3$ and we need contributions from $c_h(\mathbb{E})^2$, which as usual vanishes unless $h=0$. This is the classical contribution, i.e. at $u=0$, from Hurwitz numbers. Recall that
$$GW(0;0,0)^{\gamma}_{\alpha\beta}|_{u=0}=\mathfrak{z}(\gamma)(t_1t_2)^{\frac{1}{2}(d-\ell(\alpha)-\ell(\beta)+\ell(\gamma))}H_d^{\mathbb{P}^1}(\alpha,\beta,\gamma)$$
and the Hurwitz number controlls what happens when a transposition hits another partition: it can either merge two cycles, or split a cycle into two.

The desired linear system arises as follows. Let $(2)^r$ denote r copies of (2). Then for partitions $\mu,\nu\vdash d$, we can get ramification $(\mu,(2)^r,\nu)$ in two ways:

1. glue r copies of $(\alpha,(2),\beta)$;
2. glue the unknown (μ,γ,ν) to $(\gamma,(2)^r)$, where we know
$$GW^*(0;0,0)_{(2)^r,\gamma}=GW^*(0;0,0)^{(1^d)}_{(2)^r,\gamma}GW^*(0;0,0)_{(1^d)}$$
because of the explicit expression for the $(0,0)$ cap.

The equality we get is
$$\langle\mu|M_2^r|\nu\rangle\propto\sum_{\gamma\vdash d}GW^*(0;0,0)_{\mu\gamma\nu}\langle\gamma|M_2^r|(1^d)\rangle.$$
This ranges over all r, and we need to show the resulting system (for fixed μ,ν) is non-singular, over the field $\mathbb{Q}(t_1,t_2,q)$ where the coefficients live (by our explicit formula).

1. Note that M_2 as an operator has distinct eigenvalues, because in the limit $t_1t_2=0$ it is upper triangular with linearly independent entries on the diagonal. If $\mathcal{F}_d\subset\mathcal{F}$ is spanned by vectors of degree d, then it follows that the idempotents of the Frobenius algebra are eigenvectors of $M_2|_{\mathcal{F}_d}$. This is by picking eigenvectors $\{v_j\}$ and looking at quantum multiplication $*$, which gives
$$v_i*v_i=\sum a_jv_j\implies v_i*v_i=a_iv_i$$
by applying M_2 to both sides as follows:
$$\sum\lambda_ia_jv_j=\lambda_iv_i*v_i=M_2v_i*v_i=\sum a_jM_2v_j=\sum\lambda_ja_jv_j,$$
but the v_j are linearly independent so $\lambda_i=\lambda_j$. But eigenvalues are also distinct, so there can be only one j on the rhs.
2. Check that $|(1^d)\rangle$ is the unit in the Frobenius algebra. This is because (up to checking prefactors) of the $(0,0)$ cap being non-zero only for $\lambda=\langle(1^d)\rangle$, so



But the unit is the sum of all idempotents, and M_2 has distinct eigenvalues. Hence $\{M_2^r\,|(1^d)\rangle\}_{r\geq 0}$ spans $\mathcal{F}_d$. So the linear system of equations we got must be non-singular. Explicitly, the system is of the form $v_r\cdot x=\langle\mu|M_2^r|\nu\rangle$ ranging over all r, where $v_r=M_2^r\,|(1^d)\rangle$; because they span, solutions are unique. $\square$

1.9 Nov 13 (Melissa): 1-leg GW vertex

We will talk about multiple covers of the sphere and the disk, and then the (Gopakumar–)Marino–Vafa formula.

Example 1.42 (Multiple covers of the sphere)**.** Let $X \coloneqq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, with inclusion $i_0 \colon \mathbb{P}^1 \to X$ of the zero section. When $d > 0$, recall that

$$\bar{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) \xrightarrow{i_0} \bar{\mathcal{M}}_{g,0}(X, d)$$

is an isomorphism. Let

$$N_{g,d} \coloneqq \int_{[\bar{\mathcal{M}}_{g,0}(X,d)]^{\mathrm{vir}}} 1 = \int_{[\bar{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(V_d), \quad V_d \coloneqq R\pi_* \operatorname{ev}^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)).$$

For $d > 0$, we computed via localization that

$$F_d(\lambda) \coloneqq \sum_{g \geq 0} N_{g,d} \lambda^{2g-2} = \frac{1}{\lambda^2} \left(\sum_{g \geq 0} b_g (du)^{2g} \right)^2.$$

where $b_0 = 1$ and $b_g = \int_{\bar{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-1}$ for $g > 0$. By Faber–Pandharipande, we got

$$F_d(\lambda) = \frac{1}{d(2\sin(du/2))^2}.$$

Example 1.43 (Multiple covers of the disk)**.** Let $X \coloneqq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Let A be the antiholomorphic involution

$$(z, u, v) \mapsto (1/\bar{z}, \bar{z}\bar{v}, \bar{z}\bar{u}).$$

Let $L \coloneqq X^A \to S^1 = (\mathbb{P}^1)^A$. By writing in charts,

$$L = \{(e^{i\theta}, u, e^{-i\theta}\bar{u}) : e^{i\theta} \in S^1,\, u \in \mathbb{C}^1\},$$

and $(\mathbb{P}^1)^A = \{z = 1/\bar{z}\} = \{|z|^2 = 1\}$. So in this setup we are going to count multiple covers of the upper hemisphere. This is not completely well-defined, but let us do a heuristic computation. Let $g \geq 0$ and $\mu \coloneqq (\mu_1 \geq \mu_2 \geq \cdots \mu_\ell > 0)$ be a partition of d. We can consider maps

$$u \colon (\Sigma, \partial\Sigma) \xrightarrow{\text{holomorphic}} (X, L)$$

where $\partial\Sigma = \prod_{i=1}^{\ell} R_i$ is a product of disks, with conditions:

1. (multiple cover) $u_*[\Sigma] = d\beta \in H_2(X, L, \mathbb{Z})$;
2. (boundary conditions) $u_*[R_i] = \mu_i \gamma \in H_1(L, \mathbb{Z})$.

There is a boundary map $H_2(X, L, \mathbb{Z}) \xrightarrow{\partial} H_1(L, \mathbb{Z})$, and of course the compatibility condition is that $\sum \mu_i = d$. This defines a moduli space $\mathcal{M}_{g,\mu}$ (at least set-theoretically).

There is a $(\mathbb{C}^*)^3$-action on the total space, but L is not fixed by the whole torus. We restrict to the Calabi–Yau torus $T'_R \cong U(1)^2$. Then we should define

$$N_{g,\mu} \coloneqq \frac{1}{|\operatorname{Aut}(\mu)|} \int_{[\bar{\mathcal{M}}_{g,\mu}]^{\mathrm{vir}}} 1 = \frac{1}{|\operatorname{Aut}(\mu)|} \int_{[\bar{\mathcal{M}}_{g,\mu}(\mathbb{P}^1, D)]^{\mathrm{vir}}} e(V_{g,\mu}).$$

By localization (Katz–Liu), this is equal to

$$\frac{(\tau(\tau+1))^{\ell-1}}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{\ell} \frac{\prod_{n=1}^{\mu_i - 1} (\tau\mu_i + a_i)}{(\mu_i - 1)!} \int_{\bar{\mathcal{M}}_{g,\ell}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{\ell} (1 - \mu_i \psi_i)}.$$

This answer depends on τ, because the original integral is over a non-compact thing. We call τ the **framing**, from physics, related to the framing of the knot. So define the generating function

$$F_\mu(u,\tau) \coloneqq \sum_{g\geq 0} N_{g,\mu}(\tau) u^{2g-2+\ell(\mu)}.$$

The localization expression looks complicated, but we can set $\tau = 0$ to get a lot of vanishing:

$$F_\mu(u,0) = \begin{cases} 0 & \ell(\mu) > 1 \\ \frac{1}{d^2 u}\left(\frac{du/2}{\sin(du/2)}\right) = \frac{1}{2d\sin(du/2)} & \mu = (d) \end{cases}.$$

Question: is there a formula when $\tau \neq 0$? Introduce a new invariant

$$G_{g,\mu}(\tau) \coloneqq \frac{-\sqrt{-1}^{|\mu|+\ell(\mu)}(\tau(\tau+1))^{\ell(\mu)-1}}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \int_{\bar{\mathcal{M}}_{g,\ell}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}$$

and write the generating series

$$G_\mu(\lambda,\tau) \coloneqq \sum_{g\geq 0} G_{g,\mu}(\tau)\lambda^{2g-2+\ell(\mu)}$$

and look at the disconnected series

$$\exp\Big(\sum_{\mu\neq\emptyset} G_\mu(\lambda,\tau)p_\mu\Big) \eqqcolon 1 + \sum_{\mu\neq\emptyset} G^\bullet_\mu(\lambda,\tau)p_\mu.$$

Theorem 1.44 (Gopakumar–Marino–Vafa formula)**.**

$$G^\bullet_\mu(\lambda,\tau) = R^\bullet_\mu(\lambda,\tau) \coloneqq \sum_{|\nu|=|\mu|} \frac{\chi_\nu(C_\mu)}{\mathfrak{z}(\mu)} e^{-\sqrt{-1}(\tau+\frac{1}{2})\frac{k_i\lambda}{2}} \sqrt{-1}^{|\mu|} \frac{\dim_q R_\nu}{|\mu|!}$$

where $\dim_q R_\nu$ *is the quantum dimension.*

Remark (Framing dependence)*.* From this formula, we get a simple relation between arbitrary framing and zero framing:

$$R^\bullet_\mu(\lambda,\tau) = \sum R^\bullet_\mu(\lambda,0)\mathfrak{z}(\nu)\Phi^\bullet_{\nu\mu}(\sqrt{-1}\lambda\tau)$$

where the change of basis matrix is a generating series for disconnected double Hurwitz numbers

$$\Phi^\bullet_{\nu\mu}(\lambda) \coloneqq \sum \frac{H^\bullet_{g,\nu,\mu}}{(2g-2+\ell(\mu)+\ell(\nu))!}\lambda^{2g-2+\ell(\mu)+\ell(\nu)}.$$

We can get a formula for this via Burnside's formula.

Definition 1.45. In the GMV formula, the lhs is called the **framed 1-leg GW vertex**, and the rhs is an explicit formula for it. Define the connected version

$$\exp\Big(\sum_{\mu\neq\emptyset} R_\mu(\lambda,\tau)p_\mu\Big) \coloneqq 1 + \sum_{\mu\neq\emptyset} R^\bullet_\mu(\lambda,\tau)p_\mu.$$

Proof strategy. Our goal is to prove the GMV formula

$$G_\mu(\lambda,\tau) = R_\mu(\lambda,\tau).$$

It is easy to check at zero framing that

$$G_\mu(\lambda,0) = R_\mu(\lambda,0) = \begin{cases} 0 & \ell(\mu) > 1 \\ \frac{-\sqrt{-1}^{d+1}}{2d\sin(d\lambda/2)} & \mu = (d) \end{cases}.$$

So to prove $G_\mu = R_\mu$, it suffices to show they satisfy the same framing dependence, i.e. that

$$G^\bullet_\mu(\lambda,\tau) = \sum_{|\nu|=|\mu|} G^\bullet_\nu(\lambda,0)\mathfrak{z}(\nu)\Phi_\nu(\sqrt{-1}\lambda\tau).$$

We will define a generating function $K^\bullet_\mu(\lambda)$ of certain relative GW invariants of $(\mathbb{P}^1,\infty)$. We will compute it via localization to get an explicit formula

$$K^\bullet_\mu(\lambda) = \sum_{|\nu|=|\mu|} G^\bullet_\nu(\lambda,\tau)\mathfrak{z}(\nu)\Phi^\bullet_{\nu\mu}(-\sqrt{-1}\lambda\tau).$$

This implies $G^\bullet_\mu(\lambda,0) = K^\bullet_\mu(\lambda)$ and also the desired framing dependence. The disconnected and connected generating series are

$$K^\bullet_{g,\mu} \coloneqq \frac{\sqrt{-1}^{|\mu|+\ell(\mu)}}{|\operatorname{Aut}(\mu)|}\int_{[\bar{\mathcal{M}}^\bullet_g(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} e(V^\bullet_{g,\mu})$$
$$K_{g,\mu} \coloneqq \frac{\sqrt{-1}^{|\mu|+\ell(\mu)}}{|\operatorname{Aut}(\mu)|}\int_{[\bar{\mathcal{M}}_g(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} e(V_{g,\mu}).$$

To define $V_{g,\mu}$, form the diagram

$$\begin{array}{ccccccc} R & \longrightarrow & C & \xrightarrow{\tilde{f}} & \mathcal{T} & \xrightarrow{\tilde{\pi}} & \mathbb{P}^1 \\ & & \pi\big\downarrow & & & & \\ & & \bar{\mathcal{M}}_g(\mathbb{P}^1,\mu) & & & & \end{array}$$

where R is the universal ramification divisor, $\mathcal{T}$ is the universal target, and the map $\tilde{\pi}\colon \mathcal{T} \to \mathbb{P}^1$ is the (universal) contraction to the original $\mathbb{P}^1$. Then

$$V_{g,\mu} \coloneqq R^1\pi_*((\tilde{\pi}\tilde{f})^*\mathcal{O}(-1) \oplus \mathcal{O}_C(-R)),$$

i.e. the fiber over $\xi = [\bar{f}\colon (C,x_1,\ldots,x_\ell) \xrightarrow{f} \mathbb{P}^1[m] \to \mathbb{P}^1]$ is $H^1(C, \bar{f}^*\mathcal{O}(-1) \oplus \mathcal{O}(-x_1 - \cdots - x_\ell))$. It has rank $2g-2+\ell+d = \operatorname{vdim}\bar{\mathcal{M}}_g(\mathbb{P}^1,\mu)$. We will compute $K^\bullet_{g,\mu}$ by virtual localization.

1. (Linearizations) Take the linearization $(u,-u)$ on $T_{\mathbb{P}^1}$ and $((-\tau-1)u, -\tau u)$ on $\mathcal{O}(-1)$ and $(\tau u, \tau u)$ on $\mathcal{O}$.

2. (Fixed points) The torus fixed points in $\bar{\mathcal{M}}^\bullet_g(\mathbb{P}^1,\mu)$ are as follows. Over the original $\mathbb{P}^1$, we can have a possibly-disconnected degree-d rigid cover, and on the bubbles we can have anything. Hence

$$\sum_{\substack{g_0,g_1,\nu \\ g=g_0+g_1+\ell(\nu)+1}} \int_{\bar{\mathcal{M}}_{g_0,\ell(\nu)}} \int_{[\bar{\mathcal{M}}^\sim_{g_1}(\cdots)]^{\mathrm{vir}}} \cdots.$$

   On the connected version of the moduli space, there is a contraction map $\bar{\mathcal{M}}_g(\mathbb{P}^1,\mu) \to \bar{\mathcal{M}}_g(\mathbb{P}^1,d)$ which contracts both the target and domain with stabilization.

3. (Tangent-obstruction theory) Recall that $(\mathcal{T}^1,\mathcal{T}^2)$ is given by $\operatorname{Ext}^\bullet(\Omega_C(D),\mathcal{O}_C)$ and $H^\bullet(D^\bullet)$. The only new piece is the rubber contribution for a vertex v:

$$A_v = \prod_{e\in E_v} \frac{-u-\psi^t}{-u/d_e - \psi_{(e,v)}}$$

   arising from smoothing nodes. Since we are just a degree-d cover, this simplifies into the factor $\prod_{e\in E_v} d_e$.

Putting everything together, for a fixed locus F,

$$e(V_{g,\mu})|_F = \prod_{v\in V(\Gamma)} B_v \prod_{e\in E(\Gamma)} B_e$$
$$B_v \coloneqq \begin{cases} \Lambda_g^\vee(\tau u)\Lambda_g^\vee(-\tau u)(\tau u(-\tau-u))^{\ell-1} & v\in V^s(\Gamma) \\ \frac{\Lambda_g^\vee(\tau u)\Lambda_g^\vee(-\tau u)}{(-1)^g(\tau u)^{2g}} & \text{otherwise} \end{cases}$$
$$B_e \coloneqq \frac{\prod_{a=1}^{d_e-1}(d_e\tau+a)}{d_e^{d_e-1}}(-u)^{d_e-1}.$$

So the disconnected rubber contributions will just be disconnected double Hurwitz numbers, of the form

$$\int_{[\bar{\mathcal{M}}_g^{\bullet\sim}(\mathbb{P}^1,\nu,\mu)]^{\mathrm{vir}}} \frac{(\tau u)^{2g_1+2+\ell(\nu)+\ell(\mu)}}{-u-\psi_0}. \qquad \square$$

1.10 Nov 20 (Melissa): 2-leg GW vertex

The basic geometry is $\mathbb{C}^3$ with three legs. Let L_1 be the given by

$$L_1 \coloneqq \{(\sqrt{|u|+1}e^{i\theta}, u, e^{-i\theta}\bar{u})\} \cong S^1\times\mathbb{R}^2,$$

and define L_2, L_3 by cyclic permutation. The **Gromov–Witten vertex** can be viewed as a generating function of open GW invariants for holomorphic maps

$$(\Sigma,\partial\Sigma)\to(\mathbb{C}^3, L_1\sqcup L_2\sqcup L_3).$$

Since we have three Lagrangians, we can specify the topological type of the map by the data of $H_2(\mathbb{C}^3, L_1\sqcup L_2\sqcup L_3)$ along with $H_1(L_1\sqcup L_2\sqcup L_3) = \bigoplus_{i=1}^3 H_1(L_i)$. Note that there is a boundary map

$$\partial\colon H_2(\mathbb{C}^3, L_1\sqcup L_2\sqcup L_3)\to H_1(L_1\sqcup L_2\sqcup L_3).$$

Let μ^i be the winding number around L_i, and let $\mathcal{M}_{g,\mu^1,\mu^2,\mu^3}$ be the corresponding moduli of such maps. Compactify to get $\bar{\mathcal{M}}_{g,\mu^1,\mu^2,\mu^3}$. Let $T_{\mathbb{R}} \coloneqq \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})\}$ act on $\mathbb{C}^3$ and therefore on $\bar{\mathcal{M}}_{g,\mu^1,\mu^2,\mu^3}$. Fixed loci involve a contracted genus-g component in the source with some legs forming degree-μ_k^i covers. Then by localization we will get something like

$$\int_{[\bar{\mathcal{M}}_{g,\mu^1,\mu^2,\mu^3}]^{\mathrm{vir}}} 1 = \tilde{G}_{g,\mu^1,\mu^2,\mu^3}(w_1,w_2,w_3) \coloneqq \frac{\pm 1}{|\operatorname{Aut}|}\prod_{i=1}^3\prod_{j=1}^{\ell_i}\frac{\prod_{a=1}^{\mu_j^i}(w_i+\mu_j^i-aw_i)}{(\mu_j^i-1)!w_i^{\mu_j-1}}$$
$$\int_{\bar{\mathcal{M}}_{g,\ell_1+\ell_2+\ell_3}}\frac{\Lambda_g^\vee(w_1)\Lambda_g^\vee(w_2)\Lambda_g^\vee(w_3)(w_1w_2w_3)^{\ell_1+\ell_2+\ell_3}}{\prod_{i=1}^3\prod_{j=1}^{\ell_i}(w_i(w_i-\mu_j^i\psi_j^i))}.$$

This is how we should define the lhs.

Definition 1.46. The **3-leg GW vertex** is the generating function

$$G_{\mu^1,\mu^2,\mu^3}(\lambda,\vec{w}) \coloneqq \sum_{g\geq 0}\lambda^{2g-2+\ell_1+\ell_2+\ell_3}G_{g,\mu^1,\mu^2,\mu^3}(\vec{w})$$
$$G_{g,\mu^1,\mu^2,\mu^3}(\vec{w}) \coloneqq (-i)^{\ell_1+\ell_2+\ell_3}\tilde{G}_{g,\mu^1,\mu^2,\mu^3}(\vec{w}).$$

Let $G^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,\vec{w})$ be the disconnected version.

GW invariants of any toric CY3 (defined using the CY torus) can be obtained by gluing GW vertices $G^\bullet_{\mu^1,\mu^2,\mu^3}$, by localization. We saw from MNOP1 that DT invariants of toric CY3s can be obtained by gluing vertices C_{μ^1,μ^2,μ^3}. So GW/DT for toric CY3s is equivalent to some formula

$$G_{\mu^1,\mu^2,\mu^3}(\lambda,\vec{w}) = R^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,\vec{w})$$

matching these two vertices

Remark (1-leg vertex). This case is the GMV formula

$$G^\bullet_{\mu\emptyset\emptyset}(\lambda,\vec{w}) = R^\bullet_{\mu\emptyset\emptyset}(\lambda,\vec{w}),$$

which we showed last time. The geometry is $(\mathbb{C}^3, L_1)$, which we can embed it into the resolved conifold (X, L_1). By large N duality, open GW invariants of $(\mathbb{C}^3, L_1)$ corresponds to open GW invariants of (T^*S^3, N_K) where K is the unknot.

Remark (2-leg vertex). In general, open GW invariants of (X, L_k) should be extracted from colored HOMFLY polynomials of a knot $K \subset S^3$. For $(\mathbb{C}^3, L_1, L_2)$, its open GW invariants can be extracted from the colored HOMFLY of the Hopf link.

Remark (3-leg vertex). For the general case, Aganagic–Klem–Mariño–Vafa say something like this. By certain non-trivial transformations, we can move L_2 and L_3 to the same leg. Hence the knot K is a 3-component link. Then they compute the colored HOMFLY of K to get he formula for the 3-leg vertex. This formula will follow from MOOP. For now, let's just write the expression for $R^\bullet_{\mu^1,\mu^2,\mu^3}$ in terms of the familiar object C_{μ^1,μ^2,μ^3}:

$$R^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,\vec{w}) = \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^{3} \frac{\chi_{\nu^i}(C_{\mu^i})}{\mathfrak{z}(\mu^i)} q^{\frac{1}{2}\sum_{i=1}^3 \text{wt dependence}} C_{\nu^1,\nu^2,\nu^3}(q)$$

where $q \coloneqq e^{i\lambda}$.

Today we will prove the 2-leg case. This will give the GW/DT correspondence for local toric surfaces $K_S \to S$. Let

$$G^\bullet_{\mu^1,\mu^2}(\lambda,\tau) \coloneqq G^\bullet_{\mu^1,\mu^2,\emptyset}(\lambda,1,\tau,-\tau-1)$$
$$R^\bullet_{\mu^1,\mu^2}(\lambda,\tau) \coloneqq R^\bullet_{\mu^1,\mu^2,\emptyset}(\lambda,1,\tau,-\tau-1).$$

Theorem 1.47 (LLZ). $G^\bullet_{\mu^1,\mu^2}(\lambda,\tau) = R^\bullet_{\mu^1,\mu^2}(\lambda,\tau)$.

Proof. The proof of the 1-leg case we saw last time is a specialization of the proof for the 2-leg case we will see now. (It is not actually the proof in the 1-leg paper.) We first verify this formula in the case $\tau = -1$. In general,

$$G_{g,\mu^1,\mu^2}(\tau) = (\tau(\tau+1))^{\ell_1+\ell_2-1}(\cdots)$$

so that $G_{g,\mu^1,\mu^2}(-1) = 0$ unless (μ^1,μ^2) is $((d),\emptyset)$ or $(\emptyset,(d))$. In this case, we get an explicit expression

$$\int_{\bar{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-1)\Lambda_g^\vee(0)}{1-q\psi}.$$

This verifies the initial condition (due to Zhou)

$$G^\bullet_{\mu^1,\mu^2}(\lambda,-1) = R_{\mu^1,\mu^2}(\lambda,-1).$$

Now we look at the dependence on framing conditions. By orthogonality of characters, from the (unproved, so far) general formula for the 3-leg case, we want the framing dependence

$$R^\bullet_{\mu^1,\mu^2}(\lambda,\tau) = \sum_{|\nu^i|=|\mu^i|} R^\bullet_{\nu^1,\nu^2}(\lambda,\tau_0)\mathfrak{z}(\nu^1)\Phi^\bullet_{\nu^1,\mu^1}(i\lambda(\tau-\tau_0))\mathfrak{z}(\nu^2)\Phi^\bullet_{\nu^2,\mu^2}(i\lambda(\frac{1}{\tau}-\frac{1}{\tau_0})).$$

It suffices to show that $G^\bullet_{\mu^1,\mu^2}(\lambda,\tau)$ satisfies this framing dependence. The idea is the same as last time: define a generating function

$$K^\bullet_{\mu^1,\mu^2}(\lambda) \coloneqq \sum_g (\sqrt{-1})^{\cdots}\lambda^{2g-2+\ell_1+\ell_2}\int_{[\bar{\mathcal{M}}^\bullet_g(S/D_1\cup D_2,\beta=|\mu_1|D_1+|\mu^2|D_2,\mu^1,\mu^2)]^{\mathrm{vir}}} e(V^\bullet_{g,\mu})$$

for relative GW invariants of $(S \coloneqq \mathrm{Bl}_{2\text{ pts}}\,\mathbb{P}^2, D_1, D_2)$ where the D_i are exceptional divisors. This we compute by virtual localization, but we have to be careful because these divisors are not T-fixed. We get

$$K^\bullet_{\mu^1,\mu^2}(\lambda) = \sum_{|\nu^i|=|\mu^i|} G^\bullet_{\nu^1,\nu^2}(\lambda,\tau)\mathfrak{z}(\nu^1)\Phi^\bullet_{\nu^1,\mu^1}(-i\lambda\tau)\mathfrak{z}(\nu^2)\Phi^\bullet_{\nu^2,\mu^2}(\frac{-i\lambda}{\tau}).$$

This identity implies the desired result, as before. This calculation is analogous to the one last time. The bundle is

$$V^\bullet_{g,\mu^1,\mu^2} \coloneqq R^1\pi_*(f^*\mathcal{O}(-D_1-D_2)\otimes\mathcal{O}_C(-R))$$

with rank $\mathrm{vdim}\,\bar{\mathcal{M}}^\bullet_{g,\mu^1,\mu^2} = g-1+|\mu^1|+|\mu^2|+\ell_1+\ell_2$. A lemma tells us $e_T(V^\bullet_{g,\mu^1,\mu^2})|_F = 0$ on a fixed locus unless the image of the map is precisely the x and y axes in $\mathbb{P}^2$. This is by some clever choice of weights.

1. The contribution from the vertex contracted to $(0,0)$ gives an integral over $\bar{\mathcal{M}}^\bullet_{g,\ell_1+\ell_2}$, i.e. the term $G^\bullet_{g,\mu^1,\mu^2}$.
2. The contribution from the rubbers give double Hurwitz integrals $\Phi^\bullet_{\nu^i,\mu^i}$. □

1.11 Nov 27 (Melissa): Topological vertex

Given a toric CY3 X, its GW invariants can be obtained by gluing together local pieces at each vertex. According to the physicists, the local pieces are $F^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,n_1,n_2,n_3)$, where n_i are the *framings* at each open leg, coming from framings of 3-component links. Recall from last time that we had a generating function $G^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,w_1,w_2,w_3)$ of open GW invariants, where w_i are the weights of a CY torus. The physicists prediction for general framing is:

$$G^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,w_1,w_2,w_3) = F^\bullet_{\mu^1,\mu^2,\mu^3}(\lambda,w_2/w_1,w_3/w_2,w_1/w_3).$$

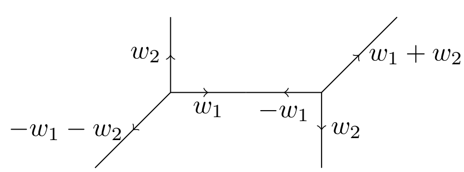
We denote $\tau \coloneqq w_2/w_1$ and hence $(-\tau-1)/\tau = w_3/w_2$ and $1/(-\tau-1) = w_1/w_3$.

Definition 1.48. Let $X^1 \subset X$ denote the 1-skeleton of X, i.e. the union of 0-dimensional and 1-dimensional orbits of the T-action. Let $T' \subset T$ be the CY subtorus. The configuration of X^1 and the T-equivariant structure gives a planar trivalent graph Γ_X associated to X, labeled by tangent weights on its edges. It will have *compact* edges and *non-compact* edges.

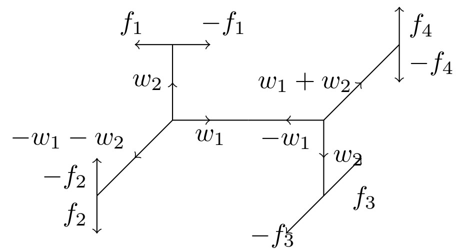
Conversely, given Γ_X, we can recover $\hat{X}$. Here $\hat{X}$ is the formal completion of X along X^1. By localization, we can define T-equivariant GW invariants of X purely using $\hat{X}$. Hence we have a procedure

$$\Gamma_X \rightsquigarrow \hat{X} \rightsquigarrow N^X_{g,\beta}.$$

Example 1.49. The resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ gives the graph



Definition 1.50 (FTCY graphs)**.** We generalize the procedure by introducing the notion of **formal toric CY (FTCY) graphs**. The idea is to add $f_i \in \mathbb{Z}^2$ to the end of each non-compact edge which specifies the equivariant structure on the normal bundle to the edge at the compactification divisor:

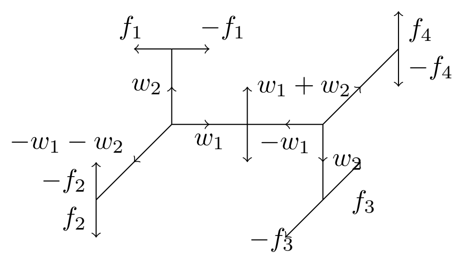


These specify framings at the compactification divisor $\hat{D}$. For example, $f_4 = -w_1 - n_4(w_1 + w_2)$ where the normal bundle to that compactified edge is $\mathcal{O}(n_4) \oplus \mathcal{O}(-n_4 - 1)$. Note that in $(\hat{Y}, \hat{D})$, the CY condition is $K_{\hat{Y}} + \hat{D} = 0$. Hence we get a procedure

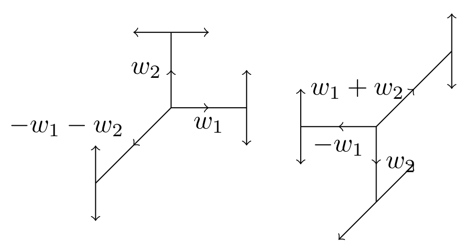
$$\text{FTCY graph} \rightsquigarrow (\hat{Y}, \hat{D}) \rightsquigarrow F^{\Gamma}_{g,\vec{d},\vec{\mu}}$$

producing T-equivariant **formal relative GW invariants**.

To introduce degenerations, we need to know how the normal bundle degenerates, equivariantly. So if we want to degenerate a FTCY graph, we must choose a framing for the node corresponding to the degeneration point, e.g.



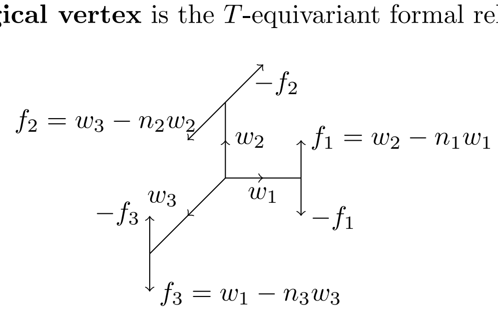
If we resolve the singularity, the result will be two graphs Γ_1, Γ_2.



The degeneration formula tells us that the original graph arises from gluing the invariants associated to Γ_1 and Γ_2:

$$F^{\bullet,\Gamma}_{d,\mu^1,\mu^2,\mu^3,\mu^4} = \sum F^{\bullet,\Gamma_1}_{\mu^1,\mu^2,\nu} \mathfrak{z}(\nu) F^{\bullet,\Gamma_2}_{\mu^3,\mu^4}.$$

Definition 1.51. The **topological vertex** is the T-equivariant formal relative GW series $F^{\bullet}_{\vec{\mu}}(\lambda; \vec{w}, \vec{n})$ corresponding to the graph



Let C_i denote the compactified leg of weight w_i. Then $N_{C_i/\hat{Y}} = \mathcal{O}(n_i) \oplus \mathcal{O}(-n_i - 1)$.

We can explicitly identify contributions to the topological vertex using our past computations.

1. The vertex will give a triple Hodge integral $G_{g_0,\nu^1,\nu^2,\nu^3}(w_1, w_2, w_3)$. Put these into a generating function $G^\bullet_{\vec{\nu}}(\lambda, \vec{w})$.
2. The legs will give double Hurwitz numbers
$$H^\bullet_{g,\mu,\nu} := \frac{1}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|} \deg(\mathrm{Br}\colon \bar{\mathcal{M}}^\bullet_{g,\mu,\nu}(\mathbb{P}^1/\{0,\infty\}) \to \mathbb{P}^r),$$
which we can compute the same way we computed the ELSV formula. Put these into a generating function $\Phi^\bullet_{g,\mu,\nu}(\lambda)$.

Proposition 1.52.

$$F^\bullet_{\vec{\mu}}(\lambda, \vec{w}, \vec{n}) = \sum_{|\nu^i|=|\mu^i|} G^\bullet_{\nu^1,\nu^2,\nu^3}(\vec{w}) \prod_{i=1}^{3} \mathfrak{z}(\nu^i)\Phi^\bullet_{\nu^i,\mu^i}\left(\sqrt{-1}\lambda(n_i - \frac{w_{i-1}}{w_i})\right).$$

Lemma 1.53 (Framing dependence, winding basis)**.**

$$F^\bullet_{\vec{\mu}}(\lambda, \vec{w}, \vec{n}) = \sum_{|\nu^i|=|\mu^i|} F^\bullet_{\vec{\mu}}(\lambda, \vec{w}, \vec{0}) \prod_{i=1}^{3} \mathfrak{z}(\nu^i)\Phi_{\nu^i,\mu^i}(\sqrt{-1}\lambda n_i).$$

Hence we can diagonalize the framing dependence if we define

$$\tilde{C}_{\vec{\mu}}(\lambda, \vec{w}, \vec{n}) := \sum_{|\nu^i|=|\mu^i|} F^\bullet_{\vec{\nu}}(\lambda, \vec{w}, \vec{n}) \prod_{i=1}^{3} \chi_{\mu^i}(\nu^i).$$

Lemma 1.54 (Framing dependence, representation basis)**.**

$$\tilde{C}_{\vec{\mu}}(\lambda, \vec{w}, \vec{n}) = q^{\frac{1}{2}\sum_{i=1}^{3} \kappa_{\mu^i} n_i/2} \tilde{C}_{\vec{\mu}}(\lambda, \vec{w}, \vec{0}).$$

Theorem 1.55 (Weight independence)**.** *The series*

$$F^\bullet_{\vec{\mu}}(\lambda, \vec{w}, \vec{0}) \in \mathbb{Q}(w_2/w_1)[[\lambda, \lambda^{-1}]]$$

does not depend on $\vec{w}$*. Hence* $F^\bullet_{\vec{\mu}}(\lambda, \vec{w}, \vec{0}) = F^\bullet_{\vec{\mu}}(\lambda, \vec{0})$.

Corollary 1.56. *The generating series for triple Hodge integrals satisfies*

$$G^\bullet_{\mu^1,\mu^2,\mu^3}(w_1, w_2, w_3) = \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(\mu^i)}{\mathfrak{z}(\mu^i)} q^{\frac{1}{2}\sum_{i=1}^{3} \kappa_{\nu^i}\frac{w_{i+1}}{w_i}} \tilde{C}_{\nu^1,\nu^2,\nu^3}(\lambda, \vec{n} = \vec{0}).$$

Lemma 1.57.

$$\begin{aligned} G_{g,\mu^1,\mu^2,\mu^3}(1,1,-2) &= (-1)^{|\mu^1|-\ell(\mu^1)} \frac{\mathfrak{z}(\mu^1 \cup \mu^2)}{\mathfrak{z}(\mu^1)\mathfrak{z}(\mu^2)} G_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(1,1,-2) \\ &\quad + \delta_{g,0} \sum_{m\geq 1} \delta_{\mu^1,(m)}\delta_{\mu^2,\emptyset}\delta_{\mu^3,(2m)} \frac{(-1)^{m-1}}{m}. \end{aligned}$$

Theorem 1.58. $\tilde{C}_{\nu^1,\nu^2,\nu^3}(\lambda) = W_{\nu^1,\nu^2,\nu^3}(q)$ *where* $q = e^{\sqrt{-1}\lambda}$.

2 DT theory

2.1 Sep 12 (Clara): GW/DT for local CY toric surfaces

Let X be a nonsingular projective CY3.

Definition 2.1 (GW side)**.** Let $N_{g,\beta} := \int_{[\bar{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1$ be GW invariants, and put them into a generating function

$$F'_{\mathrm{GW}}(X,u,v) := \sum_{g\geq 0}\sum_{\beta\neq 0} N_{g,\beta} u^{2g-2} v^{\beta}.$$

Note that we exclude constant maps. The **reduced GW partition function** is

$$Z'_{\mathrm{GW}}(X,u,v) := \exp F'_{\mathrm{GW}}(X,u,v).$$

Definition 2.2 (DT side)**.** Given a 1-dimensional subscheme Z, let Z' be the purely 1-dimensional part $[Z'] = \beta \in H_2(X,\mathbb{Z})$. Let

$$I_n(X,\beta) := \{Z \text{ at most 1-dim} : \chi(\mathcal{O}_Z) = n,\ [Z'] = \beta\}$$

and define DT invariants $D_{n,\beta} := \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1$. For example, if $\beta = 0$, we recover $I_n(X,0) = \mathrm{Hilb}^n(X)$. Define the generating function

$$Z_{\mathrm{DT}}(X,q,v) := \sum_{\beta}\sum_{n\in\mathbb{Z}} D_{n,\beta} q^n v^{\beta}.$$

We exclude constant maps by quotienting. The **reduced DT partition function** is

$$Z'_{\mathrm{DT}}(X,q,v) := \frac{Z_{\mathrm{DT}}(X,q,v)}{Z_{\mathrm{DT}}(X,\beta)_0}$$

where $Z_{\mathrm{DT}}(X,\beta)_0 := \sum_{n\in\mathbb{Z}} D_{n,0} q^n$.

Conjecture 2.3. *The change of variables $e^{iu} = -q$ equates reduced partition functions, i.e.*

$$Z'_{GW}(X,u,v) = Z'_{DT}(X,-e^{iu},v).$$

We want to understand virtual localization over $I_n(X,\beta)$. Let's first do $I_n(X,0) = \mathrm{Hilb}^n(X)$. Given $I \in I_n(X,0)$, we have $\dim \mathbb{C}[x,y,z]/I = n$ as a $\mathbb{C}$-vector space. For example, for

$$I = (x^3,y^2,z^2,xy,xz,yz), \quad \mathbb{C}[x,y,z]/I = \mathbb{C}\langle 1,x,y,z,x^2\rangle$$

we have $I \in \mathrm{Hilb}^5(X)$. We visually represent these ideals by boxes, i.e. 3D partitions. For $\beta = 0$ we have only a finite number of boxes, for 0-dimensionality of the quotient $\mathbb{C}[x,y,z]/I$.

For $\beta \neq 0$, we need an infinite number of boxes so that we have a curve, i.e. $\mathbb{C}[x,y,z]/I$ is no longer a finite-dimensional vector space. For example, $\mathbb{C}[x,y,z]/(y,z)$ is a line, and corresponds to an infinite row of boxes along the x-axis. Locally around a fixed point x_α, the ideal sheaf gives a 3D partition

$$\pi_\alpha := \{(k_1,k_2,k_3) : \prod x_i^{k_i} \notin I_\alpha\}.$$

For each leg, asymptotically the 2d partition stays the same, and we define

$$\lambda_{\alpha\beta} := \{(k_2,k_3) : \forall k_1,\ \prod x_i^{k_i} \notin I_\alpha\}.$$

So instead of specifying ideals as fixed points of our moduli space, we specify configurations of boxes around vertices and edges. Specifically, we specify:

1. a 2-dimensional partition $\lambda_{\alpha\beta}$ for each edge;

2. a 3-dimensional partition π_α for each vertex, so that its three asymptotics agree with the specified 2d partitions for the three edges.

Definition 2.4. Let π_α be a vertex partition. Define its **size** by

$$|\pi_\alpha| := \#\{\pi_\alpha \cap [0, \dots, N]^3\} - (N+1) \sum_i |\lambda_{\alpha\beta_i} \quad N \gg 0.$$

We will now apply virtual localization to do the integral:

$$\int_{[I_n(X,\beta)]^{\text{vir}}} 1 = \sum_{[I] \in I_n(X,\beta)^T} \int_{[I]} \frac{e(E_1^m)}{e(E_2)^m}.$$

Here e is the equivariant Euler class, i.e. product of weights, and m stands for "moving part" i.e. non-trivial weights. For us, the perfect obstruction theory is $E_i := \operatorname{Ext}^i(I,I)$. We will compute weights of

$$\operatorname{Ext}^1(I,I) - \operatorname{Ext}^2(I,I).$$

For X toric, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. We also have

$$\operatorname{Ext}^3(I,I) = 0, \quad \operatorname{Ext}^0(\mathcal{O},\mathcal{O}) - \operatorname{Ext}^0(I,I) = 0,$$

so that we can write

$$\operatorname{Ext}^1(I,I) - \operatorname{Ext}^2(I,I) = \chi(\mathcal{O},\mathcal{O}) - \chi(I,I).$$

This is better: we can use the Čech cover coming from fixed points to compute these Euler characteristics. By local-to-global Ext and then passing to the Čech complex,

$$\chi(I,I) = \sum_{i=0}^3 \operatorname{Ext}^i(I,I) = \sum_{i,j=0}^3 (-1)^{i+j} H^i(\mathcal{E}xt^j(I,I)) = \sum_{i,j=0}^3 (-1)^{i+j} \mathcal{C}^i(\mathcal{E}xt^j(I,I)).$$

For us, $\mathcal{C}^2(\mathcal{E}xt^j(I,I)) = 0$.

Hence we can now explicitly identify the weights of the virtual tangent space, as

$$\begin{aligned} T = &\left(\bigoplus_\alpha \Gamma(U_\alpha) - \sum (-1)^i \Gamma(U_\alpha, \mathcal{E}xt^i(I,I))\right) \\ &- \left(\bigoplus_{\alpha,\beta} \Gamma(U_{\alpha\beta}) - \sum (-1)^i \Gamma(U_{\alpha\beta}, \mathcal{E}xt^i(I,I))\right) \end{aligned}$$

Here $U_\alpha = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3]$ and $U_{\alpha\beta} = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, x_2, x_3]$. The first line is the vertex contribution, and the second line is the edge contribution.

Let $R := \mathbb{C}[x_1, x_2, x_3]$. We need to compute $R - \sum (-1)^i \operatorname{Ext}^i(I_\alpha, I_\alpha)$. This requires taking a T-equivariant free resolution of I_α

$$0 \to F_j \to F_{j-1} \to \cdots \to F_0 \to I_\alpha \to 0, \quad F_j = \bigoplus R(d_{ij}).$$

Here $d_{ij} \in \mathbb{Z}^3$ and $R(d_{ij}) := x_1^{k_1} x_2^{k_2} x_3^{k_3} R$, so that its contribution to the Euler class is

$$t^{d_{ij}} \frac{1}{(1-t_1)(1-t_2)(1-t_3)}.$$

The total contribution of I_α is therefore

$$\operatorname{tr}_{I_\alpha} = \frac{P_\alpha(t_1,t_2,t_3)}{(1-t_1)(1-t_2)(1-t_3)}, \quad P_\alpha(t_1,t_2,t_3) := \sum (-1)^i t^{d_{ij}}.$$

We also have the contribution from the actual vertex

$$Q_a = \operatorname{tr}_{R/I_\alpha}(t_1, t_2, t_3) = \sum_{(k_1,k_2,k_3)\in\pi_\alpha} t_1^{k_1} t_2^{k_2} t_3^{k_3}$$

which by the SES $0 \to I_\alpha \to R \to R/I_\alpha \to 0$ satisfies

$$Q_\alpha = \operatorname{tr}_R - \operatorname{tr}_{I_\alpha} = \frac{1 + P_\alpha(t_1, t_2, t_3)}{(1-t_1)(1-t_2)(1-t_3)}.$$

(Note that P_α begins at the -1 term of the resolution, and contains the extra minus sign.) So now we know what P_α is. We can compute

$$\begin{aligned}\sum(-1)^I \operatorname{Ext}^i(I_\alpha, I_\alpha) &= \sum_{i,j,k,l} (-1)^{i+k} R(-d_{ij} + d_{kl}) = \sum(-1)^i R(-d_{ij}) \sum(-1)^k R(d_{kl}) \\ &= \frac{P_\alpha(t_1,t_2,t_3) P_\alpha(t_1^{-1}, t_2^{-1}, t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3)}.\end{aligned}$$

Collecting all this and rewriting in terms of Q, we get

$$\operatorname{tr}_{R-\chi(I_\alpha,I_\alpha)} = Q_\alpha - \frac{\bar{Q}_\alpha}{t_1t_2t_3} + Q_\alpha \bar{Q}_\alpha \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}.$$

Here $\bar{Q}$ means we plug in t_i^{-1} instead of t_i.

2.2 Sep 19 (Clara): DT for local CY toric surfaces

First, a quick recap of what we were doing. We were computing DT invariants for smooth toric CY3s:

$$D_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 = \sum_{[I]} \frac{e(\operatorname{Ext}^2(I,I))}{e(\operatorname{Ext}^1(I,I))}.$$

So we needed the virtual character of $\operatorname{Ext}^2(I,I) - \operatorname{Ext}^1(I,I)$. We wanted to compute this on a Čech cover given by vertices and edges:

$$\bigoplus_\alpha (\Gamma(U_\alpha) - \sum(-1)^i \Gamma(U_\alpha, \mathcal{E}xt^i(I,I))) - \bigoplus_{\alpha,\beta}(\Gamma(U_{\alpha\beta}) - \sum(-1)^i\Gamma(U_{\alpha\beta}, \mathcal{E}xt^i(I,I))).$$

Last time we computed the first term and rewrote it purely in terms of the partition sitting at the vertex α:

$$\operatorname{tr}(\Gamma(U_\alpha) - \sum(-1)^i\Gamma(U_\alpha, \mathcal{E}xt^i(I,I))) = F_\alpha \coloneqq Q_\alpha - \frac{\bar{Q}_\alpha}{t_1t_2t_3} - \frac{Q_\alpha\bar{Q}_\alpha(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}.$$

Today we will do the edge computation. Here $U_{\alpha\beta} = U_\alpha \cap U_\beta$, so the ring is $R \coloneqq \Gamma(U_{\alpha\beta}) = \mathbb{C}[x_1^{\pm 1}, x_2, x_3]$. Hence

$$\operatorname{tr} R = \frac{\delta(t_1)}{(1-t_2)(1-t_3)}, \quad \delta(t_1) \coloneqq \sum_{i\in\mathbb{Z}} t_1^i.$$

If we play the same game as for the edge, we can write

$$Q_{\alpha\beta} \coloneqq \sum_{(k_2,k_3)\in\lambda_{\alpha\beta}} t_2^{k_2} t_3^{k_3}$$

and then the virtual character is

$$\operatorname{tr}(R - \chi(I_{\alpha\beta}, I_{\alpha\beta})) = \delta(t_1) F_{\alpha\beta}, \quad F_{\alpha\beta} \coloneqq -Q_{\alpha\beta} - \frac{\bar{Q}_{\alpha\beta}}{t_2t_3} + Q_{\alpha\beta}\bar{Q}_{\alpha\beta}\frac{(1-t_2)(1-t_3)}{t_2t_3}.$$

We want to split the vertex and edge contributions in such a way so that we get Laurent polynomials in the end.

The first step is to write a new vertex. The vertex x_α receives contributions from $F_{\alpha\beta_i}$ for $i = 1, 2, 3$. Pull apart

$$\operatorname{tr}(R - \chi(I_{\alpha\beta}, I_{\alpha\beta})) = \frac{F_\alpha \beta}{1 - t_1} + \frac{t_1^{-1} F_{\alpha\beta}}{1 - t_1^{-1}}.$$

Hence define the new vertex term

$$V_\alpha \coloneqq F_\alpha + \sum_{i=1}^{3} \frac{F_{\alpha\beta}}{1 - t_i}.$$

Lemma 2.5. *V_α is a Laurent polynomial.*

Proof. The character coming from the vertex is

$$Q_\alpha = \frac{Q_{\alpha\beta_1}}{1 - t_1} + \frac{Q_{\alpha\beta_2}}{1 - t_2} + \frac{Q_{\alpha\beta_3}}{1 - t_3} + \text{polynomial}.$$

Plugging this into V_α, we get the cancellations we need. □

What's left to account for:

1. (negative terms) $t_1^{-1} F_{\alpha\beta}/(1 - t_1^{-1})$;
2. (overcounting) each edge has been plugged into two different vertices.

Take $C_{\alpha\beta} \cong \mathbb{P}^1$, which has normal bundle

$$N_{C_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}).$$

Hence the transition functions look like

$$(t_1, t_2, t_3) \mapsto (t_1, t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}}),$$

and the double contributions per edge are given exactly by this change of variables. Hence define

$$E_{\alpha\beta} \coloneqq \frac{t_1^{-1} F_{\alpha\beta}}{1 - t_1^{-1}} - \frac{F_{\alpha\beta}(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}})}{1 - t_1^{-1}}.$$

This $E_{\alpha\beta}$ has no poles in t_1 because it is regular at $t_1 = 1$. Hence it is a Laurent polynomial.

Let's apply this to local CY3 surfaces. Start with a non-singular projective toric surface S, and take the total space of the canonical K_S. Do a toric compactification $\mathbb{P}(K_S \oplus 1)$. The DT invariants we define are

$$Z'_{\mathrm{DT}}(S, q)_\beta \coloneqq \frac{Z_{\mathrm{DT}}(X, q)_\beta}{Z_{\mathrm{DT}}(X, q)_0}, \quad \beta \in H_2(S, \mathbb{Z}).$$

Let $D \coloneqq X \setminus K_S$ be the divisor at infinity. Then if $I \in I_n(X, \beta)^T$, there can be only a bunch of closed points on D, and there is a purely 1-dimensional $Z' \subset S$. Split $I = \xi \oplus \alpha$, and we split off the zero-dimensional contributions from D:

$$Z'_{\mathrm{DT}}(S, q)_\beta = \frac{\sum_n q^n \sum_{I \in I_n(K_S, \beta) I} e(\operatorname{Ext}^\bullet)}{\sum_n q^n \sum_{I \in I_n(\alpha_S, 0)} e(\operatorname{Ext}^\bullet)}.$$

Pass to a 2d subtorus $\{t_1 t_2 t_3 = 1\}$ preserving the CY form. Upshot: equivariant Serre duality

$$\operatorname{Ext}^1(I, I)_0 = \operatorname{Ext}^2(I, I)_0^\vee (t_1 t_2 t_3)^{\pm 1}$$

becomes simpler. So instead of computing $\mathrm{Ext}^2 - \mathrm{Ext}^1$, we can just count the number of minus signs. We do this by writing

$$\mathrm{Ext}^2 - \mathrm{Ext}^1 = V^+ + V^-$$

such that $\bar{V}^+|_{t_1t_2t_3=1} = -V^-$. We will write

$$V^+ = \sum V_\alpha^+ + \sum E_{\alpha\beta}^+.$$

In general, there are many ways to do this splitting. But there are nice splittings that enable us to count minus signs easily.

Suppose that I_α at x_α is actually a finite 3d partition, i.e. in

$$V_\alpha = F_\alpha + \sum \frac{F_{\alpha\beta_i}}{1-t_i},$$

the second term is zero. So it suffices to split

$$F_\alpha = Q_\alpha - \bar{Q}_\alpha t_1^{-1}t_2^{-1}t_3^{-1} - Q_\alpha \bar{Q}_\alpha (1-t_1^{-1})(1-t_2^{-1})(1-t_3^{-1}).$$

Pick

$$\begin{aligned} F_\alpha^+ &= Q_\alpha - Q_\alpha\bar{Q}_\alpha(1 + t_1^{-1}t_2^{-1} + t_1^{-1}t_3^{-1} + t_2^{-1}t_3^{-1}) \\ F_\alpha^- &= -\frac{\bar{Q}_\alpha}{t_1t_2t_3} - Q_\alpha\bar{Q}_\alpha((t_1t_2t_3)^{-1} + t_3^{-1} + t_2^{-1} + t_1^{-1}). \end{aligned}$$

We are left with determining the parity of $V_\alpha^+(1,1,1)$ and $E_{\alpha\beta}^+(1,1,1)$. A computation with the splittings shows

$$\begin{aligned} V_\alpha^+(1,1,1) &\equiv |\pi_\alpha| \bmod 2 \\ E_{\alpha\beta}^+(1,1,1) &\equiv f(\alpha,\beta) + m_{\alpha\beta}|\lambda_{\alpha\beta}| \bmod 2. \end{aligned}$$

In total,

$$\frac{e(\mathrm{Ext}^2)}{e(\mathrm{Ext}^1)} = (-1)^{\chi(\mathcal{O}_Y) + \sum_{\alpha,\beta} m_{\alpha\beta}|\lambda_{\alpha\beta}|}.$$

2.3 Sep 26 (Ivan): MNOP2

MNOP2 modifies the GW/DT correspondence from MNOP1 in two ways: with insertions and with the relative theory.

Definition 2.6 (GW side insertions)**.** So far we have considered Hodge integrals

$$\int_{[\bar{\mathcal{M}}_{g,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \psi_i^{k_i}$$

possibly with λ_i. Let $\mathrm{ev}_i \colon \bar{\mathcal{M}}_{g,r}(X,\beta) \to X$ be evaluation at the i-th marked point. Denote by

$$\tau_k(\gamma_i) \coloneqq \psi_i^k \,\mathrm{ev}_i^*(\gamma_i), \quad \gamma_i \in H^*(X,\mathbb{Q}),$$

which we stick into correlators $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle$. Call:

1. $\tau_0(-)$ a **primary field**;
2. $\tau_{>0}(-)$ a **descendant field**.

Let $\bar{\mathcal{M}}'_{g,r}(X,\beta)$ be the moduli of maps from *possibly disconnected* stable curves with no collapsed connected components. Note that g for us means $g \coloneqq 1 - \chi(\mathcal{O}_C)$, even when C is disconnected. We use this to define the **reduced generating function**

$$Z_{GW}(X; u|\prod_{i=1}^r \tau_{k_i}(\gamma_i)) \coloneqq \sum_{g\in\mathbb{Z}} \langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\rangle'_{g,\beta} u^{2g-2}.$$

Remark. The sum in $g \in \mathbb{Z}$ is bounded from below, i.e. for $g \ll 0$ there are no contributions, because we fixed β and no connected components are collapsed.

Definition 2.7 (DT side insertions)**.** Let X be a non-singular projective 3-fold and I an ideal sheaf on X. Then I fits into

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where $Y \subset X$ is the subscheme associated to I. Let $I_n(X,\beta)$ denote the moduli of ideal sheaves with $[Y] = \beta$ and $\chi(\mathcal{O}_Y) = n$. Let $\mathcal{I}$ denote the universal ideal sheaf on $I_n(X,\beta) \times X$. Note that $\mathcal{I}$ has a finite resolution by locally free sheaves, so $\mathrm{ch}_* \mathcal{I}$ is well-defined. Define homology operations for $\gamma \in H^\ell(X,\mathbb{Z})$ as

$$\begin{aligned} \mathrm{ch}_{k+2}(\gamma)\colon H_*(I_n(X,\beta),\mathbb{Q}) &\to H_{*-2k+2-\ell}(I_n(X,\beta),\mathbb{Q}) \\ \xi &\mapsto \pi_{1*}(\mathrm{ch}_{k+2}(\mathcal{I}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)). \end{aligned}$$

(Here we compute the dimension shift as $+\dim X - \ell - 2k - 4 = -2k + 2 - \ell$.) For example, if we take $k = 0$, we get only integration over $\operatorname{supp} I$. The invariants are

$$\begin{aligned} \langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r)\rangle_{n,\beta} &\coloneqq \int_{[I_n(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_i) \\ &\coloneqq (\pm 1)\, \mathrm{ch}_{k_1+2}(\gamma_1) \circ \cdots \circ \mathrm{ch}_{k_r+2}(\gamma_r) [I_n(X,\beta)]^{\mathrm{vir}}. \end{aligned}$$

Define the generating function

$$Z_{DT}(X; q|\prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i))_\beta \coloneqq \sum_{n\in\mathbb{Z}} \langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i)\rangle_{n,\beta} q^n.$$

The **reduced partition function** is

$$Z'_{DT}(X, q|\cdots)_\beta \coloneqq \frac{Z_{DT}(X; q|\cdots)_\beta}{Z_{DT}(X; q|\cdots)_0}.$$

Remark. Note that

$$\operatorname{vdim} I_n(X,\beta) = \int_\beta c_1(T_X), \quad \operatorname{vdim} \bar{\mathcal{M}}_{g,r}(X,\beta) = \int_\beta c_1(T_X) + r.$$

So later if we want to compare descendants, the insertions we make have to satisfy certain degree requirements on both sides. This is partially why we take ch_{k+2} instead of ch_k.

There are a few conjectures in MNOP2.

1. (Degree 0) For $\beta = 0$,
   $$Z_{DT}(X; q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)}$$
   where $M(q) \coloneqq \prod_{n\geq 0}(1-q^n)^{-n}$ is the MacMahon function. This is known in the toric case by the computation from MNOP1, and the case for general 3-folds follows from the cobordism argument of Levine–Pandharipande.

2. (Rationality) $Z'_{DT}(X; q|\prod \cdots)$ is a rational function. This is known in the toric CY3 case.

3. (Primary fields) After the change of variables $e^{iu} = -q$,

$$(-iu)^d Z'_{GW}(X; u| \prod_{i=1}^r \tau_0(\gamma_i))_\beta = (-q)^{d/2} Z'_{DT}(X; q| \prod_{i=1}^r \tilde{\tau}_0(\gamma_i))_\beta.$$

4. (Descendant fields) The two sets

$$\mathcal{Z}'_{GW,\beta} \coloneqq \{(-iu)^{d-\sum k_i} Z'_{GW}(X; u| \prod \tau_{k_i}(\gamma_i))_\beta\}$$
$$\mathcal{Z}'_{DT,\beta} \coloneqq \{(-q)^{d/2} Z'_{DT}(X; q| \prod \tilde{\tau}_{k_i}(\gamma_i))_\beta\}$$

have the same linear span, and there is a transition matrix expressing the functions in one in terms of the other such that:

   (a) it is upper triangular with 1's along the diagonal;

   (b) it has universal coefficients depending only on classical multiplication in X.

   This has been checked in the toric case.

Definition 2.8 (Relative GW)**.** Let X be a non-singular projective 3-fold with $S \subset X$ a non-singular divisor. Let $\beta \in H_2(X, \mathbb{Z})$ be such that $\int_\beta [S] \geq 0$ and let μ be a partition of it. Define the **moduli of stable relative maps**

$$\bar{\mathcal{M}}'_{g,r}(X/S, \beta, \mu)$$

of stable relative maps $C \to X[k]$ with possibly disconnected domain and relative multiplicities μ. Think: whenever we get non-transverse intersection with S, blow up to get copies of $\Delta \coloneqq \mathbb{P}(\mathcal{O}_S \oplus N_{S/X})$, whose divisors at infinity form a sequence $S = S_0, S_1, \ldots, S_k$.

A **cohomologically weighted partition** is an unordered set

$$\eta \coloneqq \{(\eta_1, \delta_1), \ldots, (\eta_s, \delta_s)\}$$

where $\eta_i \in \mathbb{Z}_{>0}$ and $\delta_i \in H^*(S, \mathbb{Q})$. Let $\vec{\eta}$ denote the underlying partition. Using them, define relative GW invariants as

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) | \eta \rangle \coloneqq \frac{1}{|\operatorname{Aut}(\eta)|} \int_{[\bar{\mathcal{M}}'_{g,r}(X/S,\beta,\vec{\eta})]^{\mathrm{vir}}} \prod_{i=1}^r \psi_i^{k_i} \operatorname{ev}_i^*(\gamma_i) \prod_{j=1}^s \tilde{\operatorname{ev}}_j^*(\delta_j)$$

where $\tilde{\operatorname{ev}}_j$ is evaluation at pre-images of S_k.

2.4 Oct 03 (Ivan): MNOP2 II

Let X be non-singular projective threefold. Fix a non-singular divisor $S \subset X$. We first define relative DT theory.

Definition 2.9. We say I is an ideal sheaf on X **relative to** S if

$$I \otimes_{\mathcal{O}_X} \mathcal{O}_S \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_S$$

is injective. (This disallows whole components of I from lying in S.) From such an I we construct an element of $\operatorname{Hilb}(S, \int_\beta [S])$.

Remark. Being relative is an open condition on ideal sheaves, but we want a proper moduli space. To make our space proper, consider degenerations of the target space

$$X[k] \coloneqq X \cup_D \Delta \cup_D \cdots \cup_D \Delta$$

where $\Delta \coloneqq \mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$.

Definition 2.10. An ideal sheaf I on $X[k]$ is **predeformable** if for every singular divisor $S_l \subset X[k]$ (i.e. the divisor connecting Δ_{l-1} and Δ_l), the induced map

$$I \otimes_{\mathcal{O}_{X[k]}} \mathcal{O}_{S_l} \to \mathcal{O}_{X[k]} \otimes_{\mathcal{O}_{X[k]}} \mathcal{O}_{S_l}$$

is injective. In other words, if Y_{l-1} and Y_l are the subschemes associated to I on Δ_{l-1} and Δ_l, then

$$Y_{l-1} \cap S_l = Y_l \cap S_l$$

are equal scheme-theoretically.

Definition 2.11. An **isomorphism** of ideal sheaves I_1, I_2 on $X[k_1], X[k_2]$ is an isomorphism is an isomorphism $\sigma\colon X[k_1] \xrightarrow{\sim} X[k_2]$ fixing the original copy of X such that:

1. $\sigma^* \mathcal{O}_{X[k_2]}) \xrightarrow{\sim} \mathcal{O}_{X[k_1]}$ is the identity map;
2. $\sigma^* I_2 \xrightarrow{\sim} I_1$ is an isomorphism.

In particular, note that $k_1 = k_2$ necessarily. We say I is **stable** if $\operatorname{Aut} I$ is finite.

Definition 2.12 (Relative DT theory)**.** Let $I_n(X/S, \beta)$ be the **moduli space** of stable predeformable relative ideal sheaves on all possible degenerations $X[k]$ relative to S_k, such that

$$\chi(\mathcal{O}_Y) = n, \quad \pi_*[Y] = \beta \in H_2(X, \mathbb{Z})$$

where $\pi\colon X[k] \to X$ is the collapsing map.

Theorem 2.13. *This space $I_n(X/S, \beta)$ is complete DM stack with canonical perfect obstruction theory, and universal ideal sheaf $\mathcal{Y}$.*

Definition 2.14. Let $\epsilon\colon I_n(X/S, \beta) \to \operatorname{Hilb}(S, \int_\beta [S])$ be the natural map. Then ϵ^* gives relative conditions on $I_n(X/S, \beta)$. Hence relative DT invariants are of the form

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) | \eta \rangle_{n,\beta} := \int_{[I_n(X/S,\beta)]^{\text{vir}}} \prod_{i=1}^r \operatorname{ch}_{k_i+2}(\mathcal{Y})(\gamma_i) \epsilon^* \eta$$

where $\eta \in H^*(\operatorname{Hilb}(S, \int_\beta [S]))$.

Remark. Recall that $H^*(\operatorname{Hilb}(S), \mathbb{Q})$ has basis

$$c_\eta := P_{\eta_1}(\delta_1) \cdots P_{\eta_k}(\delta_k) \cdot 1$$

where η is a partition, and $P_{\eta_i}(\delta_i)$ are correspondences which insert η_i points with class $\delta_i \in H^*(S, \mathbb{Q})$. Last time we called an unordered set $\{(\eta_i, \delta_i)\}$ a cohomology-weighted partition.

Goal: for toric X, compute the degree-0 parts $Z_{DT}(X, q)_0$, i.e. $\beta = 0$. In particular, we have no relative insertions from $\operatorname{Hilb}(S, 0) = \text{pt}$. Recall that $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^3$ by standard component-wise multiplication, and the origin has tangent weights t_1^{-1}, t_2^{-1}, t_3^{-1}. For every 3d partition π, we introduced the equivariant vertex

$$V_\pi = Q_\pi - \frac{\bar{Q}_\pi}{t_1 t_2 t_3} + Q_\pi \bar{Q}_\pi \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3}$$

where

$$Q_\pi(t_1, t_2, t_3) := \sum_{(k_1,k_2,k_3) \in \pi} t_1^{k_1} t_2^{k_2} t_3^{k_3}$$

is the character of the partition π, and

$$\bar{Q}_\pi(t_1, t_2, t_3) := Q_\pi(t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

Definition 2.15. Introduce the **equivariant measure** with three parameters s_1, s_2, s_3

$$w(\pi) := \prod_{\vec{k}\in\mathbb{Z}^3} (k_1 s_1 + k_2 s_2 + k_3 s_3)^{-v_{\vec{k}}(\pi)}$$

where $v_{\vec{k}}(\pi)$ is the coefficient of $t_1^{k_1} t_2^{k_2} t_3^{k_3}$ in $V(\pi)$. This is useful because then the partition function is

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{\pi \text{ with legs } \vec{\lambda}} w(\pi) q^{|\pi|}.$$

Remark. We observe a few properties.

1. The q-coefficients of $W(\emptyset, \emptyset, \emptyset)$ are rational functions in s_1, s_2, s_3. This is because we sum finitely many rational functions.
2. $V_\pi(1,1,1) = 0$, so actually $\sum_{\vec{k}} v_{\vec{k}}(\pi) = 0$ for any π. Hence q-coefficients of $W(\emptyset, \emptyset, \emptyset)$ are degree 0 rational functions.
3. $W(\emptyset, \emptyset, \emptyset)$ is symmetric in s_1, s_2, s_3.
4. $\log W(\emptyset, \emptyset, \emptyset)|_{s_1+s_2+s_3=0} = M(-q)$, because Clara computed last time that $V_\pi|_{s_1+s_2+s_3=0} = (-1)^{|\pi|}$ and we know $M(q)$ is the generating function for finite 3d partitions.

To compute $W(\emptyset, \emptyset, \emptyset)$, consider a special geometry where it is a part. Specifically, look at $X := \mathbb{P}^1 \times \mathbb{C}^2$ with weights s_1, s_2, s_3, and fix the smooth divisor $S := \infty \times \mathbb{C}^2$. We want to compute $Z_{DT}^T(X/S, q)_0$.

Lemma 2.16. *The q-coefficients of $Z_{DT}^T(X/S, q)_0$ are rational functions in s_1, s_2, s_3 with poles in s_2, s_3 only.*

Proof. Construct a proper morphism

$$j\colon I_n(X/S, 0) \to \operatorname{Sym}^n(X) \to \operatorname{Sym}^n(\mathbb{C}^2) \to \bigoplus_{i=1}^n \mathbb{C}^2$$

where the last map is given by

$$\{(x_i, y_i)\} \mapsto ((p_i(\vec{x}), p_i(\vec{y})))_{i=1}^n$$

where p_i are power sums. So we can compute

$$\int_{[I_n(X/S,0)]^{\text{vir}}} 1 = \int_{\bigoplus_{k=1}^n \mathbb{C}^2} j_*[I_n(X/S, 0)]^{\text{vir}}.$$

But the k-th copy of $\mathbb{C}^2$ has a unique fixed point of weights $-ks_2, -ks_3$. Hence localization shows there are poles only in s_2, s_3. ☐

Localization gives two types of contributions: points over $0 \in \mathbb{P}^1$ and points over $\infty \in \mathbb{P}^1$. Hence write

$$Z_{DT}^T(X/S, q)_0 = W_0 \cdot W_\infty.$$

We know $W_0 = W(\emptyset, \emptyset, \emptyset)$, but W_∞ has contributions from the relative part.

2.5 Oct 10 (Ivan): MNOP2 III

Today we will do the computation of zero-degree contributions in relative DT for toric varieties. The way to do it is to consider a geometry with two fixed points: one on the divisor, one not. Specifically, take $T := (\mathbb{C}^*)^3$ acting on $X := \mathbb{P}^1 \times \mathbb{A}^2$, with weight $-s_1$ on $T_0\mathbb{P}^1$ and $-s_2, -s_3$ on $T_0\mathbb{A}^2$.

Recall that W_0 is the partition function for all zero-dimensional sheaves supported at $(0,0) \in \mathbb{P}^1 \times \mathbb{A}^2$. This is what we previously called $W(\emptyset, \emptyset, \emptyset)$. The new thing involves the torus-fixed divisor $D := \{\infty\} \times \mathbb{A}^2 \subset X$; we want to compute $Z^T_{\mathrm{DT}}(X/D, q)_0$. This, by localization, has contributions from T-fixed schemes supported at $(0,0)$ and D (in its bubblings). So

$$Z^T_{\mathrm{DT}}(X/D, q)_0 = W_0 \cdot W_\infty.$$

Recall that $W_0 = W(\emptyset, \emptyset, \emptyset)$ is a power series in q with rational coefficients of degree 0, symmetric in s_1, s_2, s_3, and we showed $W_0|_{s_1+s_2+s_3=0} = M(-q)$. We had a lemma last time that says $Z^T_{\mathrm{DT}}(X/D, q)_0$ has monomial poles in s_2, s_3.

To compute W_∞, we need rubber theory. What is rubber theory? Recall that the moduli in relative DT involves bubbles $R \sqcup_D R \sqcup_D \cdots \sqcup_D R$. In our case, $R = \mathbb{P}^1 \times \mathbb{A}^2$. The contribution to W_∞ is only from the moduli space I of all sheaves which are only supported on the bubbles. The space I has a description similar to that of relative DT. The difference is that on each Δ, there is a $\mathbb{G}_m$-action. Hence

$$I := I_n^\sim := I_n(R/(S_0 \cup S_\infty), 0)^\sim$$

where S_0 and S_∞ are the two divisors at $0, \infty \in \mathbb{P}^1$, and the $\sim$ means we identify by $\mathbb{G}_m$ on every $\mathbb{P}^1$. There is still an action by $\mathbb{C}^*_{s_2} \times \mathbb{C}^*_{s_3}$ on the $\mathbb{A}^2$ factor.

Question: what is the relation between $I_n^\sim$ and W_∞? Note that $I_n^\sim \hookrightarrow I_n$ with codimension 1 (where here by I_n we mean the part contributing to W_∞). The normal direction is given by deforming the node attaching the rubber to the original $\mathbb{P}^1$, i.e. it has Euler class $s_1 - \psi_0$ where $\psi_0 = c_1(\mathbb{L}_0)$. (Here the 0 is at the start of the rubber pieces, i.e. the ∞ of the original $\mathbb{P}^1$.) Hence

$$W_\infty = 1 + \sum_{n \geq 1} q^n \int_{[I_n^\sim]^{\mathrm{vir}}} \frac{1}{s_1 - \psi_0}.$$

If it weren't for this insertion, W_∞ would be a series in only s_2, s_3. We want to relate W_∞ with the simpler

$$F_\infty = \sum_{n \geq 0} q^n \int_{[I_n^\sim]^{\mathrm{vir}}} 1.$$

We know F_∞ is a power series in q with rational coefficients in s_2, s_3 only.

Lemma 2.17. *We have*

$$W_\infty = \exp(\frac{1}{s_1} F_\infty).$$

Proof. First expand W_∞ in powers of ψ_0, to get

$$W_\infty = 1 + \sum_{\ell \geq 0} \frac{1}{s_1^{\ell+1}} F_{\infty,\ell}$$

where $F_{\infty,\ell} = \sum_{n \geq 1} q^n \int_{[I_n^\sim]^{\mathrm{vir}}} \psi_0^\ell$. There is a topological recursion relation between the $F_{\infty,\ell}$ as follows. On $I_n^\sim$ we have the universal target and universal family/subscheme

$$\begin{array}{ccc} \mathcal{Y}_n & \longrightarrow & \mathcal{R} \\ & \searrow \quad \swarrow_{\pi} & \\ & I_n^\sim & \end{array}.$$

We have $\pi_*[\mathcal{Y}_n]^{\text{vir}} = n[I_n^{\sim}]^{\text{vir}}$ because $\mathcal{Y}_n \to I_n^{\sim}$ is finite flat of degree n by definition of the universal family. Rewriting in terms of $F_{\infty,\ell}$,

$$q\frac{d}{dq}F_{\infty,\ell} = \sum_{n\geq 1} q^n \int_{[\mathcal{Y}_n]^{\text{vir}}} \psi_0^\ell.$$

Using the topological recursion (see lemma below)

$$q\frac{d}{dq}F_{\infty,\ell} = F_{\infty,\ell-1}q\frac{d}{dq}F_{\infty,0},$$

we check that the unique solution is

$$F_{\infty,\ell} = \frac{F_{\infty,0}^{\ell+1}}{(\ell+1)!}.$$

Hence plugging back in we get $W_\infty = \exp(\frac{1}{s_1}F_\infty)$. □

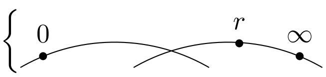
Lemma 2.18 (Topological recursion)**.** *We have*

$$q\frac{d}{dq}F_{\infty,\ell} = F_{\infty,\ell-1}q\frac{d}{dq}F_{\infty,0},$$

which in terms of integrals is

$$\int_{[\mathcal{Y}_n]^{vir}} \psi_0^\ell = \sum_{n_1+n_2=n} \int_{[I_{n_1}^{\sim}]^{vir}} \psi_0^{\ell-1} \int_{[\mathcal{Y}_{n_2}]^{vir}} 1.$$

Proof sketch. A generic point of $\mathcal{R}$ looks like a whole bunch of points on one component $\mathbb{P}^1 \times \mathbb{A}^2$, along with one more point r. We can rigidify with respect to the $\mathbb{G}_m$ action by setting $r = 1$. Explicitly, the point r lets us write a section of $\mathbb{L}_0$, by picking a coordinate z such that $z(0) = 0$, $z(\infty) = \infty$, and $z(r) = 1$, and then the section of $\mathbb{L}_0$ we get is $dz|_0$. As $r \to \infty$, we see that $dz \to 0$. Hence zeros of the section are given by bubbled components with $0, r, \infty$, i.e. the divisor in the moduli space

$D_{n_1,n_2} \coloneqq \Big\{$  $+ \text{ degenerations} \Big\}.$

Then $\psi_0 = c_1(\mathbb{L}_0) = \sum_{n_1+n_2=n} D_{n_1,n_2}$. Note that

$$D_{n_1,n_2} \cong I_{n_1}^{\sim} \times R_{n_2}$$

because the point r is on the second component. Hence (??)

$$[\mathcal{Y}_n]^{\text{vir}}|_{D_{n_1,n_2}} = [I_{n_1}^{\sim}]^{\text{vir}} \times [\mathcal{Y}_{n_2}]^{\text{vir}}.$$

Putting this all together and integrating,

$$\begin{aligned}\int_{[\mathcal{Y}_n]^{\text{vir}}} \psi_0^\ell &= \int_{[\mathcal{Y}_n]^{\text{vir}}} \psi_0 \cdot \psi_0^{\ell-1} \\ &= \sum_{n_1+n_2=n} \int_{[\mathcal{Y}_n]^{\text{vir}}|_{D_{n_1,n_2}}} \psi_0^{\ell-1} \\ &= \sum_{n_1+n_2=n} \int_{[I_{n_1}^{\sim}]^{\text{vir}}} \psi_0^{\ell-1} \int_{[\mathcal{Y}_{n_2}]^{\text{vir}}} 1.\end{aligned}$$

The last equality comes from the (??) equality of virtual classes. □

Let's return to $Z^T_{\mathrm{DT}}(X/D,q) = W_0 W_\infty$. Then

$$\log W_0 = \log Z - \log W\infty = \log Z - \frac{1}{s_1}F_\infty,$$

where F_∞ depends only on s_2, s_3. Hence all q-coefficient of $\log W_0$ is of the form

$$\frac{1}{s_1}\frac{p_1(s_1,s_2,s_3)}{p_2(s_2,s_3)}, \quad \deg p_1 = \deg p_2 + 1.$$

By symmetry of W_0 in s_1, s_2, s_3, it follows that all q-coefficients of $\log W_0$ are of the form

$$\frac{p(s_1,s_2,s_3)}{s_1 s_2 s_3}, \quad \deg p = 3.$$

Lemma 2.19 (Combinatorial lemma)**.** *(s_1+s_2) divides $p(s_1,s_2,s_3)$.*

Then by symmetry, (s_1+s_3) and (s_2+s_3) also divide p. It follows that the q-coefficients of $\log W_0$ are of the form

$$\text{constant} \cdot \frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}.$$

So for some power series $F_0(q)$,

$$\log W_0 = \frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3} F_0(q).$$

But recall that $\log Z_{\mathrm{DT}}$ has poles in s_2, s_3 only. Hence $\log W_\infty$ must be the s_1-pole part of $\log W_0$, which we can just compute to be

$$\log W_\infty = \frac{s_2+s_3}{s_1} F_0(q).$$

It remains to compute $F_0(q)$. We know $\log W_0|_{s_1+s_2+s_3=0} = \log M(-q)$, and we can plug our expression for $\log W_0$ into here to get

$$F_0(q) = -\log M(-q).$$

Hence we get explicit expressions for W_0 and W_∞.

Corollary 2.20. *$Z_{DT}(X,q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)}$.*

Proof. Taking logs,

$$\log Z_{\mathrm{DT}}(X,q)_0 = \sum_{\alpha\text{-fixed}} \frac{(s_1^\alpha + s_2^\alpha)(s_1^\alpha + s_3^\alpha)(s_2^\alpha + s_3^\alpha)}{(-s_1^\alpha)(-s_2^\alpha)(-s_3^\alpha)} \log M(-q).$$

This prefactor is exactly the localization contribution from $\int_X c_3(T_X \otimes K_X)$. □

Corollary 2.21. *$Z_{DT}(X/D,q)_0 = M(-q)^{\int_X c_3(T_X(-\log D) \otimes K_X(-\log D))}$.*

Proof. The prefactors are a little different for points in W_∞. □

2.6 Oct 17 (Anton): DT local curves

Let C be a non-singular projective curve. Let $N \to C$ be a rank 2 bundle; let N also denote the total space, which is a 3-fold. To do relative theory, we need to pick a divisor. Pick points $p_1, \ldots, p_n \in C$, and let our divisor $S = \bigcup N_{p_i}$ be the union of fibers over these points.

Theorem 2.22 (Main result)**.** *GW/DT correspondence holds for local curves.*

Consider $I_n^T(N,d)$. If N is indecomposable, then there is only a 1-dimensional torus acting on N. But every indecomposable bundle is deformation equivalent to a split bundle, and there is a 2-dimensional torus acting on $N = L_1 \oplus L_2$. The relative space $I_n(N/S,d)$ has maps

$$\epsilon_i \colon I_n(N/S,d) \to \mathrm{Hilb}(N_{p_i}, d).$$

The cohomology of $\mathrm{Hilb}(N_{p_i}, d)$ has the Nakajima basis, labeled by partitions.

Definition 2.23. Define the partition functions

$$Z(N/S)_{d,\eta^1,\dots,\eta^r} \coloneqq \sum_{n\in\mathbb{Z}} q^n \int \frac{\epsilon_i^*(C_{\eta^i})}{e(N^{\mathrm{vir}})}.$$

Let Z' denote reduced invariants. The notation will be

$$Z(g;k_1,k_2)_{\eta^1,\dots,\eta^r}$$

for genus g curve C and line bundles of degree k_1 and k_2. To abbreviate gluing terms, introduce new functions

$$DT(g;k_1,k_2)_{\eta^1,\dots,\eta^r} \coloneqq q^{-d(1-g)} Z(g;k_1,k_2)_{\eta^1,\dots,\eta^r}.$$

To raise indices, use

$$DT(g;k_1,k_2)^{\nu^1,\dots,\nu^s}_{\eta^1,\dots,\eta^r} \coloneqq DT(g;k_1,k_2)_{\nu^1,\dots,\nu^s,\eta^1,\dots,\eta^r} \prod \Delta_d(\nu^i,\nu^j)$$

where Δ is the inverse of the intersection form

$$\int C_\mu \cup C_\nu = (t_1t_2)^{-\ell(\mu)} \frac{(-1)^{d-\ell(\mu)}}{\mathfrak{z}(\mu)} \delta_{\mu,\nu}.$$

Remark. The gluing matrix is diagonal in cohomology, but is more complicated in K-theory.

Proposition 2.24 (Degeneration formulas)**.** *For $g = g' + g''$ and $k_i = k_i' + k_i''$,*

$$DT(g;k_1,k_2)^{\nu^1,\dots,\nu^t}_{\mu^1,\dots,\mu^s} = \sum_\gamma DT(g';k_1',k_2')^\gamma_{\mu^1,\dots,\mu^s} DT(g'';k_1'',k_2'')^{\nu^1,\dots,\nu^s}_\gamma$$
$$DT(g;k_1,k_2)_{\mu^1,\dots,\mu^s} = DT(g-1;k_1,k_2)^\gamma_{\mu^1,\dots,\mu^s,\gamma}.$$

Using the degeneration formula, it suffices to compute

$$DT(0;0,0)_\lambda, \quad DT(0;0,0)_{\lambda\mu}, \quad DT(0;0,0)_{\lambda\mu\nu}, \quad DT(0;0,-1)_\lambda.$$

Lemma 2.25 ($(0,0)$ tube)**.** $DT(0;0,0)^\lambda_\mu = \delta^\lambda_\mu$.

Proof. First step: show this is true modulo q. These q-constant terms just come from the intersection form. Second step: apply the degeneration formula to the tube itself to get

$$DT(0;0,0) = DT(0;0,0)^2.$$

Since the q-constant terms are invertible, $DT(0;0,0)$ is invertible, and it follows that $DT(0,0,0) = \mathrm{id}$. ☐

Lemma 2.26 ($(0,0)$ cap)**.**

$$DT(0;0,0)_\lambda = \begin{cases} \frac{1}{d!(t_1t_2)^d} & \lambda = (1^d) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Idea: look at the compactified $\mathbb{P}^1 \times \mathbb{P}^2$ geometry, and do a dimension count. Then use localization, in which one of the pieces will be the cap. □

Corollary 2.27. *We can always add or remove (1^d) insertions.*

Proof. $DT_{\mu\nu} = \sum_\gamma DT^\gamma_{\mu\nu} DT_\gamma$. But the cap is non-zero only for $\gamma = (1^d)$. Hence

$$DT^{(1^d)}_{\mu\nu} \propto DT_{\mu\nu}.$$

□

Now we have to compute $DT_{\lambda\mu\nu}$. A standard reconstruction theorem (which we will see for GW) shows $DT_{\lambda\mu\nu}$ can be reconstructed from $DT_{\lambda,(2),\nu}$. To compute this, we need descendant insertions $\sigma_k(\gamma)$.

Definition 2.28. Introduce bracket notation

$$\langle \sigma_{k_1}(\gamma_1) \cdots \sigma_{k_s}(\gamma_s) \rangle \coloneqq \int \frac{\prod \mathrm{ch}_{k_i+2}(\gamma_i)}{e(\ldots)}$$

We write things like

$$\langle \sigma_{k_1}(\gamma_1) \cdots \sigma_{k_s}(\gamma_s) | \nu^1, \ldots, \nu^s \rangle^N_{n,d}, \quad \langle \mu | \sigma_{k_1}(\gamma_1) \cdots \sigma_{k_s}(\gamma_s) | \nu \rangle^N_{n,d}$$

for relative conditions. If we omit the N, it means we take level $(0,0)$ theory, i.e. $\mathbb{P}^1 \times \mathbb{A}^2$. If we omit the n, we sum over all n with $\sum_n q^n$.

Define an operator M_σ by $\langle \mu | M_\sigma | \nu \rangle = q^{-d} \langle \mu | - \sigma_1(F) | \nu \rangle$ where $F = [N_z]$ is the fiber over $z \in \mathbb{P}^1$. The thing we want to compute is closely related to M_σ. This is because of the degeneration formula

$$\begin{aligned} \langle \mu | - \sigma_1(F) | \nu \rangle &= \sum_\gamma DT(0;0,0)^\gamma_{\mu\nu} q^{-d} \langle \gamma | - \sigma_1(F) \rangle \\ &= DT(0;0,0)^{(1^d)}_{\mu\nu} q^{-d} \langle (1^d) | - \sigma_1(F) \rangle + DT(0;0,0)^{(2)}_{\mu\nu} q^{-d} \langle (2) | - \sigma_1(F) \rangle. \end{aligned}$$

Here we use that $\langle \gamma | - \sigma_1(F) \rangle = 0$ unless $\gamma = (1^d)$ or (2).

1. (First term) We can just remove (1^d) to get the tube and use $\langle (1^d) | - \sigma_1(F) \rangle = \langle (1^d) | - \sigma_1(F) | (1^d) \rangle$ to relate it back to the operator M_σ.
2. (Second term) This involves the term $DT(0;0,0)^{(2)}_{\mu\nu}$ which we want to compute, and $\langle (2) | - \sigma_1(F) \rangle = \langle (2) | - \sigma_1(F) | (1^d) \rangle$.

It follows that once we figure out M_σ, we know $DT(0;0,0)^{(2)}_{\mu\nu}$. It turns out that if we write

$$M \coloneqq (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1} \alpha_{-k} \alpha_k + \frac{1}{2} \sum_{k,l>0} (t_1 t_2 \alpha_{k+l} \alpha_{-k} \alpha_{-l} - \alpha_{-k-l} \alpha_k \alpha_l)$$

then we have

$$M_\sigma = M - (t_1 + t_2) \varphi(q) \,\mathrm{id}, \quad \varphi(q) \coloneqq q \frac{d}{dq} \log M(-q).$$

The proof that this is the correct expression for M_σ is quite involved.

Definition 2.29. **Fock space** is generated by a vacuum vector $v_\emptyset$ by the free action of **creation** and **annihilation** operators α_{-k} and α_k for $k > 0$. A natural basis is given by

$$|\mu\rangle \coloneqq \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i} v_\emptyset.$$

The defining relations for the creation/annihilation operators are

$$[\alpha_k, \alpha_l] = k \delta_{k+l}$$

and annihilation operators kill the vacuum, i.e. $\alpha_k v_\emptyset = 0$ for $k > 0$.

In this basis for Fock space, the first term of M is diagonal, and is the only place where we have a q-dependence. The second term is off-diagonal, coming from either killing two rows and adding one or adding two rows and killing one.

Lemma 2.30 ((0, −1) cap)**.**

$$DT(0;-1,0)_\lambda = \frac{1}{\mathfrak{z}(\sigma)(t_1 t_2)^{\ell(\lambda)}} \prod \frac{1}{1-(-q)^{\lambda_i}}.$$

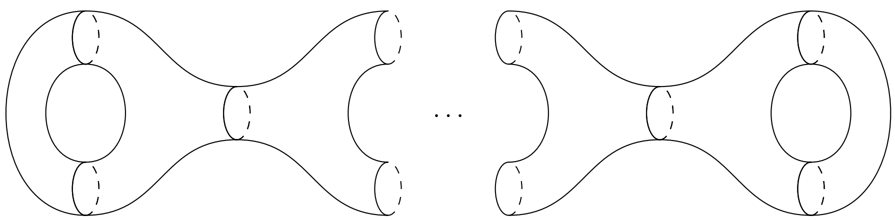
Proof. See lemma 27. □

2.7 Oct 24 (Shuai): Local curve computations

The setup is as usual: a curve C with rank-2 split vector bundle on it. We have:

1. the absolute theory $DT(g;k_1,k_2)$, but we will write the geometry explicitly, like $DT(\mathbb{P}^1 \times \mathbb{C}^2)$;
2. the relative theory $DT(g;k_1,k_2)_S$, e.g. $DT(\mathbb{P}^1 \times \mathbb{C}^2)_{0,\emptyset,\emptyset,\ldots}$.

We already know that in the associated TQFT, we only need to compute the genus-0 tube, caps, and pair of pants, because of pair of pants decompositions like



We need the DT degeneration formulas for this:

$$DT(g;k_1,k_2) = \sum_\lambda DT(g_1;k_1',k_2')_\lambda DT(g_2;k_1'',k_2'')^\lambda$$
$$DT(g;k_1,k_2)_\mu = \sum_\lambda DT(g_1;k_1',k_2')_{\mu\lambda} DT(g_2;k_1'',k_2'')^\lambda.$$

Here, to raise indices, we have

$$DT(g;k_1,k_2)^\lambda \coloneqq DT(g;k_1,k_2)_\lambda \Delta_d(\lambda,\lambda)$$

where $\Delta_d(\mu,\nu) \coloneqq \delta_{\mu,\nu}(t_1t_2)^{-\ell(\mu)}(-1)^{d-\ell(\mu)}/\mathfrak{z}(\mu)$ is the inverse of the intersection product on $\mathrm{Hilb}(\mathbb{C}^2)$.

First, how do we compute the Euler characteristic of the sheaf associated to a configuration of boxes, e.g. two 3d partitions π, π' at $0, \infty$ and a 2d partition λ along the infinite leg? Recall that in Clara's talks, we saw the *normalized volume* $|\cdot|$ of a (possibly-infinite) 3d partition. Euler characteristic is motivic, so for the $\mathbb{P}^1 \times \mathbb{C}^2$ geometry,

$$\chi = \chi(\lambda \times \mathbb{P}^1) + \chi(|\pi| + |\pi'|) = |\lambda| + |\pi| + |\pi'|.$$

In general, for a genus g curve, we will get $|\lambda|(1-g)$ instead of $|\lambda|$. This explains all the appearances of $d(1-g)$ in the paper.

The key takeaway is that the smallest Euler characteristic we can get is $d(1-g)$. In the definition of DT partition function, we shifted by $q^{-d(1-g)}$ to make the minimal case the q-constant term.

Remark (Classical contribution). We have an isomorphism

$$I_d(\mathbb{P}^1 \times \mathbb{C}^2, d) \cong \mathrm{Hilb}^d(\mathbb{C}^2).$$

Because of this isomorphism, we know the inner product on the DT TQFT corresponds to exactly the intersection pairing Δ on $\mathrm{Hilb}^d(\mathbb{C}^2)$. This is why we use Δ to raise/lower indices. Consequently, this makes the tube into the identity, as desired.

Proposition 2.31. *The level* $(0,0)$ *cap is computed by*

$$\begin{aligned} DT(0;0,0)_\lambda &= \frac{q^{-d}}{(t_1t_2)^{\ell(\lambda)}} \frac{\langle|\lambda[0]\rangle}{\langle|\emptyset\rangle} \\ &= \frac{q^{-d}}{(t_1t_2)^{\ell(\lambda)}} \frac{1}{d!} \left(\frac{\langle|\lambda[0]\rangle}{\langle|\emptyset\rangle}\right)^d \\ &= \frac{q^{-d}}{(t_1t_2)^{\ell(\lambda)}} \frac{1}{d!} q^d. \end{aligned}$$

where the second (and following) equality is non-zero only for $\lambda = (1^d)$.

Proof. The first equality comes from shifting the whole Nakajima basis to the one supported at the point $[0] = t_1t_2 \in H_T^4$. Then by linearity,

$$\frac{1}{(t_1t_2)^{\ell(\lambda)}} \langle|\lambda[0]\rangle = \langle|\rangle.$$

The rest of the expression is the definition of DT partition function.

To show that only $\lambda = (1^d)$ contributes, we do a dimension count. Recall that

$$\operatorname{vdim} I_n(\mathbb{P}^1 \times \mathbb{C}^2/\mathbb{C}^2_\infty, d) = 2d.$$

The dimension of the cycle defined by λ is $|\lambda| + \ell(\lambda)$, i.e. we need $|\lambda| + \ell(\lambda) = 2d$ for non-zero contribution. Hence $\ell(\lambda) = d$ and $\lambda = (1^d)$.

For this special partition, we have a factorization as follows. Compactify $\mathbb{P}^1 \times \mathbb{C}^2$ to get $\mathbb{P}^1 \times \mathbb{P}^2$. Idea: the absolute theory on $\mathbb{P}^1 \times \mathbb{P}^2$ can be computed in two different ways, to give the factorization identity in the third equality. Put a torus action t_1, t_2 on fibers $\mathbb{P}^2$ and s on $\mathbb{P}^1$ at 0. How do we specify the relative condition $\lambda = (1^d)$ on the additional fixed points A, B over the fiber at $\infty \in \mathbb{P}^1$? We get

$$\langle|\underbrace{1[0], \dots, 1[0]}_{d \text{ copies}}\rangle = \langle|\lambda\rangle\langle|\emptyset\rangle|_{-t_1, t_2-t_1}\langle|\emptyset\rangle_{t_1-t_2, -t_2}$$

where we put the $\emptyset$ relative condition at the extra points A, B at ∞ because we don't want our original curve to hit the infinity divisor in the fibers. ☐

2.8 Oct 31 (Shuai): Pair of pants

First goal: reduce everything to the quantum multiplication by the divisor $c_1(\mathcal{O}/I) = -(2, 1^{d-2})$. Define three operators.

1. Let M be the explicit operator

$$M \coloneqq (t_1+t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1} \alpha_{-k}\alpha_k + \frac{1}{2} \sum_{k,l>0} (t_1t_2\alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l)$$

   and then define $M_\star \coloneqq M - (t_1+t_2)\Phi(q)$ where $\Phi(q) \coloneqq (d/dq)\log Q$ where Q is the generating function for 3d partitions.

2. Define the operator M_σ by

$$\langle\mu|M_\sigma|\nu\rangle \coloneqq \langle\mu| - \sigma_1(F)|\nu\rangle.$$

3. Define the operator M_D by

$$\langle\mu|M_D|\nu\rangle \coloneqq DT(0|0,0)_{\lambda,D,\nu}.$$

We want to compare these three operators. The strategy is to argue we can focus on only a few special matrix elements.

1. (Only need terms close to the diagonal) Prove a vanishing theorem

$$|\ell(\mu) - \ell(\nu)| > 1 \implies \langle \mu| - \sigma_1(F)|\nu\rangle = 0.$$

2. (Off-diagonal terms are rational numbers) Prove another vanishing theorem

$$|\ell(\mu) - \ell(\nu)| = 1 \implies \langle \mu| - \sigma_1(F)|\nu\rangle_{n,d} = 0, \quad \forall n > d,$$

   i.e. the invariants are really just *rational* numbers.

3. (Only need certain on-diagonal terms) Prove the additivity property

$$\frac{\langle \mu|M_\sigma|\mu\rangle}{\langle \mu|\mu\rangle} = \sum q^{|\mu|-\mu_i} \frac{\langle \mu_i|M_\sigma|\mu_i\rangle}{\langle \mu_i|\mu_i\rangle}.$$

The next step is to compute these special matrix elements of M_σ.

5. To compute $M_\star$ we can use any basis we want, but for the computation of M_σ we would like to work in the *fixed point basis* J^μ instead of the Nakajima basis $|\mu\rangle$. In this basis, we have relations

$$\langle J^{(d)}|M_\sigma|J^{(d-1,1)}\rangle_n = (n-d)\langle J^{(d)}, J^{(d-1,1)}\rangle_n^\sim = (n-d)\langle J^{(d)} J^{(d-1,1)}\rangle_{n-d}^{\mathrm{Hilb}(\mathbb{C}^2)}.$$

6. Compute the low-degree term

$$\langle \mu|M_\sigma|\mu\rangle_n = (t_1+t_2)\gamma_{\mu,n}/t_1^2 \bmod (t_1+t_2)^2.$$

   The computation $\langle J^{(d)}|M_\sigma|J^{(d-1,1)}\rangle$ contains a contribution $\langle d|M_\sigma|d\rangle$. By matching the low-degree terms, it therefore suffices to show $\langle J^{(d)}|M_\sigma|J^{(d-1,1)}\rangle$ matches with $\langle J^{(d)}|M_\star|J^{(d-1,1)}\rangle$.

Proposition 2.32. $M_\sigma = M_\star$.

Proof sketch. Check that the $q=0$, $\langle\emptyset|M_\sigma|\emptyset\rangle$ and $\langle 1|M_\sigma|1\rangle$ terms match. We can explicitly compute the $M_\star$ matrix elements and show they match the following computations.

1. ($q=0$ term) This means $n=d$, i.e. our moduli space is $I_d(\mathbb{P}^1 \times \mathbb{C}^2, d) = \mathrm{Hilb}(\mathbb{C}^2, d)$. Then by a computation via an explicit resolution of I,

$$M_\sigma(q=0) \coloneqq \pi_{1*}(\mathrm{ch}_3(I)\pi_1^*(1)\pi_2^*([N_0])) = D - \frac{t_1+t_2}{2} d\,\mathrm{id}\,.$$

   This is the classical part.

2. ($\langle\emptyset|M_\sigma|\emptyset\rangle$) Degenerate to get something like

$$\langle\emptyset| - \sigma_1(F)|\emptyset\rangle = \frac{\langle\sigma_1(F)\rangle_0}{\langle\emptyset|\rangle\langle|\emptyset\rangle}.$$

   We know how to compute the top by moving the fiber class to 0 or ∞, and then using the equivariant vertex measure $W(\emptyset,\emptyset,\emptyset)$ at some specialization of weights, from MNOP2. The denominator are the usual degree-0 terms.

3. ($\langle 1|M_\sigma|1\rangle$) Degenerate again to get a similar formula. In the numerator we therefore need to compute a special case of the 1-legged vertex $W(1,\emptyset,\emptyset)$ at some specialization of weights. Then we get

$$-\frac{t_1+t_2}{2}\frac{1-q}{1+q} - (t_1+t_2)\Phi(q)$$

   as expected.

Once we show the following computation, we will be done, because we have checked the equality for $q = 0$. □

Proposition 2.33. $\langle J^d | M_\star - M_\star(q=0) | J^{(d-1,1)} \rangle = \langle J^d | M_\sigma - M_\sigma(q=0) | J^{(d-1,1)} \rangle$.

Proof. First show using some representation theory that

$$J^\lambda \equiv \frac{(-1)^{|\lambda|}|\lambda|!}{\dim \lambda} \sum_\mu \chi_\mu^\lambda t_1^{|\lambda|+\ell(\mu)} \, |\mu\rangle \bmod t_1 + t_2.$$

In the two cases (d) and $(d-1,1)$, this becomes very simple.

1. χ^d is the trivial representation.
2. χ^{d-1} is the fundamental representation $\{x \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}$.

Since the intersection pairing on $\mathrm{Hilb}(\mathbb{C}^2, d)$ is diagonal, we get

$$\langle J^{(d)} | M_\star - M_\star(0) | J^{(d-1,1)} \rangle \equiv (-1)^n (t_1 + t_2) \frac{t_1^{2n}(d!)^2}{d-1} \langle \chi^{(d-1,1)}, F \rangle_{L^2(S_n)} \bmod t_1 + t_2$$

where

$$F := -|\mu| \frac{q}{1-q} - \sum_{i=1}^{\ell(\mu)} (\mu_i)^2 \frac{(-q)^{\mu_i}}{1-(-q)^{\mu_i}}.$$

To be continued... □

2.9 Nov 14 (Yakov): DT theory of $\mathcal{A}_n$

The $\mathcal{A}_n$ surface is a minimal resolution of singularities for $\mathbb{C}^2/\mathbb{Z}_{n+1}$ with action $(z_1, z_2) \mapsto (\xi z_1, \xi^{-1} z_2)$ where ξ is a primitive $(n+1)$-th root of unity.

Example 2.34. $\mathcal{A}_1 = T^*\mathbb{P}^1$, because we have

$$\mathbb{C}^2/\mathbb{Z}_2 = \operatorname{Spec} \mathbb{C}[x^2, xy, y^2] = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2),$$

which is a quadric cone. Blowing up, we get $T^*\mathbb{P}^1$: the exceptional fiber is a $\mathbb{P}^1$, with normal bundle $\mathcal{O}(-2)$.

Let $X := \mathcal{A}_n \times \mathbb{P}^1$. Pick $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$ and form the moduli $I_\chi(X, (\beta, m))$ of ideal sheaves with

$$c_2(\mathcal{O}_Z) = (\beta, m) \in H_2(X, \mathbb{Z}) = H_2(\mathcal{A}_n, \mathbb{Z}) \oplus \mathbb{Z}.$$

Choose points $z_1, \ldots, z_k \in \mathbb{P}^1$, and consider the relative theory $I_\chi(X/S, (\beta, m))$ with respect to $S := \bigcup_i \mathcal{A}_n \times z_i$. Relative conditions are given by cohomology weighted partitions

$$\vec{\mu} := \{((\mu^{(1)}, \gamma_1), \ldots, (\mu^{(\ell)}\gamma_\ell)\}, \quad \gamma_i \in H_T^*(\mathcal{A}_n, \mathbb{C}).$$

The fibers give maps $\epsilon_i \colon I_\chi(X/S, (\beta, m)) \to \mathrm{Hilb}(\mathcal{A}_n)$. Define the DT partition function

$$Z_{\mathrm{DT}}(X)_{(\beta,m),\vec{\mu}} := \sum_\chi q^\chi \int_{[I_\chi(X/S,(\beta,m))]^{\mathrm{vir}}} \prod \epsilon_i^*(\vec{\mu}_i)$$

by residues, because the moduli is non-compact. If we omit (β, m), we take a generating series

$$Z_{\mathrm{DT}}(X)_{\vec{\mu}} \in \mathbb{C}(t_1, t_2)((q))[[s_1, \ldots, s_n]]$$

over β as well, with variable s^β. The degree m can be reconstructed from $\vec{\mu}$. The reduced partition function is

$$Z'_{\mathrm{DT}}(X)_{\vec{\mu}} := \frac{Z_{\mathrm{DT}}(X)_{\vec{\mu}}}{Z_{\mathrm{DT}}(X)_{(0,0),\vec{\emptyset}}}.$$

From MNOP2 we know $Z_{\mathrm{DT}}(X)_{(0,0),\vec{\emptyset}} = M(-q)$, the MacMahon function.

Example 2.35 (Geometry of $\mathcal{A}_n$)**.** The toric diagram for $\mathcal{A}_2$ is

Note that at any fixed point, $\wedge^2 T_p = t_1 + t_2$. This is important for us. Also, each exceptional divisor E_i is a (-2)-curve, and the collection $\{E_1, \dots, E_n\}$ span $H_2(\mathcal{A}_n, \mathbb{C})$. Their duals also span cohomology, because $\dim_{\mathbb{C}} H^*(\mathcal{A}_n, \mathbb{C})$ is the number of fixed points. The intersection pairing is

$$\langle E_i, E_{i+1}\rangle = 1, \quad \langle E_i, E_i\rangle = -2.$$

Hence $H_2(\mathcal{A}_n, \mathbb{Z})$ is the A_n root system, where $E_i \mapsto \alpha_{i,i+1}$, with effective classes $\alpha_{ij} \coloneqq E_i + \cdots + E_{j-1}$. Let ω_i be the dual basis, i.e. $\langle \omega_i, E_j\rangle = \delta_{ij}$.

Consider the rubber geometry $I_\chi(X, (\beta, m))^\sim$. These are ideal sheaves on $\mathcal{A}_n \times \mathbb{P}^1$ relative to $\mathcal{A}_n \times 0$ and $\mathcal{A}_n \times \infty$, up to a $\mathbb{C}^*$ scaling on $\mathbb{P}^1$. It has a T-equivariant perfect obstruction theory of dimension $2n - 1$. Define rubber invariants

$$\langle \mu | \nu \rangle^\sim_{\beta,\chi} \coloneqq \int_{[I_\chi(X,(\beta,m))^{T,\sim}]^{\mathrm{vir}}} \frac{\epsilon_0^*(\mu)\epsilon_\infty^*(\nu)}{e(N^{\mathrm{vir}})}, \quad = |\mu| = |\nu|.$$

Put these into generating series $\langle \mu|\nu\rangle^\sim_\beta \coloneqq \sum_\chi q^\chi \langle \mu|\nu\rangle^\sim_{\beta,\chi}$. If $\beta = 0$, these follow from the local curves case. For $\beta \neq 0$, define

$$\langle \mu|\nu\rangle^\sim_+ \coloneqq \sum_{\beta\neq 0} q^\chi s^\beta \langle \mu|\nu\rangle^\sim_{\beta,\chi} \in \mathbb{C}(t_1, t_2)((q))[[s_1, \dots, s_n]].$$

Let $F_{\mathcal{A}_n} \coloneqq \bigoplus_{m\geq 0} H_T^*(\mathrm{Hilb}_m(\mathbb{A}_n), \mathbb{C})$. We know from Nakajima that this is an irreducible representation of the Heisenberg algebra, generated by $p_k(\gamma)$ where $\gamma \in H_T^*(\mathcal{A}_n, \mathbb{C})$, with commutation relation

$$[p_k(\gamma_1), p_\ell(\gamma_2)] = -k\delta_{k+\ell}\langle \gamma_1, \gamma_2\rangle \cdot c$$

where c is a central element. There are two bases for cohomology:

1. Nakajima basis, given by cohomology weighted multi-partitions;
2. T-fixed points $J_{\vec\rho} \in \mathrm{Hilb}(\mathcal{A}_n)$, indexed using that the isolated fixed points are in charts isomorphic to $\mathbb{A}^2$, and hence
$$I_\rho = (x^{\rho_1}, yx^{\rho_2}, \dots, y^{\ell-1}x^{\rho_\ell}), \quad \ell \coloneqq \ell(\vec\rho).$$

We have an intersection pairing $\langle \vec\mu|\vec\nu\rangle$. Let $\hat{\mathfrak{g}} \coloneqq \hat{\mathfrak{g}}(n+1)$, with elements $x(k) \coloneqq xt^k$ and commutators

$$[x(k), y(l)] = [x, y](k + l) + k\delta_{k+l,0}\,\mathrm{tr}(xy) \cdot c,$$

where c is central and $[d, x(k)] = kx(k)$. Let $\hat{\mathfrak{h}} \coloneqq \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra. Embed $\mathcal{H} \hookrightarrow \hat{\mathfrak{g}} \otimes \mathbb{C}(t_1, t_2)$, called the basic representation, given by

$$p_{-k}(1) \mapsto \mathrm{id}(-k), \quad p_k(1) \mapsto \frac{-\,\mathrm{id}(k)}{(n+1)^2 t_1 t_2}, \quad p_k(E_i) \mapsto e_{ii}(k) - e_{i+1,i+1}(k).$$

The roots of $\hat{\mathfrak{gl}}(n+1)$ are

$$\Delta = \{k\delta + \alpha_{ij} : k \in \mathbb{Z}\} \cup \{k\delta : k \neq 0\}.$$

Given a weight Λ, there exists a highest weight representation V_λ such that $\rho(\mathfrak{gl}(n+1) \otimes \mathbb{C}[t_1, t_2])v = 0$ and $\rho(c)v = v$ for the highest weight vector $v \in V_\lambda$. Then we can consider the weight subspace $W \coloneqq \bigoplus_{m\geq 0} V_\lambda[\Lambda - m\delta]$. Since $W \otimes \mathbb{C}(t_1, t_2) \cong F_{\mathcal{A}_n}$, we can construct operators on $F_{\mathcal{A}_n}$ using $U(\hat{\mathfrak{g}})^{\hat{\mathfrak{h}}}$. For example, take $e_{ij}(k)e_{ji}(-k)$.

Definition 2.36. Define the operator formula

$$\langle\vec{\mu}|\Theta^{\mathrm{DT}}|\vec{\nu}\rangle = q^{-m}\langle\vec{\mu}|\vec{\nu}\rangle_{+}^{\mathrm{DT},\sim},$$

and let

$$\Omega_+ \coloneqq \sum_{i<j}\sum_{k\in\mathbb{Z}} {:}e_{ji}(k)e_{ij}(-k){:}\ \log(1-(-q)^k s_i\cdots s_{j-1}).$$

Here the normal ordering means

$$:e_{ji}(k)e_{ij}(-k): \coloneqq \begin{cases} e_{ji}(k)e_{ij}(-k) & k<0 \text{ or } k=0,\ i<j \\ e_{ij}(-k)e_{ji}(k) & \text{otherwise.}\end{cases}$$

Proposition 2.37.

$$\Theta^{DT}(q,s_1,\dots,s_m) = (t_1+t_2)(\Omega_+(q,s_1,\dots,s_n) + \sum_{i<j} F(q,s_i,\dots,s_j-1)\,\mathrm{id})$$

where $F(q,s_i) \coloneqq \sum_{k\geq 0}(k+1)\log(1-(-q)^{k+1}s)$.

To prove this, we do some geometry. Let $\mathcal{J}\to I_\chi(X,(\beta,m))\times X$ be the universal sheaf. Let $\gamma\in H_T^*(X,\mathbb{C})$ and define the insertions

$$\mathrm{ch}_{k+2}(\gamma)(\xi) \coloneqq \pi_{1*}(\mathrm{ch}_{k+2}(\mathcal{J})\pi_2^*(\gamma)\cap\pi_1^*(\xi))$$

as homology operations $H_*^T(I_n,\mathbb{C})\to H^T_{*-2k+2-\ell}(I_n,\mathbb{C})$.

Proposition 2.38. *For $\beta\neq 0$, the descendants $\langle\sigma_{\ell_1}(\gamma_{\ell_1})\cdots\sigma_{\ell_k}(\gamma_{\ell_k})|\nu_1,\dots,\nu_b\rangle_{\beta,\chi}$ have positive valuation with respect to (t_1+t_2). For $\beta=0$, they vanish mod t_1+t_2 for $\chi>m$.*

Now let's talk about rigidification. Let $\delta_0 \coloneqq i_*(\omega_1+\cdots+\omega_n)\in H_T^*(\mathcal{A}_n\times\mathbb{P}^1,\mathbb{C})$ where $icolon\mathcal{A}_n\to X$ is the inclusion.

Lemma 2.39. *For $\omega\in H_T^2(\mathcal{A}_n,\mathbb{C})$ a divisor,*

$$\langle\vec{\mu}|\sigma_0(i_*\omega)|\vec{\nu}\rangle = (\beta,\omega)\langle\vec{\mu}|\vec{\nu}\rangle_\beta^\sim.$$

In particular, for $\omega=\delta_0$, we get

$$\langle\vec{\mu}|\sigma_0(\delta_0)|\vec{\nu}\rangle = \sum s_k\partial_{s_k}\langle\vec{\mu}|\vec{\nu}\rangle^\sim.$$

Proof. Let $\pi\colon R\to I_\chi(X/S,(\beta,m))^\sim$ be the universal target, and take the universal sheaf $\mathcal{J}\to R$. Then R is the set of pairs of an ideal sheaf and a point r on the target X disjoint from relative divisors and singular points. By projection, we get a point on $\mathbb{P}^1$. We can rigidify the $\mathbb{C}^*$-action by insisting that $r\mapsto 1$. Then we can exhibit R as a substack of the universal target for the rigidified theory $I_\chi(X/S,(\beta,m))$. The virtual class are equal:

$$[R]^{\mathrm{vir}} \coloneqq \pi^*[I_\chi^\sim]^{\mathrm{vir}} = \phi^*[I_\chi]^{\mathrm{vir}}.$$

By projection formula, we therefore get

$$(\omega,\beta)\langle\vec{\mu}|\vec{\nu}\rangle_\beta = \langle\vec{\mu}|\,\mathrm{ch}_2(\mathcal{J})f^*\omega|\vec{\nu}\rangle_\beta^{R,\sim} = \langle\vec{\mu}|\sigma_0(i_*\omega)|\vec{\nu}\rangle_\beta.$$ □

Now introduce a new $\mathbb{C}^*$ acting on $\mathbb{P}^1$. Let ρ be a box configuration for X. Given λ, define

$$\mathrm{rank}(\lambda) \coloneqq \frac{1}{2}\sum_{r\in\mathbb{Z}}(c_r(\lambda)-c_{r+1}(\lambda))$$

where $c_r(\lambda)$ is the number of boxes of content r. Let

$$\mathrm{rank}_{t_3}(\rho) \coloneqq \sum_k \mathrm{rank}(\rho^k/\rho^{k+1}).$$

Lemma 2.40. *The multiplicity of $t_1 + t_2$ in $w(\pi)$ is $\sum \operatorname{rank}_{t_3}(\pi_i)$ over all fixed points π_i.*

Proof. In general, we have terms like

$$\frac{F(z_1, z_2)}{1 - z_3} + \frac{F(z_1/z_3^a, z_2/z_3^b)}{1 - z_3^{-1}}.$$

In the $\mathcal{A}_n$ geometry, we have $a = -2$ and $b = 0$. Plug in $z_3 = 1/z_1$ so that the terms $(z_1 z_3)^k$ become constant terms. Then it suffices to write some combinatorial expression for the constant terms. □

Corollary 2.41. *When $\beta \neq 0$, the descendants $\langle \sigma_{k_1}(\gamma_{\ell_1}) \cdots \sigma_{k_r}(\gamma_{\ell_r}) \rangle_{\beta,m} = 0$ mod $t_1 + t_2$.*

2.10 Nov 28 (Andrei): Toric GW/DT correspondence

Let X be a toric variety. We would like to prove that GW counts are really equal to DT counts in X. The first thing to use is localization: to X corresponds some polyhedron (its toric diagram) consisting of 1-dimensional T-orbits and fixed points. T-fixed curves can have some features along compact edges, and at vertices.

1. (Edges) In DT theory, edges carry (constant) 2d partitions $\lambda(e)$. In GW theory, edges carry multiple covers $z \mapsto z^{\mu^i}$. The size of the partitions is the degree. Note that

$$[C] = \sum_{\text{edges}} |\lambda(e)| [\text{edge}] \in H_2(X, \mathbb{Z}).$$

2. (Vertices) In DT theory, vertices carry 3d partitions with asymptotics λ, μ, ν. In GW theory, vertices involve source curves whose genus-g component is collapsed to a point, and multiple covers of open edges.

To do localization, we need the deformation theory $(\mathrm{Def} - \mathrm{Obs})^{\text{moving}}$ over the T-fixed locus. In general, we split this as

$$(\mathrm{Def} - \mathrm{Obs})^{\text{mov}} = \sum_{\text{edges } e} (\mathrm{Def}(C_e) - \mathrm{Obs}(C_e)) + \sum_{\text{vertices } v} (\text{contributions of } v),$$

in the sense that we *define* the contributions of a vertex to be the difference. To compute this vertex contribution, we can put it into a $(\mathbb{P}^1)^3$ geometry; alternatively, in DT theory, we can even just use $\mathbb{C}^3$, where both terms of the difference are infinite-dimensional but their difference is finite. Then

$$\text{curve counts on } X = \sum_{\text{partitions } \lambda(e)} Q^{[C]} \text{weights of}(C_e) \prod \text{Vertices}(q).$$

The function Vertices(q) will be a sum in DT theory and an integral in GW theory (and a smaller integral in PT theory. In general it is some transcendental function of q. The first step is to break this transcendental function into rational functions, and match those between GW and DT.

Definition 2.42. Take the $\mathbb{C}^2 \times \mathbb{P}^1$ geometry, and look at the curve counts for a relative condition at 0 and a non-singular condition at ∞. Then the evaluation map goes to $\mathrm{Hilb}(\mathbb{C}^2, d) \times \mathrm{Hilb}(\mathbb{C}^2, d)$, which is an operator. This operator Ψ is the **fundamental solution** to the quantum differential equation (QDE)

$$q \frac{d}{dq} \Psi = M(q)\Psi - \Psi M(0).$$

The hard part is the matrix $M(q)$. The essential point is that this matrix is the same for both GW and DT.

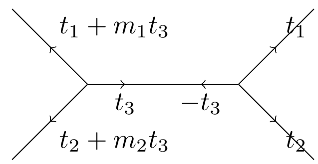
If we take this fundamental solution and insert it into the localization picture, the contributions break up into new pieces, where each bare leg is composed with the operator Ψ. This gives the **capped vertex** $\hat{V}(\lambda, \mu, \nu, t_1, t_2, t_3, q)$ and the **capped edge** $S(m_1, m_2, t_1, t_2, t_3, q)$.

Theorem 2.43. *Both capped vertices and capped edges*

1. *are rational functions of q, and*
2. *satisfy GW/DT.*

Remark. This is good because now we have distributed the complexity across vertices and edges: both now depend on q and are rational, but originally we had one depending on q transcendentally and the other independent of q. Rationality is good especially because eventually we want to do the substitution $q = e^{-iu}$.

Let's do the edge first. Its toric diagram is



By abstract nonsense, it is a shift operator for the QDE, and satisfies

$$\nabla S(t_1, t_2) = S(t_1 + m_1 t_3, t_2 + m_2 t_3)\nabla \quad \nabla := t_3 q \frac{d}{dq} - M(q).$$

Hence S is uniquely determined from its initial condition at $q = 0$. Since the matrix $M(q)$ matches in GW and DT, the operators S are the same as well. So it is enough to prove rationality on either side. There are some very general statements that can be made about rationality in the Kähler variable.

Now let's do vertices. Take $X = \mathcal{A}_2 \times \mathbb{P}^1$. Its toric diagram contains a trivalent vertex, so if we put a relative condition at $0 \in \mathbb{P}^1$ and decompose its contributions, we will get the desired vertex. In general, we want to do curve counts in $S \times \mathbb{P}^1$ relative to $D := S \times \{0\}$, but for us we only need $S = \mathbb{C}, \mathcal{A}_1, \mathcal{A}_2$. In particular, in all these cases, $c_1(S) = 0$. Fix a curve class (β, d), so that evaluation map goes to $\operatorname{Hilb}(S, d)$. Then we have the following non-trivial result.

Theorem 2.44.

$$\operatorname{ev}_*(\sum_{\beta,\chi} Q^\beta q^\chi [-]^{vir}) = q^d [\operatorname{Hilb}(S, d)].$$

Proof. We stated this result in DT, but we will prove it in GW. This is all for simplicity.

Since $c_1(S) = 0$, the virtual dimension is $2d = \dim \operatorname{Hilb}$. Once we fix how the curve hits the divisor, ev is proper. Hence we can compute it equivariantly or non-equivariantly, i.e. the lowest-degree term in the pushforward of the lhs is some multiple of $[\operatorname{Hilb}(S, d)]$ and the whole expression does not depend on the equivariant weight ϵ on the $\mathbb{P}^1$. To prove that all other terms are zero, it suffices to prove they are zero in the $\epsilon \to \infty$ limit, since they do not depend on ϵ. Look at the localization formula. The only contributions that depend on ϵ are:

1. from the original $\mathbb{P}^1$ component (which has weight ϵ);
2. the smoothing term $1/(-\epsilon + \text{tangent line})$ from the bubble(s);
3. the smoothing $\prod 1/(\epsilon/\mu_i - \psi_i)$ of the nodes in $\bar{\mathcal{M}}_{g,n}(S)$, from the other divisor;
4. the obstruction $H^1(C, \mathcal{O}_C \otimes \epsilon)$ to deforming the contracted component.

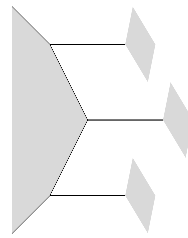
In total, these contributions give

$$\int_{[\bar{\mathcal{M}}_{g,n}(S,\beta)]^{\text{vir}}} \frac{\Lambda_g(\epsilon)}{\epsilon \prod(\epsilon/\mu_i - \psi_i)} = \epsilon^{g-1-n} \int_{[\bar{\mathcal{M}}_{g,n}(S,\beta)]^{\text{vir}}} \frac{1 - \lambda_1/\epsilon + \cdots \pm \lambda_g/\epsilon^g}{\prod(1/\mu_i - \psi_i/\epsilon)}.$$

The virtual dimension is $(g-1)+n$, so we need a class that is at least this dimension to get a non-zero contribution. Hence in the integrand, we will pick up a factor of at least ϵ^{-g+1-n}. So the total exponent of ϵ is always negative. ☐

Hence the evaluation pushforward is trivial, and it trivially satisfies GW/DT. But now we need to take this trivial thing and break it up. Again, we do everything capped. So we get a capped rubber near the relative divisor.

Definition 2.45. Take $S \times \mathbb{P}^1$. A **capped rubber** is a curve count of the form



In other words, we can have rubbers on both sides, but on the capped legs we only allow features of degree 0 in S, i.e. degree $(0, d)$ in total. This picture says that the capped rubber is

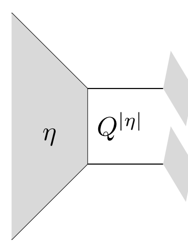
$$\hat{\Psi} := \Psi(q, Q, \ldots)\Psi(q, 0, \ldots)^{-1}$$

where Ψ is the fundamental solution we had earlier.

Corollary 2.46. *The capped rubber satisfies GW/DT and lies in* $\mathbb{Q}(q)[[Q]]$.

Proof. GW/DT follows from Ψ satisfying GW/DT. The second claim comes from Ψ actually satisfying a system of QDEs in $q(d/dq)$ and $Q_i(d/dQ_i)$, with $\Psi(0,0) = 1$. So the capped rubber is still a fundamental solution in $Q_i(d/dQ_i)$, and $\hat{\Psi}(q, 0) = 1$. The coefficients of the QDE are rational functions of q, and the desired claim follows. ☐

The conclusion is that the whole $\mathcal{A}_2 \times \mathbb{P}^1$ diagram is rational in q and satisfies GW/DT. The corresponding diagram for $\mathcal{A}_1 \times \mathbb{P}^1$ is



and can be uniquely solved for the 2-leg vertices with conditions η and μ, as follows. Fix a term $Q^{|\eta|}$, i.e. fix η. Wlog assume $|\eta| \leq |\mu|$, by symmetry. So we are free to pick $|\lambda| < |\eta|$. Hence

$$\operatorname{rank}(\hat{V}(\lambda, \eta, \emptyset))_{|\lambda|<k,\, |\eta|=k} = \#(\text{partitions of } \eta).$$

If this matrix has maximal rank then we are done. But to compute the rank of a matrix we can pick any specialization of its variables, so we just pick a stupid specialization $t_1 + t_2 + t_3 = 0$. It follows that the 2-leg vertex satisfies GW/DT. Then we go back to the $\mathcal{A}_2 \times \mathbb{P}^1$ geometry and undo the 2-leg vertices to get the 3-leg vertices.

It is a conjecture that $\hat{V}$ is a polynomial in q, not just a rational function.

2.11 Dec 04 (Henry): Capping and QDE

Let X be a toric 3-fold. The main idea is that *capped* localization is much better than regular localization for DT theory. We illustrate the derivation of both to compare and contrast.

Definition 2.47 (Regular localization, MNOP1)**.** We can directly localize $Z(X, q)$ (non-reduced partition function):

$$Z(X,q) = \sum_n q^n \sum_{I \in I_n(X,\beta)} e(T_{[I]}).$$

1. (Compute) For a given T-fixed $[I]$, split $T_{[I]} I_n(X, \beta)$ via Čech cohomology into contributions from vertices and edges:

$$\begin{aligned} T_{[I]} = &(\Gamma(U_\alpha) - \chi(I|_{U_\alpha}, I|_{U_\alpha})) \\ &- \left(\Gamma(U_{\alpha\beta}) - \chi(I|_{U_{\alpha\beta}}, I|_{U_{\alpha\beta}})\right). \end{aligned}$$

2. (Rearrange to cancel poles) Realize that these expressions are horrible because they are infinite power series in t_1, t_2, t_3. So we formally rearranged into nicer data

$$T_{[I]} = \sum_\alpha V_\alpha + \sum_{\alpha\beta} E_{\alpha\beta}$$

   where both V_α and $E_{\alpha\beta}$ are Laurent *polynomials* in $\mathbb{Q}[t_1^\pm, t_2^\pm, t_3^\pm]$.

Then after taking equivariant Euler class, we get a product of contributions from vertices and edges. In particular, the contribution from the vertex is the **equivariant vertex**

$$W(\lambda,\mu,\nu) \coloneqq \sum_{\pi \in \Pi(\lambda,\mu,\nu)} q^{|\pi|} w(\pi), \quad w(\pi) \coloneqq \prod_{\square=(a,b,c)\in\pi} (s_1 a + s_2 b + s_3 c)^{-v(\square)}.$$

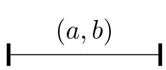
Here $v(\square)$ is the coefficient of $t_1^a t_2^b t_3^c$ in V_π. Note that this whole process does not involve relative GW/DT theory. Also, remember we have to normalize by degree-0 invariants to get reduced partition functions Z', so we often consider

$$W(\lambda,\mu,\nu)/W(\emptyset,\emptyset,\emptyset).$$

Remark. Remember that the TQFT DT$(-)$ is built from a q-shift of the *reduced* partition functions Z'. So if we want to use degeneration, we must work with Z'.

Definition 2.48 (Capped localization, MOOP)**.** Use degeneration to split the toric graph of X into vertices and edges with relative conditions.

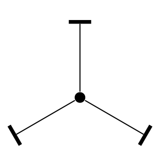
1. Take the $X_e \coloneqq \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathbb{P}^1$ geometry. The **capped edge** is



$$\coloneqq E(\lambda, \mu, t_1, t_2, t_3, t_1', t_2', q) \coloneqq Z'(X_e/F_0 \cup F_\infty; q)_{\lambda,\mu},$$

   i.e. an edge with two relative conditions λ and μ. Here $t_1' = t_1 - a t_3$ and $t_2' = t_2 - b t_3$.

2. Take $U \coloneqq (\mathbb{P}^1)^3 - \cup_{i=1}^3 L_i$ where L_i are the three T-invariant lines at (∞,∞,∞). Let D_i be the divisor with i-th coordinate ∞. The **capped vertex** is



$$\coloneqq C(\lambda,\mu,\nu,t_1,t_2,t_3,q) \coloneqq Z'(U/\cup_i D_i q)_{\lambda,\mu,\nu}.$$

There are also contributions $G(\lambda, \mu, t_1, t_2, t_3, q)$ from gluing two relative conditions (i.e. raising/lowering indices).

Remark. In contrast to *capped* versions of objects, the regular versions will be called *bare*. (For example, we will refer to the equivariant vertex as the bare vertex.) Capped versions of objects are generally more nicely behaved as functions of q, because they arise as proper pushforwards from a compact moduli.

The point of MOOP is to match capped vertices and edges in GW and DT, and therefore prove that up to $q = -e^{iu}$ the two theories are equivalent.

Why do we care about the bare vertex $W(\lambda, \mu, \nu)$? Following the logic in MNOP1, if we take the Calabi–Yau torus $t_1 t_2 t_3 = 1$, Serre duality says $T_{[I]}$ is an odd function under

$$(t_1, t_2, t_3) \mapsto (t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

Explicitly, if $t_1^a t_2^b t_3^c$ appears, so does $-t_1^{-a} t_2^{-b} t_3^{-c}$. Then terms in $w(\pi)$ appear in pairs, of the form

$$t_1^a t_2^b t_3^c - t_1^{-a} t_2^{-b} t_3^{-c} \mapsto \frac{-s_1 a - s_2 b - s_3 c}{s_1 a + s_2 b + s_3 c} = -1.$$

We computed the total parity to be $w(\pi) = (-1)^{|\pi|}$ in MNOP1. Hence

$$W(\lambda, \mu, \nu)|_{s_1+s_2+s_3=1} = \sum_{\pi \in \Pi(\lambda,\mu,\nu)} (-q)^{|\pi|}$$

is just an enumeration of 3d partitions.

Example 2.49. Recall from MNOP1 that we had the formula

$$W(\emptyset, \emptyset, \emptyset) = M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}} \rightsquigarrow W(\emptyset, \emptyset, \emptyset)|_{s_1+s_2+s_3=1} = M(-q),$$

agreeing with $M(-q) = \sum_{\pi \in \Pi(\emptyset,\emptyset,\emptyset)} (-q)^{|\pi|}$.

Theorem 2.50 (Okounkov–Reshetikhin–Vafa)**.**

$$\sum_{\pi \in \Pi(\lambda,\mu,\nu)} (-q)^{|\pi|} = (\textit{prefactor}) s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

where $s_{\lambda/\mu}$ are skew Schur functions and $\rho = (-1/2, -3/2, -5/2, \ldots)$.

Definition 2.51. The reduced vertex in the CY limit

$$W'(\lambda, \mu, \nu)|_{s_1+s_2+s_3=1}$$

is called the **topological vertex** (up to some prefactors).

Because an explicit formula exists, a common argument in DT problems is to show that an object is independent of weights t_1, t_2, t_3, and to evaluate the desired object in the CY limit $t_1 + t_2 + t_3 = 0$. (For example, this is exactly the strategy for matching the capped vertex.)

Remark. Historically, the AKMV paper first proposed the topological vertex from physical arguments involving large N duality. Melissa's lectures on her and her collaborators' work showed that, up to a conjectural identity $\tilde{W}_{\vec{\mu}}(q) = W_{\vec{\mu}}(q)$, the GW topological vertex computes GW theory. To complete the argument for the validity of the topological vertex, note that:

1. MNOP1 + the formula for the DT topological vertex matches the GW topological vertex with the DT topological vertex;
2. MOOP proves the full equivariant GW/DT correspondence.

To see the relationship between the capped vertex and the bare vertex, we focus on the simple case of one relative leg instead of three relative legs. These are the 1-leg vertices. For simplicity, and also to match with notation in Andrei's K-theory notes, write

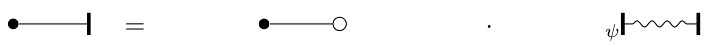
This is the setting of local curves: we called the capped 1-leg vertex the $(0,0)$ cap in that context, corresponding to the reduced series

$$\langle\ |\lambda\rangle'_{(0,0)} = \frac{\langle\ |\lambda\rangle}{\langle\ |\emptyset\rangle}.$$

We can view the bare 1-leg vertex as a pushforward from the *non-singular* part of the relative moduli $I_n(X/D, \beta)$ where there is nothing in the bubbles. In this way, we think of the lack of relative conditions on the bare leg as a *non-singularity condition*, denoted by an empty circle. Hence the localization contributions to the capped vertex $\langle\ |\lambda\rangle'$ split into three pieces:

1. a bare vertex term $W'(\mu, \emptyset, \emptyset)|'_{s,t_1,t_2}$ coming from contributions outside the bubble(s);
2. a node smoothing term $1/(-s-\psi_\infty)$ from where the bubble(s) connect to the rigid $\mathbb{P}^1$;
3. a rubber integral with relative conditions $[J_\mu]$ and λ.

We write this pictorially as



$$\langle\ |\lambda\rangle' = \sum_{|\mu|=|\lambda|} W'(\mu, \emptyset, \emptyset)|_{s,t_1,t_2} \cdot (\text{gluing term}) \cdot \langle [J_\mu]| \frac{1}{-s-\psi_\infty} |\lambda\rangle^{\sim\prime}.$$

Remark. When we write rubber integral series, in general we mean

$$\langle \mu|\mathcal{F}|\nu\rangle^{\sim} := q^d \langle \mu|\nu\rangle_{d,d} + \sum_{n>d} q^n \langle \mu|\mathcal{F}|\nu\rangle^{\sim}_{n,d}.$$

Definition 2.52. Define the **capping operator** Ψ on Fock space $\mathcal{F}$ by its matrix elements

$$\langle \mu|\Psi|\nu\rangle_{\mathcal{F}} := q^{-d} \langle \mu| \frac{1}{1-\psi_\infty} |\nu\rangle^{\sim\prime} = q^{-d} M(-q)^{-(t_1+t_2)} \langle \mu| \frac{1}{1-\psi_\infty} |\nu\rangle^{\sim}.$$

In degree 0, this is essentially what we called W_∞ in MNOP2, where we computed $W_\infty = M(-q)^{(t_1+t_2)/s}$.

Theorem 2.53 (QDE)**.**

$$q\frac{d}{dq}\Psi = \mathbf{M}\Psi - \Psi\mathbf{M}(0).$$

Here $\mathbf{M} = \mathbf{M}(q)$ *is an explicit operator on Fock space given by*

$$\mathbf{M}(q) := (t_1+t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1} \alpha_{-k}\alpha_k + \frac{1}{2} \sum_{k,l>0} (t_1 t_2 \alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l).$$

Corollary 2.54. Ψ *is invertible.*

Proof. If the inverse exists, it is uniquely determined by

$$0 = q\frac{d}{dq}(\Psi\Psi^{-1}) = (\mathbf{M}\Psi - \Psi\mathbf{M}(0))\Psi^{-1} + \Psi q\frac{d}{dq}\Psi^{-1},$$

i.e. $q(d/dq)\Psi^{-1} = \mathbf{M}(0)\Psi^{-1} - \Psi^{-1}\mathbf{M}$. As long as the q-constant term in Ψ is invertible, this has a solution. This term is proportional to $M(-q)^{-(t_1+t_2)}$, so it is invertible as long as q is not a root of unity. □

Hence if we know the capped 1-leg vertex and the capping operator, we get a square system of linear equations for the unknown $W'(\mu, \emptyset, \emptyset)$, and therefore we can solve for the bare 1-leg vertex.

The QDE for the capping operator essentially arises from rubber calculus. This refers to (a slightly more general version with ψ_0 of) the following. Let $N := \mathcal{O} \oplus \mathcal{O} \to \mathbb{P}^1$ be the level $(0,0)$ geometry.

Lemma 2.55. *Define the DT insertion on fibers*

$$\sigma_1 := \pi_*(\mathrm{ch}_3(\mathcal{J}_\infty)) \in A^1_T(\mathrm{Hilb}(D_\infty, d), \mathbb{Q}).$$

Then

$$(d-n)\langle \mu|\psi_\infty^a|\nu\rangle^\sim_{n,d} = \langle \mu|\sigma_1(F)\psi_\infty^a|\nu\rangle^N_{n,d} - \langle \mu|\psi_\infty^{a-1}|\sigma_1 \cdot \nu\rangle^\sim_{n,d}.$$

Proof. The rubber moduli has a universal target

$$\pi\colon \mathcal{R} \to I_n(R/R_0 \cup R_\infty, d)^\sim,$$

where we define $[\mathcal{R}]^{\mathrm{vir}} := \pi^*[I_n^\sim]^{\mathrm{vir}}$. The rubber calculus comes from computing

$$\langle \mu|\,\mathrm{ch}_3(\mathcal{J})\pi^*(\psi_\infty^a)|\nu\rangle^{\mathcal{R}}_{n,d}$$

in two different ways.

1. Apply push-pull with π to get

   $$\langle \mu|\,\mathrm{ch}_3(\mathcal{J})\pi^*(\psi_\infty^a)|\nu\rangle^{\mathcal{R}}_{n,d} = (d-n)\langle \mu|\psi_\infty^a|\nu\rangle^\sim_{n,d}.$$

   This is because we have $\pi_*(\mathrm{ch}_3(\mathcal{J}) \cap [\mathcal{R}]^{\mathrm{vir}}) = (d-n)[I_n]^{\mathrm{vir}}$ by a fiberwise calculation via GRR:

   $$\mathrm{ch}_3(I_Z) = -\,\mathrm{ch}_3(i_*\mathcal{O}_Z) = (d-n)[\mathrm{pt}] \in A_0(X), \quad \forall I \in I_n(X,d).$$

2. Use the rigidification map

   $$\phi\colon \mathcal{R} \to I_n(N/N_0 \cup N_\infty, d)$$

   given by rigidifying the $\mathbb{P}^1$ component carrying the extra target point. By comparing deformation theories,

   $$\phi^*[I_n]^{\mathrm{vir}} = [\mathcal{R}]^{\mathrm{vir}}.$$

   On $\mathcal{R}$, we also have a comparison relation

   $$\pi^*\psi_\infty = \phi^*\psi_\infty - \phi^*D_\infty$$

   where $D_\infty \subset I_n$ is the virtual boundary divisor where the rubber over 0 carries Euler characteristic n. Hence again by push-pull,

   $$\langle \mu|\,\mathrm{ch}_3(\mathcal{J})\pi^*(\psi_\infty^a)|\nu\rangle^{\mathcal{R}}_{n,d} = \langle \mu|\,\mathrm{ch}_3(\mathcal{J})\psi_\infty^a|\nu\rangle^N_{n,d} - \langle \mu|\psi_\infty^{a-1}|\sigma_1 \cdot \nu\rangle^\sim_{n,d}.$$

   Now note that $\mathrm{ch}_3(\mathcal{J})$ in $\langle - \rangle^N$ is what we called the primary insertion $\sigma_1(F)$. □

When we put this into a generating series, we get the identity

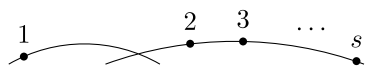
$$q\frac{d}{dq}q^{-d}\langle\mu|\frac{1}{1-\psi_\infty}|\nu\rangle^\sim = q^{-d}\langle\mu|\frac{-\sigma_1(F)}{1-\psi_\infty}|\nu\rangle - q^{-d}\langle\mu|\frac{1}{1-\psi_\infty}|(-\sigma)\cdot\nu\rangle^\sim.$$

We will see that $\mathbf{M}(0)$ is essentially multiplication by $-\sigma$, so the second term, up to normalization, is $\Psi\mathbf{M}(0)$. To relate the first term to Ψ, we apply topological recursion.

Lemma 2.56 (Topological recursion)**.** *For the relative theory of $N/N_0 \cup N_\infty$,*

$$\langle\mu|\psi_0^a|\nu\rangle^N_{n,d} = \sum_{\eta\vdash d}\sum_{n_1+n_2=n+d}\langle\mu|\eta\rangle^N_{n_1,d}\Delta_d(\eta,\eta)\langle\eta|\psi_\infty^{a-1}|\nu\rangle^\sim_{n_2,d}.$$

Proof. The class ψ_0 is dual to the boundary stratum δ_0 consisting of non-trivial bubbles at 0:

$\delta_0 =$ 

.

Hence we can exchange one ψ_∞ for pulling back the integral to δ_0. But δ_0 factors via degeneration into the rhs. ☐

2.12 Dec 05 (Henry): Capping and QDE II

We are in the middle of proving the capping operator Ψ satisfies the QDE

$$q\frac{d}{dq}\Psi = \mathbf{M}\Psi - \Psi\mathbf{M}(q=0)$$

using rubber calculus and topological recursion. From rubber calculus, we got

$$q\frac{d}{dq}q^{-d}\langle\mu|\frac{1}{1-\psi_\infty}|\nu\rangle^\sim = q^{-d}\langle\mu|\frac{-\sigma_1(F)}{1-\psi_\infty}|\nu\rangle - q^{-d}\langle\mu|\frac{1}{1-\psi_\infty}|(-\sigma)\cdot\nu\rangle^\sim.$$

From topological recursion, the first term becomes

$$q^{-d}\langle\mu|\frac{-\sigma_1(F)}{1-\psi_\infty}|\nu\rangle = q^{-2d}\sum_\eta\langle\mu|-\sigma_1(F)|\eta\rangle\Delta_d(\eta,\eta)\langle\eta|\frac{1}{1-\psi_\infty}|\nu\rangle^\sim.$$

This is some operator $-\sigma_1(F)$ composed with Ψ, up to some factors. This operator has a name, from local curves, and is related to $\mathbf{M}(q)$.

Proposition 2.57. *Define the operator M_σ by*

$$\langle\mu|M_\sigma|\nu\rangle_{\mathcal{F}} \coloneqq q^{-d}\langle\mu|-\sigma_1(F)|\nu\rangle.$$

Then

$$M_\sigma = \mathbf{M}(q) - (t_1+t_2)\Phi(q)\cdot\mathrm{id}, \quad \Phi(q) \coloneqq q\frac{d}{dq}\log M(-q).$$

It remains to put everything together and arrive at the QDE for Ψ. This is purely an algebra exercise. We see that M_σ is the operator which plays a crucial role in deriving the QDE. So we should try to match it with its analogue on the GW side.

To avoid confusion, write M_σ^{DT} for what we have been discussing so far. On the GW side, the equivalent of M_σ is the following.

Definition 2.58. Define the operator M_σ^{GW} by

$$\langle\mu|M_\sigma^{\mathrm{GW}}|\nu\rangle_{\mathcal{F}} \coloneqq \mathrm{GW}^*(-\tau_1(F)|0;0,0)_{\mu\nu} = (-iu)^{\ell(\mu)+\ell(\nu)}Z'_{\mathrm{GW}}(-\tau_1(F)|0;0,0)_{\mu\nu}.$$

The key to computing M_σ^{GW} is to recall that from GW local curves, we know a different operator M_D.

Definition 2.59. Define the operator M_D in both GW and DT as

$$\langle\mu|M_D^{\mathrm{DT}}|\nu\rangle_{\mathcal{F}} \coloneqq \mathrm{DT}(0;0,0)_{\mu,D,\nu} = -\,\mathrm{DT}(0;0,0)_{\mu,(2),\nu}$$
$$\langle\mu|M_D^{\mathrm{GW}}|\nu\rangle_{\mathcal{F}} \coloneqq (-1)^{|\mu|}\,\mathrm{GW}^*(0;0,0)_{\mu,D,\nu}.$$

Recall that $D = -(2) \coloneqq -(2,1^{d-2})$.

Remark. On the DT side, we know M_σ^{DT} but not M_D^{DT}. On the GW side, we know M_D^{GW} but not M_σ^{GW}. On both sides, the following formula holds.

Proposition 2.60.

$$M_\sigma = M_D - (t_1+t_2)\left(\frac{d}{2}\frac{(-q)+1}{(-q)-1}|\cdot| - \Phi(q)\right).$$

Remark. This is essentially Proposition 26 in DT local curves, but structurally that argument is GW/DT-agnostic. Its only DT input is two particular invariants. So we give the GW details instead. Since we always work in the $(0,0)$ geometry, omit writing this information.

Proof sketch. First apply the degeneration formula to M_σ^{GW}:

$$\mathrm{GW}^*(-\tau_1(F))_{\mu\nu} = \sum_\gamma \mathrm{GW}^*_{\mu\gamma\nu}\,\Delta_d(\gamma,\gamma)\,\mathrm{GW}^*(-\tau_1(F))_\gamma.$$

(Here $\Delta_d(-,-)$ refers to the *non-standard* pairing on Fock space, because we use GW^* instead of GW.) By a dimension argument, $\mathrm{GW}^*(-\tau_1(F))_\gamma = 0$ unless $\gamma = (1^d)$ or $\gamma = (2,1^{d-2})$. (The analogous vanishing on DT side is much harder; see section 4.6.) So this sum only has two terms.

1. ($\gamma = (2)$ term) Here we have terms

   $$\mathrm{GW}^*(0;0,0)_{\mu,(2),\nu} = (-1)^{|\mu|}\langle\mu|-M_D^{\mathrm{GW}}|\nu\rangle$$

   and $\mathrm{GW}^*(-\tau_1(F)|0;0,0)_{(2)}$.
2. ($\gamma = (1^d)$ term) Here we have terms

   $$\mathrm{GW}^*(0;0,0)_{\mu,(1^d),\nu} = \mathrm{GW}^*(0;0,0)_{\mu,\nu} = \Delta_d(\mu,\nu)$$

   and $\mathrm{GW}^*(-\tau_1(F)|0;0,0)_{(1^d)}$.

In terms of connected invariants, this means we must compute

$$\int_{[\bar{\mathcal{M}}_{g,1}(\mathbb{P}^1,1)]^{\mathrm{vir}}} \lambda_g\lambda_{g-1}\tau_1(p), \quad \int_{[\bar{\mathcal{M}}_{g,1}(\mathbb{P}^1,(2))]^{\mathrm{vir}}} \lambda_g\lambda_g\tau_1(p).$$

These we can easily compute by localization and known series for Hodge integrals. □

Remark. The same proof works on the DT side, but with the input

$$q^{-d}\langle -\tau_1(F)|\mu\rangle' = \langle(1^d)|M_\sigma|\mu\rangle, \quad \mu = (1^d),(2).$$

Fortunately, what we know so far is:

$$M_\sigma^{\mathrm{DT}} = \mathbf{M}(q) - (t_1+t_2)\Phi(q)\cdot\mathrm{id}, \quad M_D^{\mathrm{GW}} = \mathbf{M}(q) - \frac{t_1+t_2}{2}\frac{(-q)+1}{(-q)-1}|\cdot|.$$

Here $|\cdot|$ is the energy operator, which returns d at degree d. So on DT side we can just use our formula for M_σ to compute the two required inputs.

Since we have a formula for M_D^{GW} from GW local curves, we now have a formula for M_σ^{GW}. One can easily check the GW vs DT formulas are the same, so

$$M_\sigma^{\mathrm{GW}} = M_\sigma^{\mathrm{DT}}.$$

This also matches up $M_D^{\mathrm{GW}} = M_D^{\mathrm{DT}}$. Since M_D fully determines the level $(0,0)$ pair of pants in both theories, we have matched the final piece of the level $(0,0)$ local curves theories. (We have not matched the $(-1,0)$ cap.)

Andrei told us in his talk that to match capped edges, we write them as solutions to the same QDE on the GW and DT sides. Note that we only need to match capped edges of level $(0,0)$ and $(0,-1)$ by the usual degeneration argument. We know the capped edge of level $(0,0)$ is the tube, which is the identity map. So it suffices to match $(0,-1)$ edges. View them as operators by

$$\begin{aligned}\langle\lambda|\mathbf{O}_{\mathrm{GW}}|\mu\rangle &\coloneqq (-iu)^{\ell(\lambda)+\ell(\mu)-d} Z'_{\mathrm{GW}}(0;0,-1;u)_{\lambda,\mu}\\ \langle\lambda|\mathbf{O}_{\mathrm{DT}}|\mu\rangle &\coloneqq (-q)^{-d/2} Z'_{\mathrm{DT}}(0;0,-1;q)_{\lambda,\mu}.\end{aligned}$$

The GW/DT correspondence requires $\mathbf{O}_{\mathrm{GW}} = \mathbf{O}_{\mathrm{DT}}$ with $q = -e^{iu}$.

Proposition 2.61. *The following DE holds for both $\mathbf{O}_{\mathrm{GW}}$ and $\mathbf{O}_{\mathrm{DT}}$:*

$$-t_3 q\frac{d}{dq}\mathbf{O} = -\mathbf{M}(t_1,t_2)\mathbf{O} + \mathbf{O}\mathbf{M}(t_1',t_2').$$

Proof sketch. This follows by fairly general arguments. The key idea is to compute a $\sigma_1(1)$ or $\tau_1(1)$ insertion in two different ways.

1. (GW side) The dilaton equation says for connected invariants, we can pull out $\tau_1(1)$ insertions and multiply by $2g-2+n$ instead. Hence

$$t_3\mathbf{O}_{\mathrm{GW}}(\tau_1(1)) = t_3\left(u\frac{d}{du} + d\right)\mathbf{O}_{\mathrm{GW}}.$$

2. (DT side) Because $\mathrm{ch}_3(I) = d-n$ is constant for all I, we have

$$Z_{\mathrm{DT}}(\sigma_1(1);0,-1)_{\lambda,\mu} = (-q\frac{d}{dq} + d)Z_{\mathrm{DT}}(0,-1)_{\lambda,\mu}.$$

From MNOP2, we know degree-0 series $Z_{\mathrm{DT}}(0;0,-1;q)_{\emptyset,\emptyset}$. So normalize to give

$$t_3\mathbf{O}_{\mathrm{DT}}(\sigma_1(1)) = \left(-t_3q\frac{d}{dq} + (t_1+t_2-t_1'-t_2')\Phi(q)\right)\mathbf{O}_{\mathrm{DT}}.$$

The rest of the argument works for both GW and DT in general. By localization, $t_3\sigma_1(1) = \sigma_1(F_0) - \sigma_1(F_\infty)$ and similarly for τ_1. But by degeneration,

$$\begin{aligned}\mathbf{O}(\sigma_1(F_0))^\lambda_\mu &= \sum (\mathbf{O}^{t_1,t_2}(\sigma_1(F))^\sim)^\lambda_\nu \cdot \mathbf{O}^\nu_\mu\\ \mathbf{O}(\sigma_1(F_\infty))^\lambda_\mu &= \sum \mathbf{O}^\lambda_\nu \cdot (\mathbf{O}^{t_1',t_2'}(\sigma_1(F))^\sim)^\nu_\mu.\end{aligned}$$

The rubber integrals can be rigidified: $\mathbf{O}(\sigma_1(F))^\sim = \mathbf{O}(\sigma_1(F))$. Hence

$$t_3\mathbf{O}(\sigma_1(1)) = [\mathbf{O}(\sigma_1(F)), \mathbf{O}],$$

plugging in (t_1,t_2) or (t_1',t_2') as appropriate. But we have shown

$$\mathbf{O}_{\mathrm{DT}}(\sigma_1(F)) = M_\sigma = \mathbf{O}_{\mathrm{GW}}(\tau_1(F)).$$

Putting everything together gives the DE. ∎

Now we repeat the exact same argument for the capped rubber. Recall that the only differences between capped rubbers and capped edges is that:

1. capped rubbers are only allowed to have degree-0 features on the fiber F_0;
2. capped rubbers are in the $\mathcal{A}_n \times \mathbb{P}^1$ geometry instead of $\mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathbb{P}^1$.

Hence the degeneration and rigidification argument still works, up to some details.

Proposition 2.62. *The capped rubber* $\mathbf{O}(CR)$ *satisfies the DE*

$$-t_3 q \frac{d}{dq}\mathbf{O}(CR) = -\mathbf{M}_{\mathcal{A}_n}(q=0; t_1, t_2)\mathbf{O}(CR) + \mathbf{O}(CR)\mathbf{M}_{\mathcal{A}_n}(q; t_1', t_2').$$

Proof sketch. There are two modifications.

1. On F_0, where we used to have $\mathbf{M}(t_1, t_2)$, we now have $\mathbf{M}(q=0; t_1, t_2)$ for only degree-0 features.
2. These matrices $\mathbf{M}_{\mathcal{A}_n}$ are the analogues of $\mathbf{M}$ (for $(0,0)$ geometry) for the $\mathcal{A}_n$ geometry. Matching them on the GW and DT sides is the crucial input from the $\mathcal{A}_n$ papers. (See corollary 8.5 in Maulik–Oblomkov's DT theory of $\mathcal{A}_n \times \mathbb{P}^1$.) □