Notes for Arithmetic and Algebraic Geometry
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Abstract

These are my live-texed notes for the Spring 2017 offering of MATH GR6262 Arithmetic and Algebraic Geometry. Let me know when you find errors or typos. I'm sure there are plenty.

Chapter 1

Affine schemes and quasi-coherent sheaves

1.1 Affine schemes and schemes

Definition 1.1.1. A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A **morphism** of locally ringed spaces $(f, f^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$, the map of stalks $f^\#_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism, i.e. $(f^\#_x)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$. Let RS (resp. LRS) denote the category of ringed spaces (resp. locally ringed spaces).

Definition 1.1.2. There is a **global sections functor**

$$\begin{gathered}\Gamma\colon \mathsf{LRS} \to \mathsf{Rings}\\ (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)\\ ((f, f^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)) \mapsto (f^\#\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)).\end{gathered}$$

Example 1.1.3. Let $(M, \mathcal{C}^\infty)$ be a manifold M with the sheaf of smooth functions $\mathcal{C}^\infty$. This is a locally ringed space because $\mathcal{C}^\infty_x$ consists of all the germs vanishing at $x \in M$. The global sections functor $\Gamma(-,-)\colon \mathsf{LRS}^{\mathrm{op}} \to \mathsf{Rings}$ has a left adjoint $\operatorname{Spec}\colon \mathsf{Rings} \to \mathsf{LRS}^{\mathrm{op}}$, i.e.

$$\operatorname{Mor}_{\mathsf{LRS}}(X, \operatorname{Spec}(A)) = \operatorname{Mor}_{\mathsf{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

We will prove this later. Yoneda's lemma says this determines $\operatorname{Spec}(A)$ in LRS up to unique isomorphism.

Definition 1.1.4. Recall that points in the topological space $\operatorname{Spec} A$ are prime ideals $\mathfrak{p}$ in the ring A. Given $f \in A$, there is a **standard open**

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\}.$$

These form a basis for the topology. The **structure sheaf** $\mathcal{O}$ associated to $\operatorname{Spec} A$ obeys these "rules":

1. $\mathcal{O}(D(f)) = A_f$,
2. $\mathcal{O}_\mathfrak{p} = A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$,
3. if $D(f) \subset D(g)$, then the restriction $\mathcal{O}(D(g)) \to \mathcal{O}(D(f))$ commutes with the localizations $\mathcal{O}(D(f)) = A_f \to A_{fg}$ and $\mathcal{O}(D(g)) = A_g \to A_{fg}$.

To construct $\mathcal{O}$ properly, we do one of the following:

1. use the fact that given a topological space X and a basis $\mathcal{B}$ and a "sheaf $\mathcal{F}$ on $\mathcal{B}$" (i.e. specified on and satisfying the sheaf condition for open sets in $\mathcal{B}$), there is a unique extension of $\mathcal{F}$ to a sheaf on X;

2. (Hartshorne) for an open U, set

$$\mathcal{O}(U) \coloneqq \{(s_x)_{x\in U} : s_x \in \mathcal{O}_{X,x} \text{ and } \star \text{ holds}\}$$

   where $\star$ is the condition that U can be covered by opens V such that there exists $a, b \in A$ such that for every $\mathfrak{p} \in V$ we have $b \notin \mathfrak{p}$ and $s_{\mathfrak{p}} = a/b$ in $A_{\mathfrak{p}}$.

Remark. Note that $A_{\mathfrak{p}} = \operatorname{colim}_{f\in A, f\notin\mathfrak{p}} A_f =: \mathcal{O}_{\mathfrak{p}}$.

Lemma 1.1.5. *Let $X \in \mathrm{Ob}(\mathsf{LRS})$ and $f \in \Gamma(X, \mathcal{O}_X)$. Then*

$$U = \{x \in X : f \text{ does not vanish at } x\}$$

is open in X, and there exists a unique $g \in \mathcal{O}_X(U)$ such that $f|_U g = 1$.

Proof. Since f does not vanish at x, it has an inverse in the stalk $\mathcal{O}_{X,x}$. But then it has an inverse in a neighborhood of x. Hence U is open. □

Theorem 1.1.6. *Let X be a locally ringed space and A be a ring. Then the map*

$$\mathrm{Mor}_{\mathsf{LRS}}(X, \mathrm{Spec}(A)) \to \mathrm{Mor}_{\mathsf{Rings}}(A, \Gamma(X, \mathcal{O}_X)), \quad (f, f^\#) \mapsto (f^\# \colon A \to \Gamma(X, \mathcal{O}_X))$$

is a bijection of sets.

Proof. We will construct the corresponding $(f, f^\#)$ given $\varphi \colon A \to \Gamma(X, \mathcal{O}_X)$.

1. On points $x \in X$, let $f(x)$ be the prime ideal $\mathfrak{p}$ given by the inverse image of $\mathfrak{m}_x \in \mathcal{O}_{X,x}$ in the composition

$$A \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}.$$

   We must show such an f is continuous. If $a \in A$, then

$$f^{-1}(D(a)) = \{x \in X : \varphi(a) \text{ does not vanish at } x\}.$$

   This set is open by the preceding lemma.

2. We also need to construct a map $\mathcal{O}_{\mathrm{Spec}\,A}(D(a))\mathcal{O}_X(f^{-1}(D(a)))$. We know $\mathcal{O}_{\mathrm{Spec}\,A}(D(a)) = A_a$. By the preceding lemma, the universal property of localization is applicable to the diagram

$$\begin{array}{ccc} A_a & \xrightarrow{?} & \mathcal{O}_X(f^{-1}(D(a))) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & \Gamma(X, \mathcal{O}_X), \end{array}$$

   giving the desired map.

There is more to check, but we will skip that. □

Corollary 1.1.7. *The category* Rings *of rings is anti-equivalent to the category of affine schemes, i.e. the full subcategory of* LRS *consisting of objects isomorphic to* $\mathrm{Spec}\,A$ *for some ring A.*

Definition 1.1.8. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that every $x \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A **morphism** of schemes is a morphism of locally ringed spaces. The **category** of schemes is denoted Sch, and the category of schemes over a fixed scheme S is denoted Sch/S.

Remark. Given a category $\mathcal{C}$ and a $U \in \mathrm{Ob}(\mathcal{C})$, the **comma category** $\mathcal{C}/U$ is the category whose objects are arrows $V \to U$ for $V \in \mathcal{C}$, and morphisms are morphisms $V_1 \to V_2$ commuting with the arrows $V_1 \to U$ and $V_2 \to U$. So Sch/S is a comma category. If we write Sch/R for a ring R, we mean "schemes over $\operatorname{Spec} R$." The scheme $\operatorname{Spec} \mathbb{Z}$ is the final object in Sch, so $\mathsf{Sch}/\mathbb{Z} \cong \mathsf{Sch}$.

Lemma 1.1.9. *If $(X, \mathcal{O}_X)$ is a scheme and $U \subset X$ is open, then $(U, \mathcal{O}_U) \coloneqq (U, \mathcal{O}_X|_U)$ is a scheme.*

Proof. Every point in X has an open neighborhood which is an affine scheme. Given $x \in U$, let $V = \operatorname{Spec} A$ be its affine open neighborhood. Since V is affine, there exists a principal open $D(a)$ around x in V that is contained in U. So it suffices to show $(D(a), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)})$ is an affine scheme. Well, $D(a) \cong \operatorname{Spec}(A_a)$, and it is straightforward to check the structure sheaves are isomorphic using the fact that $(A_f)_g = A_{fg}$. $\square$

1.2 Quasi-coherent $\mathcal{O}_X$-modules

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is an $\mathcal{O}_X$-module having:

Name of property	every point has open nbhd U s.t.
locally generated by sections	$\exists \bigoplus_{n \in I} \mathcal{O}_U \twoheadrightarrow \mathcal{F}|_U$ for some I
finite type	$\exists (\mathcal{O}_U)^{\bigoplus n} \twoheadrightarrow \mathcal{F}|_U$ for some n
quasi-coherent	$\exists \bigoplus_{j \in J} \mathcal{O}_U \to \bigoplus_{i \in I} \mathcal{O}_U \to \mathcal{F}|_U \to 0$ exact
finite presentation	$\exists (\mathcal{O}_U)^{\oplus m} \to (\mathcal{O}_U)^{\oplus n} \to \mathcal{F}|_U \to 0$ exact
locally free	$\exists \bigoplus_{i \in I} \mathcal{O}_U \cong \mathcal{F}|_U$ for some I
finite locally free	$\exists (\mathcal{O}_U)^{\oplus n} \cong \mathcal{F}|_U$ for some n.

Warning: these notions may behave unexpectedly.

Remark. The **direct sum** $\bigoplus_i \mathcal{F}_i$ is the sheafification of the presheaf $U \mapsto \bigoplus_{i \in I} \mathcal{F}_i(U)$. (This sheafification is only necessary for infinite index sets.) Alternatively, it is the coproduct, so

$$\operatorname{Hom}_{\mathcal{O}_X}(\bigoplus_{i \in I} \mathcal{F}_i, \mathcal{G}) = \prod_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G}).$$

Since $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$, we can view, for example, $\bigoplus_{i \in I} \mathcal{O}_U \twoheadrightarrow \mathcal{F}|_U$ as a family of sections $s_i \in \mathcal{F}(U)$.

Definition 1.2.2. Let U be an open in X, and $j \colon U \hookrightarrow X$ be the inclusion. Then the **extension by zero** map $j_! \colon \mathsf{Ab}(U) \to \mathsf{Ab}(X)$ is given by

$$(j_! \mathcal{F})_x \coloneqq \begin{cases} \mathcal{F}_x & x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.2.3. Let k be a field and $n \geq 1$. Let $X = \mathbb{A}_k^n \coloneqq \operatorname{Spec}(k[x_1, \ldots, x_n])$. Let $0 \in X$ correspond to $(x_1, \ldots, x_n)$. Let $j \colon X \setminus \{0\} \hookrightarrow X$ be the inclusion. Look at the extension by zero $\mathcal{F} = j_! \mathcal{O}_{X \setminus \{0\}}$, so that

$$\mathcal{F}(U) = \begin{cases} 0 & 0 \in U \\ \mathcal{O}_X(U) & 0 \notin U. \end{cases}$$

The claim is that $\mathcal{F}$ is not locally generated by sections. The idea is that any section of $\mathcal{F}$ is 0 in any neighborhood of 0, but $\mathcal{F}$ is not 0 when restricted to any neighborhood of 0.

Example 1.2.4. Let $i \colon \{0\} \hookrightarrow X$ be the inclusion. Then there is a short exact sequence

$$0 \to j_!(\mathcal{O}_{X \setminus \{0\}}) \to \mathcal{O}_X \to i_* \mathcal{O}_{X,0} \to 0.$$

We call the pushforward of a stalk, e.g. $i_*\mathcal{O}_{X,0}$, a **skyscraper sheaf**. (In general,

$$0 \to j_! j^{-1}\mathcal{G} \to \mathcal{G} \to i_* i^{-1}\mathcal{G} \to 0$$

is exact, and this often lets us do Noetherian induction.) The claim is that $i_*\mathcal{O}_{X,0}$ is of finite type, but not quasi-coherent.

Theorem 1.2.5. *Let A be a ring. Let M be an A-module. There exists a unique sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules $\tilde{M}$ on $\operatorname{Spec} A$ together with a map $M \to \Gamma(\operatorname{Spec} A, \tilde{M})$ of A-modules characterized by each of the following properties:*

1. *(construction) $\tilde{M}(D(f)) = M_f$ (and when $f = 1$, this should be the given map) and $\tilde{M}(D(f)) \to \tilde{M}(D(fg))$ is the map $M_f \to M_{fg}$;*
2. *(Yoneda) the map $\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}}(\tilde{M}, \mathcal{F}) \to \operatorname{Hom}_A(M, \Gamma(\operatorname{Spec} A, \mathcal{F}))$ is bijective for all $\mathcal{O}_X$-modules $\mathcal{F}$;*
3. *(Qcoh) $\tilde{M}$ is quasi-coherent and $M \to \Gamma(\operatorname{Spec} A, \tilde{M})$ is an isomorphism;*
4. *(Stalks) the map $M \to \Gamma(\operatorname{Spec} A, \tilde{M})$ induces isomorphisms $M_\mathfrak{p} \to (\tilde{M})_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$ (the remark below explains where this map comes from).*

Remark. The map $\Gamma(X, \mathcal{F}) \to \mathcal{F}_\mathfrak{p}$ is a map from an A-module to an A_p-module. So there is a unique factorization to $\Gamma(X, \mathcal{F})_\mathfrak{p}$ giving a canonically induced map $\Gamma(X, \mathcal{F})_\mathfrak{p} \to \mathcal{F}_\mathfrak{p}$.

Corollary 1.2.6. *Let X be a scheme, and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The following are equivalent:*

1. *$\mathcal{F}$ is quasi-coherent;*
2. *there exists an affine open covering $X = \bigcup_{i \in I} U_i$ with $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some A_i-module M_i;*
3. *for any affine open $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U \cong \tilde{M}$ for some A-module M.*

Definition 1.2.7. Let $\mathsf{Qcoh}(\mathcal{O}_X)$ denote the category of quasi-coherent $\mathcal{O}_X$-modules.

Lemma 1.2.8. *The functor $\mathsf{Mod}_A \to \mathsf{Qcoh}(\mathcal{O}_{\operatorname{Spec} A})$ given by $M \mapsto \tilde{M}$ is an equivalence of categories, and the functor $\mathsf{Mod}_A \to \mathsf{Mod}(\mathcal{O}_X)$ given by $M \mapsto \tilde{M}$ is exact.*

Remark. It is possible to have an abelian category and a full subcategory which is also abelian, but the inclusion functor is not exact. For example, take the category of finitely generated modules over a DVR, and the subcategory of torsion-free such modules. This is why it is important in this lemma to carefully specify the targets of functors.

Proof. Taking stalks and localizing are exact. ☐

Lemma 1.2.9. *On a scheme X, the kernel and cokernel of a map of quasi-coherent sheaves are also quasi-coherent.*

Proof. We know now that quasi-coherence is a local property, so it is enough to show this for when X is affine. By the equivalence of categories, the map $\tilde{M} \to \tilde{N}$ is actually $\tilde{\phi}$ for some map $\phi \colon M \to N$ of A-modules. By exactness of $\sim$, we get

$$\operatorname{coker}(\tilde{\phi}) = \widetilde{\operatorname{coker}(\phi)}, \quad \ker(\tilde{\phi}) = \widetilde{\ker(\phi)}.$$ ☐

Remark. We have shown that if X is a scheme, then $\mathsf{Qcoh}(\mathcal{O}_X)$ is abelian and the inclusion $\mathsf{Qcoh}(\mathcal{O}_X) \hookrightarrow \mathsf{Mod}(\mathcal{O}_X)$ is exact.

Lemma 1.2.10. *Let X be a scheme. Let*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If two of the three are quasi-coherent, so is the third.

Proof. We will assume $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent. Again, we may assume $X = \operatorname{Spec} A$ is affine. Set $M_i := \Gamma(X, \mathcal{F}_i)$. So we get an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to \cdots$$

where $\cdots$ consists of higher cohomologies (since Γ is left-exact only). Apply $\sim$ to get

$$\begin{tikzcd} 0 \arrow[r] & \tilde{M}_1 \arrow[r] \arrow[d, "\cong"'] & \tilde{M}_2 \arrow[r] \arrow[d] & \tilde{M}_3 \arrow[d, "\cong"'] & \\ 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma, it is enough to show that $\tilde{M}_2 \twoheadrightarrow \tilde{M}_3$. Take $m \in M_2$ and set

$$I := \{f \in A : fm \in \operatorname{im}(M_2 \to M_3)\},$$

which is an ideal of A. We want to show $I = A$, so it suffices to show for all $\mathfrak{p} \in A$ prime, $I \not\subset \mathfrak{p}$. Let $x \in X$ correspond to $\mathfrak{p}$. Since $\mathcal{F}_2 \twoheadrightarrow \mathcal{F}_3$, there exists $x \in U \subset X$ open such that $m|_U$ comes from $s \in \mathcal{F}_2(U)$. We can assume $U \subset D(f)$ for $f \in A$ and $f \notin \mathfrak{p}$.

Claim: $f^N \in I$ for some $N \gg 0$. Let $z \in V(f)$, and say z corresponds to the prime $\mathfrak{q} \subset A$. Then we can find $g \in A \setminus \mathfrak{q}$ and $s' \in \mathcal{F}_2(V(g))$ lifting $m|_{D(g)}$. Set

$$m' := s|_{D(fg)} - s'|_{D(fg)} \in \mathcal{F}_1(D(fg)) = (M_1)_{fg}.$$

By clearing the denominator, there exists n (which depends on g) such that $m_1' := f^n m'$ is in $(M_1)_g$. Look at $f^n s$ on $D(f)$ and $f^n s' + m_1'$ on $D(g)$. These agree on $D(f) \cap D(g) = D(fg)$ as sections of $\mathcal{F}_2$, so by the sheaf condition they glue to give a section $\sigma \in \mathcal{F}_2(D(f) \cup D(g))$ mapping to $f^n m \in \mathcal{F}_3(D(f) \cup D(g))$. (Just check on both opens $D(f)$ and $D(g)$.) Since $V(f)$ is quasi-compact, we can find $g_1, \ldots, g_m \in A$ such that $V(f) \subset D(g_1) \cup \cdots \cup D(g_m)$, and there exists $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ mapping to $f^n m|_{D(f) \cup D(g_j)}$ in $\mathcal{F}_3$ and $\sigma_j|_{D(f)} = f^n s$ (independent of j).

Claim: $f^N \sigma_j$ satisfy the gluing condition for $N \gg 0$. Read the proof on the Stacks Project. But now we are done. $\square$

Example 1.2.11. $\mathsf{Qcoh}(\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}) = \mathsf{Ab}$.

Example 1.2.12. Let $X := \operatorname{Spec} k[x, y]$ where k is a field, and let $0 \in X$ be the point corresponding to (x, y). Let $U := X \setminus \{0\}$, so that $U = D(x) \cup D(y)$. Using the sheaf condition, we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(U) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(D(x)) \times \mathcal{O}_X(D(y)) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(D(x) \cap D(y)) \arrow[d, equal] \\ 0 \arrow[r] & k[x, y] \arrow[r] & k[x, y, 1/x] \times k[x, y, 1/y] \arrow[r] & k[x, y, 1/xy] \end{tikzcd}$$

where the map $\mathcal{O}_X(D(x)) \times \mathcal{O}_X(D(y)) \to \mathcal{O}_X(D(x) \cap D(y))$ is given by $(s, t) \mapsto s - t$.

1.3 Fiber products of schemes

Theorem 1.3.1. *The category of schemes has all fiber products.*

Proof. Johan: "do not read the proof of this!" □

Remark. The category of schemes has a final object, so it has products and therefore all limits.

Lemma 1.3.2. *If $X \to S \leftarrow Y$ are morphisms of affine schemes corresponding to ring maps $A \leftarrow R \to B$, then $X \times_S Y$ is affine too, corresponding to $A \otimes_R B$ (which is the pushout in the category of rings).*

Proof. For any locally ringed space T, we have

$$\begin{aligned}
\mathrm{Mor}_{\mathsf{LRS}}(T, X \times_S Y) &= \mathrm{Mor}_{\mathsf{LRS}}(T, X) \times_{\mathrm{Mor}_{\mathsf{LRS}}(T,S)} \mathrm{Mor}_{\mathsf{LRS}}(T, Y) \\
&= \mathrm{Mor}_{\mathsf{Rings}}(A, \mathcal{O}_T(T)) \times_{\mathrm{Mor}_{\mathsf{Rings}}(R,\mathcal{O}_T(T))} \mathrm{Mor}_{\mathsf{Rings}}(B, \mathcal{O}_T(T)) \\
&= \mathrm{Mor}_{\mathsf{Rings}}(A \otimes_R B, \mathcal{O}_T(T)) \\
&= \mathrm{Mor}_{\mathsf{LRS}}(T, \mathrm{Spec}(A \otimes_R B)).
\end{aligned}$$

By Yoneda's lemma, $X \times_S Y = \mathrm{Spec}(A \otimes_R B)$. (Since the category of schemes Sch is a full subcategory, this works for schemes too.) □

Lemma 1.3.3. *Consider a fiber product diagram*

$$\begin{array}{ccc}
X \times_S Y & \xrightarrow{q} & Y \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & S
\end{array}$$

in Sch. If $U \subset S$, $V \subset X$, and $W \subset Y$ are open such that $f(V) \subset U \supset g(W)$, then

$$V \times_U W = p^{-1}(V) \cap q^{-1}(W).$$

Proof. We have

$$\begin{aligned}
\mathrm{Mor}_{\mathsf{Sch}}(T, p^{-1}(V) \cap q^{-1}(W)) &= \{c\colon T \to X \times_S Y : (p \circ c)(T) \subset V,\, (q \circ c)(T) \subset W\} \\
&= \{(T \xrightarrow{a} X, T \xrightarrow{b} Y) : f \circ a = g \circ b,\, a(T) \subset V,\, b(T) \subset W\} \\
&= \mathrm{Mor}_{\mathsf{Sch}}(T, V) \times_{\mathrm{Mor}_{\mathsf{Sch}}(T,U)} \mathrm{Mor}_{\mathsf{Sch}}(T, W).
\end{aligned}$$

Again we are done by Yoneda's lemma. □

Remark. Choose affine open covers

$$S = \bigcup_{i \in I} U_i, \quad f^{-1}(U_i) = \bigcup_{j \in J_i} V_j, \quad g^{-1}(U_i) = \bigcup_{k \in K_i} W_k.$$

Then we construct $X \times_S Y$ using affine fiber products:

$$X \times_S Y = \bigcup_{i \in I} \bigcup_{(j,k) \in J_i \times K_i} = V_j \times_{U_i} W_k.$$

Corollary 1.3.4. *If $f\colon X \to S$ is an open immersion and $S' \to S$ is any morphism of schemes, then the base change $f'\colon X' \to S'$ is an open immersion. (Here $X' := X \times_S S'$.)*

Proof. Apply previous lemma. □

Remark. This corollary is usually stated as "open immersions are preserved by arbitrary base change."

Example 1.3.5. Let X be a scheme over $\mathbb{R}$, i.e. X has a morphism to $\mathrm{Spec}\,\mathbb{R}$. Then $X_{\mathbb{C}}$ means the "base change to $\mathbb{C}$" given by $X_{\mathbb{C}} = X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$, viewed as a scheme over $\mathbb{C}$.

1.4 Quasi-compact morphisms

Definition 1.4.1. A morphism of schemes $f\colon X \to Y$ is **quasi-compact (qc)** if the underlying map of topological spaces is quasi-compact, i.e. $f^{-1}(V)$ is quasi-compact for all $V \subset Y$ quasi-compact and open.

Lemma 1.4.2. *Let $f\colon X \to Y$ be a morphism of schemes. TFAE:*

1. *f is qc;*
2. *$f^{-1}(V)$ is qc for all $V \subset Y$ affine open;*
3. *there exists an affine open covering $Y = \bigcup_{i \in I} V_i$ such that $f^{-1}(V_i)$ is qc for all i.*

Proof. Let $Y = \bigcup_{i \in I} V_i$. Let $W \subset Y$ be qc open; we must show $f^{-1}(W)$ is qc. We can find an affine open covering $W = W_1 \cup \cdots \cup W_m$ such that for each $1 \leq j \leq m$, there exists $i \in I$ with $W_j \subset V_i$. Note that $f^{-1}W_j = W_j \times_{V_i} f^{-1}V_i$ by the previous lemma. Hence this has a finite affine open covering, i.e. is qc. Then $f^{-1}W = f^{-1}W_1 \cup \cdots \cup f^{-1}W_m$ is qc. □

Remark. This lemma shows that being qc is local on Y, i.e. it suffices to pick a point $y \in Y$ and find a quasi-compact open around y such that $f^{-1}(V)$ is quasi-compact.

Lemma 1.4.3. *Being qc is a property of morphisms which is preserved by arbitrary base change.*

Lemma 1.4.4. *Composition of qc morphisms is qc.*

1.5 Separation axioms

Definition 1.5.1. Let $f\colon X \to S$ be a morphism of schemes. Let $\Delta_{X/S}\colon X \to X \times_S X$ be the **diagonal morphism** defined (uniquely) by $\mathrm{pr}_i \circ \Delta_X = \mathrm{id}_X$ for $i = 1, 2$.

Definition 1.5.2. We say:

1. f is **separated** iff $\Delta_{X/S}$ is a closed immersion;
2. f is **quasi-separated (qs)** iff $\Delta_{X/S}$ is qc.

Lemma 1.5.3. *TFAE:*

1. *f is quasi-separated;*
2. *for all $U \subset S$ affine open and $V, W \subset X$ affine open with $f(V) \subset U \supset f(W)$, the intersection $V \cap W$ is qc;*
3. *something about open covers (literally what Johan wrote).*

Proof. We have $X \times_S X = \bigcup_{U,V,W} V \times_U W$, and

$$\Delta_{X/S}^{-1}(V \times_U W) = V \cap W.$$

So by the lemma characterizing qc morphisms, (1) and (2) are equivalent. □

1.6 Functoriality of quasi-coherent modules

For this section, let $\varphi\colon A \to B$ be a ring map and $f\colon X = \operatorname{Spec}(A) \to Y = \operatorname{Spec}(B)$ be the induced map. Let $\mathsf{Mod}_B \to \mathsf{Mod}_A$ given by $N \mapsto N_A$, where N_A is N thought of as an A-module using φ.

Lemma 1.6.1. $f_*(\tilde{N}) = \widetilde{N_A}$.

Proof. Compute that

$$f_*(\tilde{N})(D(a)) = \tilde{N}(f^{-1}D(a)) = \tilde{N}(D(\varphi(a))) = N_{\varphi(a)} = (N_A)_a. \qquad \square$$

Lemma 1.6.2. $f^*(\tilde{M}) = \widetilde{B \otimes_A M}$.

Proof. Take $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_Y}(f^*(\tilde{M}), \mathcal{G}) &= \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, f_*\mathcal{G}) = \operatorname{Hom}_{\mathsf{Mod}_A}(M, \Gamma(X, f_*\mathcal{G})) \\ &= \operatorname{Hom}_A(M, \mathcal{G}(Y)_A) = \operatorname{Hom}_B(B \otimes_A M, \mathcal{G}(Y)) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\widetilde{B \otimes_A M}, \mathcal{G}). \qquad \square\end{aligned}$$

Corollary 1.6.3. *For any morphism of schemes $f\colon X \to Y$, the pullback f^* preserves quasi-coherence.*

Example 1.6.4. Warning: this is not true for f_*. Take $X \coloneqq \coprod_{n\in\mathbb{Z}} \mathbb{A}^1_k \xrightarrow{f} \mathbb{A}^1_k =: Y$. Then $\mathcal{F} \coloneqq f_*\mathcal{O}_X$ is not quasi-coherent, because

$$\mathcal{F}(\mathbb{A}^1_k) = \prod_{n\in\mathbb{Z}} k[x], \quad \mathcal{F}(D(x)) = \prod_{n\in\mathbb{Z}} k[x, 1/x],$$

and it is not true that $\prod_{n\in\mathbb{Z}} k[x, 1/x]$ is the localization of $\prod_{n\in\mathbb{Z}} k[x]$ at x. For example, the element $(1/x, 1/x^2, 1/x^3, \ldots)$ is not in the localization.

Proposition 1.6.5. *For a qcqs morphism of schemes $f\colon X \to Y$, the pushforward f_* preserves quasi-coherence.*

Proof. Let $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_X)$. It suffices to show $f_*\mathcal{F}$ is quasi-coherent on an affine open covering. (We generally just say "we may assume Y is affine.") Note that:

1. f is qc implies $X = \bigcup_{i=1}^n U_i$ with U_i affine open;
2. f is qs implies $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ with U_{ijk} affine open.

Let $V \subset Y$ be open. Then the sheaf condition for $\mathcal{F}$ says

$$0 \to (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V) \to \bigoplus_{i=1}^n \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus_{i\le j} \bigoplus_k \mathcal{F}(f^{-1}V \cap U_{ijk}).$$

This is true for all V, and maps are compatible with restrictions. Write $f_{ijk} \coloneqq f|_{U_{ijk}}$. Then

$$0 \to f_*\mathcal{F} \to \bigoplus_{i=1}^n f_{i*}(\mathcal{F}|_{U_i}) \to \bigoplus_{i\le j,k} f_{ijk*}(\mathcal{F}|_{U_{ijk}}).$$

Pushforward on affine schemes and direct sum preserve quasi-coherence, so these last two terms are quasi-coherent. Now $f_*\mathcal{F}$ is the kernel of a map between quasi-coherent sheaves, and is therefore quasi-coherent. $\square$

Lemma 1.6.6. *Let X be a scheme. Then $\mathsf{Qcoh}(\mathcal{O}_X)$ has arbitrary direct sums (and arbitrary colimits), and they are the same as in $\mathsf{Mod}(\mathcal{O}_X)$.*

Lemma 1.6.7. *If $f\colon X \to Y$ is qcqs, then $f_*\colon \mathsf{Qcoh}(\mathcal{O}_X) \to \mathsf{Qcoh}(\mathcal{O}_Y)$ commutes with arbitrary direct sums (and arbitrary colimits).*

Definition 1.6.8. A scheme X is **quasi-affine** iff it is qc and isomorphic to an open subscheme of an affine scheme.

Lemma 1.6.9. *Let X be a scheme. Let $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$. Assume that:*

1. *$U_i \coloneqq X_{f_i} \coloneqq \{x \in X : f_i$ does not vanish at $x\}$ is affine;*
2. *$X = \bigcup_{i=1}^n U_i$.*

Then X is quasi-affine.

Proof. Set $A = \Gamma(X, \mathcal{O}_X)$. Consider the canonical map $c\colon X \to \operatorname{Spec} A =: S$. Then:

1. c is qc because X has a finite open cover such that $c|_{U_i}\colon U_i \to S$ is qc;
2. c is qs because X has an affine open cover such that $U_i \cap U_j = U_i \cap X_{f_j} = (U_i)_{f_j|_{U_i}}$, which is a principal open of the affine U_i.

Hence $c_*\mathcal{O}_X \in \mathsf{Qcoh}(\mathcal{O}_S)$, i.e.

$$c_*\mathcal{O}_X = \Gamma(\widetilde{S, c_*\mathcal{O}_X}) \text{ and } \Gamma(S, c_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = A.$$

Hence $c_*\mathcal{O}_X = \mathcal{O}_S$. Now $U_i = \operatorname{Spec} B_i$ where

$$B_i = \Gamma(U_i, \mathcal{O}_{U_i}) = \mathcal{O}_X(c^{-1}(D(f_i))) = (c_*\mathcal{O}_X)(D(f_i)) = \mathcal{O}_S(D(f_i)) = A_{f_i}.$$

So $U_i = \operatorname{Spec}(A_{f_i}) \to S = \operatorname{Spec}(A)$ is an open immersion, Then c is an isomorphism

$$X \to \bigcup_{i=1}^n D(f_i) \subset S.$$

□

Lemma 1.6.10. *Let X be a scheme. Let $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$. If X_{f_i} is affine and $f_1, \dots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$, then X is affine.*

Proof. The first (resp. second) condition here implies condition (1) (resp. condition (2)) in the previous lemma. So X is quasi-affine with $X \to \bigcup_{i=1}^n D(f_i) \subset S$. But with these stronger hypotheses, $\bigcup_{i=1}^n D(f_i) = \operatorname{Spec} A$, hence $X \cong \operatorname{Spec} A$. □

Definition 1.6.11. A morphism $f\colon X \to Y$ of schemes is **affine** iff $f^{-1}(V)$ is affine for all $V \subset Y$ affine open.

Lemma 1.6.12. *Let $f\colon X \to Y$ be a morphism of schemes. TFAE:*

1. *f is affine;*
2. *there exists an affine open cover $Y = \bigcup_j V_j$ such that $f^{-1}V_j$ is affine for every j.*

Proof. Look up the "affine communication lemma" in Vakil's notes to reduce the statement to:

Given $Y = \operatorname{Spec} B = \bigcup_{j=1}^m D(b_j)$ with $f^{-1}(D(b_j))$ affine, show that X is affine.

Now apply the previous lemma with $f_j \coloneqq f^\#(b_j)$. □

Definition 1.6.13. Let $f\colon X \to Y$ be a morphism of schemes. Then we say:

1. f is a **closed immersion** iff f is affine and for all $V \subset Y$ affine open, $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ is surjective;

2. f is **finite** iff in addition, $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ is a finite ring map.

Example 1.6.14. A standard example of a closed immersion is $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. A standard example of a finite morphism is $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where B is finite an A-module.

Lemma 1.6.15. *Compositions and base change of affine (resp. finite, or closed immersions) morphisms are affine (resp. finite, or closed immersions).*

Chapter 2

Sheaf cohomology

2.1 Preliminaries

Definition 2.1.1. An **abelian category** $\mathcal{A}$ is an additive category where there are kernels and cokernels, and coim = im. An object $I \in \mathrm{Ob}(\mathcal{A})$ is **injective** if given a mono $N \to M$, any map $N \to I$ extends to a map $M \to I$. An object is **projective** if the dual diagram holds. We say $\mathcal{A}$ **has enough injectives** (resp. **has enough projectives**) if every object M has a mono $M \hookrightarrow I$ into an injective object (resp. every object M has an epi $M \twoheadrightarrow I$ into a projective object).

Example 2.1.2. Let R be a ring and take $\mathcal{A} = \mathsf{Mod}_R$. Then there are enough projectives because every free module is projective. Similarly, there are enough injectives because for an R-module F, the R-module $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is injective.

Example 2.1.3. Let $(X, \mathcal{O}_X)$ be a ringed space and take $\mathcal{A} = \mathsf{Mod}(\mathcal{O}_X)$. Then there are enough injectives. One example of an injective is $\prod_{x \in X} i_{x*}(I_x)$, where I_x is an injective $\mathcal{O}_{X,x}$-module, for the immersion $i_x \colon (\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. **Warning**: $\mathsf{Mod}(\mathcal{O}_X)$ does not have enough projectives in general.

Definition 2.1.4. Given an object $M \in \mathrm{Ob}(\mathcal{A})$, an **injective resolution** is an exact complex

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

with I^n an injective object for all n. We will think of this as a map of complexes $M \to I^\bullet$. A **projective resolution** is an exact complex

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to M \to 0$$

with P^n a projective object for all n. We will think of this as a map of complexes $P^\bullet \to M$.

Definition 2.1.5. The **n-th cohomology** of a complex $K^\bullet$ is

$$H^n(K^\bullet) \coloneqq \frac{\ker(K^n \xrightarrow{d_K^n} K^{n+1})}{\mathrm{im}(K^{n-1} \xrightarrow{d_K^{n-1}} K^n)}.$$

Definition 2.1.6. A **map of complexes** $\alpha \colon K^\bullet \to L^\bullet$ is a collection of maps $\alpha^n \colon K^n \to L^n$ such that each square in

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r] \arrow[d, "\alpha^{n-1}"'] & K^n \arrow[r] \arrow[d, "\alpha^n"'] & K^{n+1} \arrow[r] \arrow[d, "\alpha^{n+1}"'] & \cdots \\
\cdots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

We say α is a **quasi-isomorphism (qis)** if for all n, the induced map $H^n(\alpha) \colon H^n(K^\bullet) \to H^n(L^\bullet)$ is an isomorphism.

Remark. If the original abelian category $\mathcal{A}$ is abelian, the category of complexes of objects in $\mathcal{A}$ is still abelian. Short exact sequences therefore still give rise to long exact sequences of cohomology.

Definition 2.1.7. Two maps $\alpha, \beta\colon K^\bullet \to L^\bullet$ are **homotopic** iff there exist $h^n\colon K^n \to L^{n-1}$ such that

$$(\alpha^n - \beta^n) = d_L^{n-1} \circ h^n + h^{n+1} \circ d_K^n.$$

A map of complexes is **nil-homotopic** if it is homotopic to 0. Homotopic maps define the same map on cohomology.

Lemma 2.1.8. *If $\mathcal{A}$ has enough injectives, then injective resolutions exist and are functorial up to homotopy. In other words, given $\varphi\colon M \to N$ and injective resolutions $M \to I^\bullet$ and $N \to J^\bullet$, then there is an arrow $I^\bullet \to J^\bullet$, unique up to homotopy, making the following diagram commute:*

$$\begin{array}{ccc} M & \longrightarrow & I^\bullet \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow \\ N & \longrightarrow & J^\bullet. \end{array}$$

The dual statement holds for projective resolutions.

2.2 Derived functors

Definition 2.2.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between them.

1. If $\mathcal{A}$ has enough injectives, define the **n-th right derived functor of** F by

   $$R^n F(M) \coloneqq H^n(F(I^\bullet))$$

   where $M \to I^\bullet$ is an injective resolution.

2. If $\mathcal{A}$ has enough projectives, define the **n-th left derived functor of** F by

   $$L^{-n} F(M) \coloneqq H^{-n}(F(P^\bullet))$$

   where $P^\bullet \to M$ is a projective resolution.

$R^n F, L^n F\colon \mathcal{A} \to \mathcal{B}$ are well-defined functors.

Lemma 2.2.2. *Given a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of $\mathcal{A}$ and injective resolutions $M_1 \to I_1^\bullet$ and $M_3 \to I_3^\bullet$, there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & I_3^\bullet & \longrightarrow & 0 \end{array}$$

such that $M_2 \to I_2^\bullet$ is also an injective resolution, and $0 \to I_1^n \to I_2^n \to I_3^n \to 0$ are short exact sequences (of injective objects, and therefore split) for all n.

Remark. Note that $F(I_1^\bullet) \to F(I_2^\bullet) \to F(I_3^\bullet)$ is term-wise split (because we can take F of the splitting). Hence in particular, this is a short exact sequence of complexes of B.

Corollary 2.2.3. *In the situation of the lemma, there exists a long exact cohomology sequence*

$$0 \to R^0F(M_1) \to R^0F(M_2) \to R^0F(M_3) \xrightarrow{\delta} R^1F(M_1) \to R^1F(M_2) \to R^1F(M_3) \xrightarrow{\delta} R^2F(M_1) \to \cdots.$$

The boundary maps $\delta\colon R^nF(M_3) \to R^{n+1}F(M_1)$ are well-defined.

Remark. $F = R^0F$ if F is left exact, and $F = L^0F$ if F is right exact. This is because left exact functors will give an exact sequence

$$0 \to F(M) \to F(I^0) \to F(I^1) \to \cdots$$

but $R^0F(M) = \ker(F(I^0) \to F(I^1)) = F(M)$ by exactness.

Example 2.2.4. Some common derived functors:

Functor	R/L	Derived functor
$\mathrm{Hom}_R(-, N)\colon \mathsf{Mod}_R \to \mathsf{Mod}_R^{\mathrm{op}}$	L	$\mathrm{Ext}^i_R(-, N)$
$M \otimes_R -\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$	L	$\mathrm{Tor}^R_i(M, -)$
$\Gamma(X, -)\colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}_{\Gamma(X,\mathcal{O}_X)}$	R	$H_i(X, -)$
$\Gamma(U, -)\colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}_{\Gamma(U,\mathcal{O}_X)}$	R	$H_i(U, -)$
$f_*\colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}(\mathcal{O}_Y)$	R	R^if_*
$\mathrm{Hom}_X(\mathcal{F}, -)\colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}_{\Gamma(X,\mathcal{O}_X)}$	R	$\mathrm{Ext}^i_X(\mathcal{F}, -)$
$\mathcal{H}om_X(\mathcal{F}, -)\colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}(\mathcal{O}_X)$	R	$\mathcal{E}xt^i_X(\mathcal{F}, -)$.

Remark. We need a good way to compare different ways of computing cohomology. For example, is $\mathrm{Tor}^i_R(M, N)$

$$(L^{-i}(M \otimes_R -))(N) \text{ or } (L^{-i}(- \otimes_R N))(M)?$$

Example 2.2.5. Let $\mathsf{Psh}(\mathsf{Ab})$ (resp. $\mathsf{Sh}(\mathsf{Ab})$) be the category of abelian sheaves (resp. abelian presheaves). Let $i\colon \mathsf{Sh}(\mathsf{Ab}) \to \mathsf{Psh}(\mathsf{Ab})$ be the forgetful functor. This is left exact but not right exact. We have

$$R^n i(\mathcal{F})(U) = \frac{\ker(iI^n(U) \to iI^{n+1}(U))}{\mathrm{im}(iI^{n-1}(U) \to iI^n(U))} = H^n(U, \mathcal{F})$$

because in $\mathsf{Psh}(\mathsf{Ab})$, the sections-over-$U$ functor is exact. Hence $R^n i(\mathcal{F})$ is the "n-th cohomology presheaf." After sheafification, it will become 0.

2.3 Spectral sequences

Disclaimer: this section is probably typo-ridden. Check all indices yourself before using anything from here!

Definition 2.3.1. A **double complex** of an abelian category $\mathcal{A}$ is a bigraded object $A^{\bullet,\bullet}$ with differentials $d_1^{p,q}\colon A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q}\colon A^{p,q} \to A^{p,q+1}$. Then there is an **associated total complex**

$$\mathrm{Tot}^n(A) \coloneqq \bigoplus_{p+q=n} A^{p,q}, \quad d \coloneqq \sum_{p+q=n} d_1^{p,q} + (-1)^p d_2^{p,q}.$$

(The sign is so that $d^2 = 0$.) Our goal is to compute $H^n(\mathrm{Tot}^\bullet(A))$.

Remark. The direct sum totalization is not the same as the product totalization, but from now on we will always assume $A^{\bullet,\bullet}$ lies in the first quadrant.

Definition 2.3.2. Let $\xi \in H^2(\mathrm{Tot}(A))$ be the class of $(x^{0,2}, x^{1,1}, x^{2,0})$. This is an example of what we will call a **zig-zag**. We can modify a zig-zag by adding elements of the form $d_2(y^{p,q-1}) + d_1(x^{p-1,q})$ (but note that each element we add affects two elements of the zig-zag). So $H^n(\mathrm{Tot}^\bullet(A)) = \{\text{zig-zags}\}/\{\text{allowed modifications}\}$. Define two **filtrations**

$$F_I^q H^\bullet \coloneqq \{\text{zig-zags represented by } (x^{0,n}, \ldots, x^{n,0}) : x^{i,j} = 0\ \forall j < q\}$$
$$F_{II}^p H^\bullet \coloneqq \{\text{zig-zags represented by } (x^{0,n}, \ldots, x^{n,0}) : x^{i,j} = 0\ \forall i < p\}.$$

There are two important cases:

$$
\begin{aligned}
F_I^n H^n &= \frac{\{x \in A^{0,n} : d_1(x) = d_2(x) = 0\}}{\{d_2(y) : y \in A^{0,n-1},\, d_1(y) = 0\}} \\
&= H^n((V^m := \ker(A^{0,m} \xrightarrow{d_1} A^{1,m}), d_2)) \\
F_{II}^n H^n &= H^n((U^m := \ker(A^{m,0} \xrightarrow{d_2} A^{m,1}), d_1)).
\end{aligned}
$$

We get maps of complexes $V^\bullet \to \mathrm{Tot}^\bullet(A) \leftarrow U^\bullet$. Trivially, if $F_I^n H^n = H^n$ (resp. $F_{II}^n H^n = H^n$), then $V^\bullet \to \mathrm{Tot}^\bullet(A)$ (resp. $U^\bullet \to \mathrm{Tot}^\bullet(A)$) is a qis. There are two types of **partial zig-zags of length r at** (p, q):

1. (type I) for $r = 0$, they are elements $x^{p,q} \in A^{p,q}$; for $r = 1$, they are elements $x^{p,q} \in A^{p,q}$ such that $d_1(x^{p,q}) = 0$; for $r = 2$, they are elements $x^{p-1,q+1} \xrightarrow{d_1}$ same $\leftarrow x^{p,q} \xrightarrow{d_1} 0$; we continue analogously as follows:

$$
\begin{tikzcd}
x^{p-r+1,q+r-1} \arrow[r] & \text{same} & & \\
& \ddots \arrow[u] \arrow[r] & \text{same} & \\
& & x^{p,q} \arrow[u] \arrow[r] & 0;
\end{tikzcd}
$$

2. (type II) they are elements

$$
\begin{tikzcd}
0 & & \\
x^{p,q} \arrow[u] \arrow[r] & \text{same} & \\
& \ddots \arrow[u] \arrow[r] & \text{same} \\
& & x^{p+r-1,q-r+1}. \arrow[u]
\end{tikzcd}
$$

The **(p,q)-spot in the r-th page of the first spectral sequence** is

$$
{}_I E_r^{p,q} := \frac{{}_I Z_r^{p,q}}{{}_I B_r^{p,q}},
$$

where ${}_I Z_r^{p,q}$ is the set of heads $x^{p,q}$ of partial zig-zags of type I of length r. Similarly, define the **second spectral sequence** ${}_{II} E_r^{p,q}$. Define the **differentials**

$$
\begin{aligned}
{}_I d_r^{p,q} &\colon {}_I E_r^{p,q} \to {}_I E_r^{p-r+1,q+r} \\
{}_{II} d_r^{p,q} &\colon {}_{II} E_r^{p,q} \to {}_{II} E_r^{p+r,q-r+1}
\end{aligned}
$$

Facts: ${}_I d_r^2 = {}_{II} d_r^2 = 0$, and

$$
{}_I E_{r+1}^{p,q} = \text{cohomology of } {}_I d_r \text{ at } E_r^{p,q}.
$$

Furthermore, $\mathrm{Gr}_{F_I}^p(H^n(\mathrm{Tot})) = E_r^{p,q}$ for $r \geq n$. Analogous facts hold for type II. Define

$$
{}_I B_0^{p,q} := 0, \quad {}_I B_1^{p,q} := \mathrm{im}(d_1 \colon {}_I E_1^{p-1,q} \to {}_I E_1^{p,q}) = \mathrm{im}({}_I d_0^{p-1,q}), \quad \ldots .
$$

Example 2.3.3. We do some examples for small r.

1. ($r = 0$) Here ${}_IE_0^{p,q} = A^{p,q} = {}_{II}E_0^{p,q}$.
2. ($r = 1$) Here ${}_IE_1^{p,q}$ is the cohomology of d_1 at (p, q), and ${}_{II}E_1^{p,q}$ is the cohomology of d_2 at (p, q). Note that if rows are exact except in degree 0, then $V^\bullet \to \mathrm{Tot}(A^{\bullet,\bullet})$ is qis. (Similarly, if columns are exact except in degree 0, then $U^\bullet \to \mathrm{Tot}(A^{\bullet,\bullet})$ is qis.)

Remark. Assume ${}_Id_r^{p,q}(x_{p,q}) = 0$ in $E_r^{p-r+1,q+r}$. This implies by definition that $d_2(x^{p-r+1,q+r-1}) = d_1(w^{p-r,q+r-1}) + d_2(y^{p-r+1,q+r-1})$ where $y^{p-r+1,q+r-1}$ is the tail of a partial zig-zag. This allows us to check that ${}_IE_{r+1}^{p,q}$ is indeed the cohomology of ${}_IE_r^{p,q}$ using ${}_Id_r$, as claimed.

Example 2.3.4. Let R be a ring, M, N be R-modules, and $P^\bullet \to N$ be a projective resolution, and $I^\bullet \to M$ be an inductive resolution. Take the double complex $A^{p,q} := \mathrm{Hom}_R(P^{-q}, I^p)$, with induced differential. Since P^{-q} is projective, the rows $\mathrm{Hom}_R(P^{-q}, I^\bullet)$ are exact except in degree 0, and

$$V^\bullet = \ker(\mathrm{Hom}_R(P^{-\bullet}, I^0) \to \mathrm{Hom}_R(P^{-\bullet}, I^1)) = \mathrm{Hom}_R(P^{-\bullet}, N).$$

Similarly, $\mathrm{Hom}(M, I^\bullet) = U^\bullet$. Hence their cohomologies, which are Exts computed by different resolutions, are isomorphic by the example above. In fact, this isomorphism is functorial, because the double complex construction is functorial. Analogously, $\mathrm{Tor}_R(M, N)$ can be computed by resolving either variable.

Example 2.3.5. Let $f\colon X \to Y$ be a continuous map of compact topological manifolds. Then there is a **Leray spectral sequence**

$$E_2^{p,q} = H^p(Y, R^qf_*\underline{\mathbb{Q}}) \Rightarrow H^{p+q}(X, \underline{\mathbb{Q}}).$$

This may not degenerate, but, for example, taking cohomology does not change the Euler characteristic (assuming bounded cohomological dimension). Hence this allows us to show $\chi(X)$ is made up of Euler characteristics of certain sheaves $R^qf_*\underline{\mathbb{Q}}$ on Y.

2.4 Čech cohomology

Definition 2.4.1. Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be an open covering of X and $\mathcal{F}$ be an abelian sheaf on X. Let $U_{i_0\cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$. Then there is a **Čech complex** with terms

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p}) \ni \alpha = (\alpha_{i_0\cdots i_p})$$

and boundary map $d\colon \check{\mathcal{C}}^p \to \check{\mathcal{C}}^{p+1}$

$$d(\alpha)_{i_0\cdots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0\cdots\widehat{i_j}\cdots i_p}|_{U_{i_0\cdots i_{p+1}}}.$$

The **p-th Čech cohomology** is $\check{H}^p(\mathcal{U}, \mathcal{F}) := H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$.

Remark. The natural category for which we define Čech cohomology is presheaves, since the only thing we need are the restriction maps. Also, $\mathcal{F}$ is a sheaf implies $\mathcal{F}(X) = \check{H}^0(\mathcal{U}, \mathcal{F})$. (It is not an if and only if because we only took an open cover of X, instead of an arbitrary open set U.)

Remark. The problem with Čech cohomology is that a short exact sequence of (abelian) sheaves does not give rise to a short exact sequence of Čech complexes. Also, refinement of open covers does not induce a canonical map of Čech complexes, because there is a choice involved. So we cannot directly take a colimit over open covers of Čech cohomology.

Lemma 2.4.2. *If $\mathcal{I}$ is an injective abelian sheaf or $\mathcal{O}_X$-module, then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for $p > 0$.*

Proof. Consider the complex of sheaves

$$K^\bullet_{\mathcal{U}} := \cdots \to \bigoplus_{i_0<i_1} j_{i_0 i_1 !}\underline{\mathbb{Z}} \to \bigoplus_{i_0} j_{i_0 !}\underline{\mathbb{Z}}$$

where $j_{i_0\cdots i_p}\colon U_{i_0\cdots i_p} \to X$ is the inclusion. (Recall for $j\colon U \to X$ an inclusion, $j_!$ is left adjoint to j^{-1}, i.e. $\operatorname{Hom}_U(\mathcal{G}, \mathcal{F}|_U) = \operatorname{Hom}_X(j_!\mathcal{G}, \mathcal{F})$ and $(j_!\mathcal{G})_x = \mathcal{G}_x$ for $x \in U$ and is 0 otherwise.) Then $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) = \operatorname{Hom}(K^\bullet_{\mathcal{U}}, \mathcal{F})$ by adjunction. It suffices to show $K^\bullet_{\mathcal{U}}$ is exact and $H^0(K^\bullet_{\mathcal{U}}) = \underline{\mathbb{Z}}$. Since Hom is exact on injectives, the Čech complex is exact and we are done.

To check $K^\bullet_{\mathcal{U}}$ is exact, it suffices to check on stalks. Say $x \in U_1, \ldots, U_p$ but $x \notin U_{i+1}, \ldots, U_n$. Then

$$(K^\bullet_{\mathcal{U}})_x\colon \ \cdots \to \bigoplus_{1\le i_0<i_1\le p} \mathbb{Z} \to \bigoplus_{i_0=1,\ldots,p} \mathbb{Z}$$

is exact, by viewing it as a Koszul complex. ☐

Theorem 2.4.3 (Čech to sheaf cohomology spectral sequence)**.** *For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules (or abelian sheaf), there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H^q}(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

where $\underline{H^q}(\mathcal{F})$ denotes the presheaf $U \mapsto H^q(U, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and look at the double complex $A^{p,q} := \mathcal{C}^p(\mathcal{U}, \mathcal{I}^q)$. By the lemma, the rows $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^q)$ are exact in positive degree, and in degree 0 we get $\Gamma(X, \mathcal{I}^q)$, so

$$\Gamma(X, \mathcal{I}^\bullet) \to \operatorname{Tot}(A^{\bullet,\bullet})$$

is a qis. By definition, $\Gamma(X, \mathcal{I}^\bullet)$ computes sheaf cohomology of $\mathcal{F}$, and therefore so does $\operatorname{Tot}(A^{\bullet,\bullet})$. But the columns are not exact, and so the (vertical) spectral sequence is

$$\begin{aligned} E_1^{p,q} &= \prod_{i_0<\cdots<i_p} H^q(U_{i_0\cdots i_p}, \mathcal{F}) = \mathcal{C}^p(\mathcal{U}, \underline{H^q}(\mathcal{F})) \\ E_2^{p,q} &= \check{H}^p(\mathcal{U}, \underline{H^q}(\mathcal{F})). \end{aligned}$$

☐

Corollary 2.4.4. *If $H^q(U_{i_0\cdots i_p}, \mathcal{F}) = 0$ for all $q > 0$ and i_k, then $H^n(X, \mathcal{F}) = \check{H}^n(\mathcal{U}, \mathcal{F})$.*

Corollary 2.4.5. *Let $\mathcal{B}$ be a collection of opens and* Cov *be a set of open coverings of opens of X. Assume:*

1. *if $\mathcal{U} \in \mathrm{Cov}$, then $\mathcal{U}\colon U = \bigcup_{i=1}^n U_i$ and $U, U_i, \ldots, U_{i_0\cdots i_p} \in \mathcal{B}$;*
2. *for $U \in \mathcal{B}$, the coverings of U in* Cov *are cofinal;*
3. *$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$ and $\mathcal{U} \in \mathrm{Cov}$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $U \in \mathcal{B}$ and $p > 0$.

Lemma 2.4.6. *Let X be affine and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^n(X, \mathcal{F}) = 0$ for $n > 0$.*

Proof. Let $\mathcal{B}$ be the collection of affine opens $\operatorname{Spec} A$ of X. Let Cov be the standard coverings $\bigcup_{i=1}^m D(f_i)$ of affine opens $\operatorname{Spec} A$ of X. So to apply the previous corollary, it suffices to check $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. But $\mathcal{F}|_{\operatorname{Spec} A} = \tilde{M}$ for some A-module M, and therefore it suffices to show the complex

$$K^\bullet\colon 0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0<i_1} M_{f_{i_0} f_{i_1}} \to \cdots,$$

of A-modules is exact in positive degree. It is enough to show exactness after localizing at some prime $\mathfrak{p}$, i.e. if f_q is a unit for some q. We will show $K^\bullet$ is homotopic to 0. The homotopy is defined by

$$h\colon K^{p+1} \to K^p, \quad h(s)_{i_0\cdots i_p} := \begin{cases} s_{1 i_0\cdots i_p} & 1 < i_0 \\ 0 & \text{otherwise.} \end{cases}$$

This definition makes sense because $M_{f_1 f_{i_0}\cdots f_{i_p}} = M_{f_{i_0}\cdots f_{i_p}}$ when f_1 is a unit. Now compute $(dh + hd)(s)_{i_0\cdots i_p} = s_{i_0\cdots i_p}$, and we are done. □

Corollary 2.4.7. *If X is a scheme which has a finite affine open cover $X = \bigcup_{i=1}^n U_i$ such that $U_{i_0\cdots i_p}$ is affine for all p, then $\check{H}^n(\mathcal{U}, \mathcal{F}) = H^n(X, \mathcal{F})$ for all $\mathcal{F}$ quasi-coherent.*

Remark. For a separated scheme, the intersection of affines is affine. However there are interesting non-separated schemes where the intersection can still be affine.

2.5 Cohomology of projective space

Johan: "this is the most important calculation in algebraic geometry."

Theorem 2.5.1. *Let A be a ring. Then*

$$H^p(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = \begin{cases} A[T_0, \ldots, T_n]_d & p = 0 \\ 0 & p \neq 0, n \\ E_d & p = n \end{cases}$$

where $E = A[T_0, \ldots, T_n][1/T_0 \cdots T_n] / \sum_i A[T_0, \ldots, T_n][1/T_0 \cdots \hat{T}_i \cdots T_n]$ is a graded module.

Proof. We have $\mathbb{P}^n_A = D_+(T_0) \cup \cdots \cup D_+(T_n)$, and the intersection of two D_+s is another D_+, so all intersections are affine. Let $\mathcal{U}$ be this covering. So $H^p(\mathbb{P}^n_A, \mathcal{O}(d)) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}(d)))$. Call the Čech complex $\check{\mathcal{C}}^\bullet$. Recall that $\mathcal{O}(d) = A[T_0, \ldots, T_n](d)^\sim$. Unwinding definitions,

$$\check{\mathcal{C}}^\bullet_d \colon \left(\prod_{i_0} A[T_0, \ldots, T_n][1/T_{i_0}]\right)_d \to \prod_{i_0 < i_1} \left(A[T_0, \ldots, T_n][1/T_{i_0} T_{i_1}]\right)_d \to \cdots .$$

Now we do a trick from Hartshorne: take the direct sum over all d. Set

$$K^\bullet := \bigoplus_{d \in \mathbb{Z}} \check{\mathcal{C}}^\bullet_d \colon \prod_{i_0} A[T_0, \ldots, T_d][1/T_{i_0}] \to \prod_{i_0 < i_1} A[T_0, \ldots, T_d][1/T_{i_0} T_{i_1}] \to \cdots$$

where all the maps are the obvious ones and commute with the $\mathbb{Z}^{n+1}$-grading. It is clear that

$$H^0(K^\bullet) = A[T_0, \ldots, T_n], \quad H^n(K^\bullet) = E.$$

It suffices now to show $H^i(K^\bullet) = 0$ for $0 < i < n$. We do this by induction on n. Consider the short exact sequence

$$0 \to K^\bullet \xrightarrow{T_n} K^\bullet \to K^\bullet / T_n K^\bullet \to 0.$$

Note that $K^\bullet / T_n K^\bullet$ is exactly $K^\bullet$ for $n - 1$. By induction we know its cohomology. So we want to show

$$0 \to H^0(K^\bullet) \xrightarrow{T_n} H^0(K^\bullet) \to H^0(K^\bullet / T_n K^\bullet) \to 0, \quad 0 \to H^{n-1}(K^\bullet / T_n K^\bullet) \to H^n(K^\bullet) \xrightarrow{T_n} H^n(K^\bullet) \to 0$$

are exact. For example, take $1/T_0 \cdots T_{n-1}$ in $H^{n-1}(K^\bullet/T_n K^\bullet)$. Via the boundary map, it becomes $1/T_0 \cdots T_n \in H^n(K^\bullet)$, by tracing out the snake lemma:

$$\begin{array}{ccccc} & & K^{n-1} & \longrightarrow & K^{n-1}/T_n K^{n-1} \\ & & \partial \downarrow & & \\ K^n & \xrightarrow{T_n} & K^n & & \end{array}$$

and noting that the boundary ∂ leaves the term unchanged. Conclusion: $H^i(K^\bullet) \xrightarrow{T_n} H^i(K^\bullet)$ is an isomorphism for $i \neq 0, n$. Think of $H^i(K^\bullet)$ as a module over $A[T_0, \ldots, T_n]$. Hence $H^i(K^\bullet) \cong H^i(K^\bullet)_{T_n}$. Localization is exact, so this is also $H^i((K^\bullet)_{T_n})$. But $(K^\bullet)_{T_n}$ is the Čech complex for $\mathcal{O}$ on $\operatorname{Spec} A[T_0, \ldots, T_n][1/T_n]$ covered by $D(T_0), \ldots, D(T_n)$, which we have seen is exact because it is a quasi-coherent sheaf over an affine scheme. □

2.6 Coherent $\mathcal{O}_X$-modules

Definition 2.6.1. Let $f\colon X \to Y$ be a morphism of schemes.

1. If Y is affine, we say X is **projective over** Y iff there exists $n \geq 0$ and a closed immersion $X \to \mathbb{P}^n_Y$ over Y.
2. We say f is **locally projective** iff for all $V \subset Y$ affine open, $f^{-1}(V)$ is projective over V.

Lemma 2.6.2. *If $a\colon W \to Z$ is an affine morphism of schemes, then $H^n(W, \mathcal{F}) = H^n(Z, a_*\mathcal{F})$ for any $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_W)$.*

Proof. Show that $R^p a_* \mathcal{F} = 0$ for $p > 0$. Then conclude immediately by the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^q a_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$. Recall that $R^p a_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^p(a^{-1}V, \mathcal{F}|_{a^1 V})$. But since a is affine, we can pick V such that $a^{-1}V$ is affine, so that stalks of $R^p a_* \mathcal{F}$ are 0. □

Example 2.6.3. If $X = \operatorname{Spec} k$, we have $\mathsf{Mod}(\mathcal{O}_X) = \mathsf{Qcoh}(\mathcal{O}_X) = \mathsf{Vect}_k$. But an interesting subcategory is the subcategory of finite-dimensional vector spaces, and this corresponds to $\mathsf{Coh}(\mathcal{O}_X)$, the category of coherent $\mathcal{O}_X$-modules.

Definition 2.6.4. A scheme X is **locally Noetherian** iff every affine open is of the form $\operatorname{Spec} A$ for A Noetherian. Equivalently, there exists an affine open cover by spectra of Noetherian rings.

Lemma 2.6.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. TFAE:*

1. *$\mathcal{F}$ is coherent;*
2. *$\mathcal{F}$ is quasi-coherent and for any affine open $U = \operatorname{Spec} A$, we have $\mathcal{F}|_U = \tilde{M}$ where M is a finite A-module;*
3. *$\mathcal{F}$ is quasi-coherent and there exists an affine open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} = \tilde{M}_i$ with M_i a finite $\mathcal{O}_X(U_i)$-module.*

Definition 2.6.6. The category of coherent $\mathcal{O}_X$-modules is denoted $\mathsf{Coh}(\mathcal{O}_X)$, and is a full subcategory of $\mathsf{Qcoh}(\mathcal{O}_X)$, which in turn is a full subcategory of $\mathsf{Mod}(\mathcal{O}_X)$.

Theorem 2.6.7. *Let $f\colon X \to Y$ be a proper morphism (e.g. locally projective) of schemes with Y (and therefore X, by finite type) locally Noetherian. Then*

$$(\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)) \implies (R^p f_* \mathcal{F} \in \mathsf{Coh}(\mathcal{O}_Y)).$$

Example 2.6.8. For example, let $Y = \operatorname{Spec} k$ for k a field. Let $\mathcal{F}$ on $\mathbb{P}^n_k$ be coherent. Then $\dim_k H^p(\mathbb{P}^n_k, \mathcal{F}) < \infty$, because H^p is the pushforward $R^p f_* \mathcal{F}$ for $f\colon \mathbb{P}^n_k \to Y$.

2.7 Cohomology of coherent sheaves on Proj

Definition 2.7.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. An **invertible $\mathcal{O}_X$-module** is an $\mathcal{O}_X$-module $\mathcal{L}$ which is locally free of rank 1. Equivalently, the functor $-\otimes_{\mathcal{O}_X} \mathcal{L} \colon \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}(\mathcal{O}_X)$ is an equivalence of categories. The set of isomorphism classes of invertible $\mathcal{O}_X$-modules is denoted $\operatorname{Pic} X$, called the **Picard group**, and is an abelian group under the group law $(\mathcal{L}, \mathcal{N}) \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$. An invertible $\mathcal{O}_X$-module is **trivial** if it is isomorphic to $\mathcal{O}_X$.

Let A be a graded ring, and $X = \operatorname{Proj} A$. Let $\mathcal{O}_X(d) = \widetilde{A(d)}$. Shorthand: write $\mathcal{F}(d) \coloneqq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$.

Lemma 2.7.2. *If A is generated in degree 1 over degree 0, then $\mathcal{O}_X(d)$ is invertible and $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) = \mathcal{O}_X(d+e)$.*

Remark. In general, there is a map $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) \to \mathcal{O}_X(d+e)$, but it is not always an isomorphism.

Proof. Write $X = \bigcup_{f \in A_1} D_+(f)$, so it suffices to restrict to $D_+(f)$. Recall that $\mathcal{O}_X(d)|_{D_+(f)} = A(d)_f = (A_f)_d$ as $A_{(f)}$-modules, and there exists an isomorphism of $A_{(f)}$-modules $(A_f)_0 \cong (A_f)_d$ given by $x \mapsto f^d x$. (This is because $f \in A_1$ and is invertible.) This map is compatible with tensor product. ☐

Lemma 2.7.3. *If A_+ is finitely generated as an ideal, then the functor $M \mapsto \tilde{M}$ from graded A-modules to $\mathsf{Qcoh}(\mathcal{O}_X)$ is essentially surjective.*

Proof sketch. Key fact: given $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_X)$, taking $M = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{F}(d))$ gives a graded A-module such that there is a canonical map $\tilde{M} \to \mathcal{F}$ which is an isomorphism. The A-module structure on M comes from the composition

$$A_d \times \Gamma(X, \mathcal{F}(e)) \to \Gamma(X, \mathcal{O}_X(d)) \times \Gamma(X, \mathcal{F}(e)) \to \Gamma(X, \mathcal{F}(e) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)) \to \Gamma(X, \mathcal{F}(d+e)).$$

The proof of this fact goes by showing $\mathcal{F}(D_+(f)) = M_{(f)}$, with this choice of M. This in turn is very similar to the same statement for quasi-coherents on affines. ☐

Lemma 2.7.4. *If A_+ is a finitely generated ideal and $\mathcal{F} \in \mathsf{Qcoh}(X)$ is of finite type, then there exists $r \geq 0$ and integers $d_1, \ldots, d_r \in \mathbb{Z}$ and a surjection $\bigoplus_{j=1}^r \mathcal{O}_X(d_j) \twoheadrightarrow \mathcal{F}$.*

Proof. Write $\mathcal{F} = \tilde{M}$ by the previous lemma. Then $\mathcal{F}$ finite type and X quasi-compact implies there exists $N \subset M$ a finite graded A-module such that $\tilde{N} = \mathcal{F}$. Now take a surjection $\bigoplus_{j=1}^r A(d_j) \twoheadrightarrow N$. ☐

Theorem 2.7.5. *Let A_0 be a Noetherian ring, $A \coloneqq A_0[T_0, \ldots, T_n]$, and let $X \coloneqq \mathbb{P}^n_{A_0} \coloneqq \operatorname{Proj} A$. Take a coherent sheaf $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$. Then:*

1. *$H^i(X, \mathcal{F})$ is a finite A_0-module;*
2. *$H^i(X, \mathcal{F}(d)) = 0$ for $i > 0$ when $d \gg 0$;*
3. *$H^i(X, \mathcal{F}(d)) = 0$ for $i \geq n+1$ and all d;*
4. *$\bigoplus_{d \geq 0} H^0(X, \mathcal{F}(d))$ is a finite graded A-module.*

Proof. (3) is clear by Čech cohomology. In particular, (3) implies (1) and (2) hold for $i \geq n+1$. We induct downward on i. Choose a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus_{j=1}^r \mathcal{O}_X(d_j) \to \mathcal{F} \to 0.$$

The kernel of a map of coherent sheaves is coherent. Twist the sequence by d:

$$0 \to \mathcal{G}(d) \to \bigoplus_{j=1}^r \mathcal{O}_X(d_j + d) \to \mathcal{F}(d) \to 0.$$

By the cohomology computation over $\mathbb{P}^n_A$ previously, statements (1) and (2) are true for the middle term. The long exact sequence is

$$\to \cdots \to H^i(\bigoplus \mathcal{O}_X(d_j + d)) \to H^i(\mathcal{F}(d)) \to H^{i+1}(\mathcal{G}(d)) \to \cdots .$$

The induction hypothesis says $H^{i+1}(\mathcal{G}(d))$ is finite, so $H^i(\mathcal{F}(d))$ is finite too. Similarly, the first and last term vanish for $d \gg 0$, so does the middle one. Finally, for (4), combine (1) and (2) to show $H^0(X, \mathcal{F}(d))$ is finite for $d \gg 0$, and only finitely many modules remain, so the entire sum is a finite graded A-module. □

Theorem 2.7.6. *If A is a Noetherian graded ring generated by A_1 over A_0, then the theorem above holds for $X = \operatorname{Proj} A$ too.*

Proof. Choose a surjection $B := A_0[T_0, \ldots, T_n] \twoheadrightarrow A$, which gives an embedding $\operatorname{Proj} A \overset{i}{\hookrightarrow} \mathbb{P}^n_B$. Then we have $H^i(\operatorname{Proj} A, \mathcal{F}) = H^i(\mathbb{P}^n_B, i_*\mathcal{F})$, and the previous theorem applies. □

Remark. We showed earlier that $M \mapsto \tilde{M}$ has a right inverse $\mathcal{F} \mapsto \bigoplus_{d \geq 0} \Gamma(X, \mathcal{F}(d))$. This is also a left inverse once we take into account torsion modules, which get killed. In other words,

$$\mathsf{GrMod}_{fg}(A_0[T_0, \ldots, T_n])/\text{torsion modules} \cong \mathsf{Coh}(\mathcal{O}_{\mathbb{P}^n_{A_0}})$$

where GrMod_{fg} means finitely generated graded modules. (For example, $\mathsf{Mod}(k[x])/\text{tors} \cong \mathsf{Mod}(k(x)) \cong \mathsf{Vect}(k(x))$.)

2.8 Higher direct images

Lemma 2.8.1 (Induction principle)**.** *Let $\mathcal{P}$ be a property of qcqs schemes. Assume*

1. *$\mathcal{P}$ holds for affine schemes, and*
2. *if $X = U \cup V$ with $X, U, V, U \cap V$ qcqs and U affine and $\mathcal{P}$ holds for $U, V, U \cap V$, then $\mathcal{P}$ holds for X.*

Then $\mathcal{P}$ holds for all qcqs schemes.

Proof. In this proof, all schemes are qcqs. Write $\mathcal{P}(X)$ to mean "$\mathcal{P}$ holds for X." We first show $\mathcal{P}$ holds for all separated X by induction on $t(X)$, the minimal number of affines needed to cover X. Base case: if $t(X) = 0, 1$, this is obvious. For the induction step, write $X = U \cup V$ with U affine and $t(V) < t(X)$. Suppose $V = V_1 \cup \cdots \cup V_{t(V)}$ is an affine open cover. Then $U \cap V = (U \cap V_1) \cap \cdots \cap (U \cap V_{t(V)})$, and since X is separated, this is an affine open cover. Clearly $t(U \cap V) \leq t(V)$. By the induction hypothesis, we have the properties $\mathcal{P}(U), \mathcal{P}(V), \mathcal{P}(U \cap V)$. Hence we get $\mathcal{P}(X)$.

Now we show $\mathcal{P}$ holds for all X (not necessarily separated) by induction on $t(X)$ again. Write $X = U \cup V$ with U affine and $t(V) < t(X)$. Then $U \cap V \subset U$ is open, and hence is separated. Hence we get $\mathcal{P}(U)$ by the affine case, $\mathcal{P}(V)$ by induction, and $\mathcal{P}(U \cap V)$ by the separated case. Again we get $\mathcal{P}(X)$. □

Proposition 2.8.2. *Let $f \colon X \to Y$ be a qcqs morphism of schemes. Then $R^i f_*$ sends $\mathsf{Qcoh}(\mathcal{O}_X)$ into $\mathsf{Qcoh}(\mathcal{O}_Y)$.*

Proof. The question is local on Y and so we may assume Y is affine. We will apply the induction principle with the property $\mathcal{P}(X)$ given by "the proposition holds for $f \colon X \to Y$ with Y affine." We check the necessary properties.

1. If X is affine, then $f \colon X \to Y$ is affine and therefore $R^i f_* \mathcal{F} = 0 \in \mathsf{Qcoh}(\mathcal{O}_Y)$ for $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_X)$ and $i > 0$.

2. If $X = U \cup V$, then let $a \coloneqq f|_U$ and $b \coloneqq f|_V$ and $c \coloneqq f|_{U\cap V}$. Then Mayer–Vietoris gives a long exact sequence

$$0 \to f_*(\mathcal{F}) \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*(\mathcal{F}) \to R^1 a_*(\mathcal{F}|_U) \oplus R^1 b_*(\mathcal{F}|_V) \to \cdots .$$

This long exact sequence gives short exact sequences

$$0 \to \operatorname{coker}(R^{i-1}a_* \oplus R^{i-1}b_* \to R^i c_*) \to R^i f_* \to \ker(R^i a_* \oplus R^i b_* \to R^i c_*) \to 0,$$

but Qcoh is closed under ker and coker and extensions, so $R^i f_*(\mathcal{F}) \in \mathsf{Qcoh}(\mathcal{O}_Y)$. □

Corollary 2.8.3. *Suppose $f\colon X \to Y$ is qcqs, Y affine, and $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_X)$. Then*

1. $H^i(X, \mathcal{F}) = H^0(Y, R^i f_*\mathcal{F})$*, and*
2. $R^i f_*\mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$*.*

Proof. Statement (2) is a consequence of the proposition and statement (1), so it suffices to prove (1). Use the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$. By the proposition, $R^q f_*\mathcal{F}$ are quasi-coherent, and by hypothesis Y is affine. So there is only one non-zero column in $E_2^{p,q}$ and the spectral sequence degenerates to give (1). □

Corollary 2.8.4. *If $f\colon X \to Y$ is locally projective and Y is locally Noetherian and $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$. Then $R^i f_*\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_Y)$.*

Proof. Being coherent is a local property, so there is an immediate reduction $Y = \operatorname{Spec}(A_0)$, where A_0 is a Noetherian ring, and $X \overset{i}{\hookrightarrow} \mathbb{P}^n_{A_0}$ is a closed immersion. Then $R^i f_*\mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$, so it suffices to show $H^i(X, \mathcal{F})$ is a finite A_0-module. This is true because $H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_{A_0}, i_*\mathcal{F})$, which is finite as the cohomology of a coherent sheaf over Proj. (Pushforwards of coherents along closed immersions are coherent, and this is easy to prove.) □

Example 2.8.5. Take $\mathbb{P}^1_k$. Let $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O} \oplus \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(-2)$. Fact: if X is qcqs, then $H^i(X, -)$ commutes with all direct sums. Then $\dim H^0(\mathbb{P}^1_k, \mathcal{F}) = \dim H^1(\mathbb{P}^1_k, \mathcal{F}) = \infty$. So the above results do not hold for quasi-coherent sheaves.

Definition 2.8.6. Let X be proper over a field k. If $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$, then the **Euler characteristic** of $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Lemma 2.8.7. *On $\mathbb{P}^n_k$, the function $d \mapsto \chi(\mathbb{P}^n_k, \mathcal{F}(d))$ is a numerical polynomial.*

Proof sketch. We induct on $\dim \operatorname{supp} \mathcal{F}$. Pick $s \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ cutting $\operatorname{supp}\mathcal{F}$ by a hyperplane. Then $\mathcal{F} \xrightarrow{s} \mathcal{F}(1)$ has a kernel $\mathcal{K}$ and cokernel $\mathcal{Q}$ that are both coherent, and have lower dimension. By induction, we get $d \mapsto \chi(\mathbb{P}^n, \mathcal{K}(d))$ and $d \mapsto \chi(\mathbb{P}^n, \mathcal{Q}(d))$ are numerical polynomials. By the additivity of $\chi(\mathbb{P}^n, -)$, we get

$$0 = \chi(\mathbb{P}^n, \mathcal{K}(d)) - \chi(\mathbb{P}^n, \mathcal{F}(d)) + \chi(\mathbb{P}^n, \mathcal{F}(d+1)) - \chi(\mathbb{P}^n, \mathcal{Q}(d)).$$

Hence $d \mapsto \chi(\mathbb{P}^n, \mathcal{F}(d))$ is a numerical polynomial. □

Corollary 2.8.8. *The function $d \mapsto \dim H^0(\mathbb{P}^n_k, \mathcal{F}(d))$ is a numerical polynomial.*

Proof. By vanishing, $\dim H^0(\mathbb{P}^n_k, \mathcal{F}(d)) = \chi(\mathbb{P}^n_k, \mathcal{F}(d))$ for $d \gg 0$. □

2.9 Serre duality

Let A be a Noetherian ring. Let X be a scheme projective over A, i.e. there is a closed immersion $X \hookrightarrow \mathbb{P}^n$ over $\operatorname{Spec} A$. Let d be the maximum dimension of a fiber of $f\colon X \to \operatorname{Spec} A$. Fact: if $\mathcal{F} \in \mathsf{Qcoh}(\mathcal{O}_X)$, then $H^i(X, \mathcal{F}) = 0$ for $i \geq d+1$. (This is not an easy fact; we proved it for $X = \mathbb{P}^d_A$.) We are interested in top-dimensional cohomology $H^d(X, -)\colon \mathsf{Coh}(\mathcal{O}_X) \to \mathsf{Coh}(A)$. This is a right exact functor, since given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, we get $H^d(X, \mathcal{F}_1) \to H^d(X, \mathcal{F}_2) \to H^d(X, \mathcal{F}_3) \to 0$ by the fact. However instead of looking at this functor, we look at the contravariant functor $\mathcal{F} \mapsto H^d(X, \mathcal{F})^\vee$. This functor is "bad": it forgets stuff, is not exact, and does not even have finite cohomological dimension.

Definition 2.9.1 (Bad definition)**.** In this situation, a **relative dualizing module** $\omega_{X/A}$ is an object of $\mathsf{Coh}(\mathcal{O}_X)$ which represents this functor $H^d(X, -)^\vee$, such that there are isomorphisms

$$\operatorname{Hom}_X(\mathcal{F}, \omega_{X/A}) \xrightarrow{\alpha_{\mathcal{F}}} H^d(X, \mathcal{F})^\vee$$

functorial in $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$. In particular, we can define the **trace map**

$$\operatorname{tr}_{X/A} \coloneqq \alpha_{\omega_{X/A}}(\operatorname{id}_{\omega_{X/A}})\colon H^d(X, \omega_{X/A}) \to A.$$

Remark. By Yoneda's lemma, if $\omega_{X/A}$ exists, it is unique up to unique isomorphism. However $\omega_{X/A}$ does not behave well unless $f\colon X \to \operatorname{Spec} A$ is flat with fibers CM and equi-dimensional of dimension d.

Remark. Think of the above as a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{F}}\colon \operatorname{Hom}_X(\mathcal{F}, \omega_{X/A}) \times H^d(X, \mathcal{F}) \to A, \quad (\varphi, \xi) \mapsto \alpha_{\mathcal{F}}(\varphi)(\xi).$$

Warning: that $\alpha_{\mathcal{F}}$ is an isomorphism only guarantees the pairing is perfect on the $\operatorname{Hom}_X(\mathcal{F}, \omega_{X/A})$ side. It is not true that the pairing is perfect in general. (However over a field the pairing is always perfect.) Using the trace map, we can write

$$\langle \varphi, \xi \rangle_{\mathcal{F}} = \operatorname{tr}_{X/A}(\varphi(\xi)).$$

Theorem 2.9.2. *$X \coloneqq \mathbb{P}^d_A$ has a relative dualizing module $\omega \cong \mathcal{O}(-d-1)$.*

Proof. Set $\omega = \mathcal{O}(-d-1)$. Pick an isomorphism $\operatorname{tr}\colon H^d(\mathbb{P}^d_A, \omega) \to A$ (using the computation of cohomology of projective space earlier). Now define $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ as above. This in turn gives $\alpha_{\mathcal{F}}$'s functorial in $\mathcal{F}$, for all $\mathcal{F}$ coherent.

Step 1: show that $\alpha_{\mathcal{F}}$ is an isomorphism if $\mathcal{F} = \mathcal{O}_X(a)$ for some a. Namely we get

$$\begin{array}{ccc} \operatorname{Hom}_X(\mathcal{O}_X(a), \mathcal{O}_X(-d-1)) & \xrightarrow{\alpha_{\mathcal{F}}} & H^d(X, \mathcal{O}_X(a))^\vee \\ \| & & \| \\ \Gamma(X, \mathcal{O}_X(-a-d-1)) = S_{-a-d-1} & \longrightarrow & (E_a)^\vee \end{array}$$

where recall $S = A[T_0, \ldots, T_d]$ and $E = S[1/T_0 \cdots T_d] / \sum_{i=0}^d S[1/T_0 \cdots \hat{T}_i \cdots T_d]$. So the map $\alpha_{\mathcal{F}}$ arises from the multiplication

$$S_{-a-d-1} \times E_a \to E_{-d-1} = H^d(\mathbb{P}^d_A, \omega) \xrightarrow{\operatorname{tr}} A,$$

which is a perfect pairing of free A-modules.

Step 2: show that $\alpha_{\mathcal{F}}$ is always an isomorphism. Pick an exact sequence

$$(\mathcal{E}_1 \coloneqq \bigoplus_{j=1}^s \mathcal{O}_X(b_j)) \to (\mathcal{E}_0 \coloneqq \bigoplus_{i=1}^r \mathcal{O}_X(a_i)) \to \mathcal{F} \to 0$$

coming from the coherence of $\mathcal{F}$. Then apply the right-exact functor to get

$$\begin{array}{ccccccc}
\operatorname{Hom}(\mathcal{E}_1,\omega) & \longleftarrow & \operatorname{Hom}(\mathcal{E}_0,\omega) & \longleftarrow & \operatorname{Hom}(\mathcal{F},\omega) & \longleftarrow & 0 \\
{\scriptstyle\alpha_{\mathcal{E}_1}}\downarrow & & {\scriptstyle\alpha_{\mathcal{E}_0}}\downarrow & & {\scriptstyle\alpha_{\mathcal{F}}}\downarrow & & \\
H^d(X,\mathcal{E}_1)^\vee & \longleftarrow & H^d(X,\mathcal{E}_0)^\vee & \longleftarrow & H^d(X,\mathcal{F})^\vee & \longleftarrow & 0,
\end{array}$$

using the functoriality of $\alpha_{\mathcal{F}}$. So each of the squares commute. But from step 1 we know the first two vertical arrows $\alpha_{\mathcal{E}_1}$ and $\alpha_{\mathcal{E}_0}$ are isomorphisms, so by the five lemma $\alpha_{\mathcal{F}}$ is also an isomorphism. $\square$

Theorem 2.9.3 ("Correct" duality statement)**.** *If X is flat over A, then there exists $\omega^\bullet_{X/A} \in D^b_{coh}(\mathcal{O}_X)$ such that*

$$R\operatorname{Hom}_X(\mathcal{F},\omega^\bullet_{X/A}) = R\operatorname{Hom}_A(R\Gamma(X,\mathcal{F}),A).$$

Remark. We recover the relative dualizing module as $\omega_{X/A} = \mathcal{H}^{-d}(\omega^\bullet_{X/A})$, where $\mathcal{H}$ denotes the cohomology sheaf.

2.10 δ-functors

Definition 2.10.1. Let $\mathcal{A},\mathcal{B}$ be abelian categories. A **δ-functor** from $\mathcal{A}$ to $\mathcal{B}$ is a family $\{F^n\}_{n\geq 0}$ of additive functors from $\mathcal{A}$ to $\mathcal{B}$ and for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, maps $\delta\colon F^n(A_3) \to F^{n+1}(A_1)$ such that:

1. there is a long exact sequence $\cdots \to F^n(A_2) \to F^n(A_3) \xrightarrow{\delta} F^{n+1}(A_1) \to F^{n+1}(A_2) \to \cdots$;
2. the δ's are compatible with morphisms of short exact sequences.

Given two δ-functors $(\{F^n\},\delta)$ and $(\{G^n\},\delta)$, a **map of δ-functors** is a family of functors $\{h^n\colon F^n \to G^n\}$ commuting with the δ maps.

Example 2.10.2. The only example we know so far of δ-functors is the derived functors of a left or right exact functor.

Definition 2.10.3. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called **effaceable** iff for all $A \in \operatorname{Ob}(\mathcal{A})$, there exists an injective map $A \xrightarrow{u} A'$ in $\mathcal{A}$ such that $F(u)\colon F(A) \to F(A')$ is zero.

Example 2.10.4. Let $\mathcal{A} = \mathsf{Mod}(\mathcal{O}_X)$ and $\mathcal{B} = \mathsf{Ab}$. Then $F = H^i(X,-)$ is effaceable iff $i \geq 1$. This is because there exists enough injectives in $\mathsf{Mod}(\mathcal{O}_X)$, and $H^i(X,\mathcal{I}) = 0$ for all $i \geq 1$.

Lemma 2.10.5. *Let $(\{F^n\},\delta)$ and $(\{G^n\},\delta)$ be δ-functors. Let $t^0\colon F^0 \to G^0$ be a transformation of functors. If F^i is effaceable for $i \geq 1$, then t^0 extends uniquely to a map of δ-functors.*

Proof sketch. Suppose we have already constructed t^i for $0 \leq i \leq n$. Pick an object $A \in \operatorname{Ob}(\mathcal{A})$. We want to construct $t^{n+1}\colon F^{n+1}(A) \to G^{n+1}(A)$. By assumption, we can find $0 \to A \xrightarrow{u} A' \to A'' \to 0$ such that $F^{n+1}(u) = 0$. Then

$$\begin{array}{ccccccc}
F^n(A') & \longrightarrow & F^n(A'') & \xrightarrow{\delta} & F^{n+1}(A) & \xrightarrow{0} & F^{n+1}(A') \\
{\scriptstyle t^n}\downarrow & & {\scriptstyle t^n}\downarrow & & & & \\
G^n(A') & \longrightarrow & G^n(A'') & \xrightarrow{\delta} & G^{n+1}(A) & \longrightarrow & G^{n+1}(A').
\end{array}$$

Since the first square commutes, define $t^{n+1}\colon F^{n+1}(A) \to G^{n+1}(A)$ by lifting elements, since $F^{n+1}(A)$ is a cokernel. Then show it is independent of choice of lift, $\square$

2.11 Back to Serre duality

Example 2.11.1. Let X/k be a projective scheme of dimension d over a field. We can consider two δ-functors:

1. $\{\mathrm{Ext}^i_X(-,\mathcal{G})\colon \mathsf{Coh}(\mathcal{O}_X)\to \mathsf{Vect}_k\}_{i\geq 0}$ for any $\mathcal{O}_X$-module $\mathcal{G}$;
2. $\{H^{d-i}(X,-)^\vee\}_{i\geq 0}$, which is a δ-functor by taking the dual of the long exact sequence for $H^{d-i}(X,-)$ (this is why we work over a field).

Lemma 2.11.2. *For any coherent $\mathcal{G}$ on X/k projective, the functors $\mathrm{Ext}^i_X(-,\mathcal{G})$ are coeffaceable for $i\geq 1$.*

Proof. Recall that for any $\mathcal{F}\in\mathsf{Coh}(\mathcal{O}_X)$, there exists a surjection $\bigoplus_{i=1}^r \mathcal{O}(a_i)\twoheadrightarrow \mathcal{F}$. Note that we can do this with $a_i \ll 0$ (but the r might change), because $\mathrm{Hom}(\mathcal{O}(-a),\mathcal{F})=\Gamma(\mathcal{F}(a))$, which gets bigger as a becomes more negative. So it suffices to show $\mathrm{Ext}^i_X(\mathcal{O}_X(a),\mathcal{G})=0$ for $a\ll 0$. But

$$\mathrm{Ext}^i_X(\mathcal{O}_X(a),\mathcal{G}) = H^i(X,\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{O}_X(-a)) = H^i(X,\mathcal{G}(-a))=0 \quad a\ll 0. \qquad \square$$

Corollary 2.11.3. *If k is a field and $\omega_{X/k}$ is a relative dualizing module on X/k, we get canonical maps*

$$t^i_{\mathcal{F}}\colon \mathrm{Ext}^i_X(\mathcal{F},\omega_{X/k})\to H^{d-i}(X,\mathcal{F})^\vee$$

arising from the map $\alpha_{\mathcal{F}}\colon \mathrm{Hom}^i_X(\mathcal{F},\omega_{X/k})\to H^d(X,\mathcal{F})^\vee$ in the definition of the relative dualizing module. Moreover, these maps are isomorphisms iff the functors $\{H^{d-i}(X,-)^\vee\}$ are also coeffaceable for all $i\geq 1$, by the uniqueness of the lifted maps.

Lemma 2.11.4. *On $X=\mathbb{P}^d_k$, the functors $\{H^{d-i}(X,-)^\vee\}$ are coeffaceable.*

Proof. By exactly the same arguments as in the proof of coeffaceability of $\mathrm{Ext}^i_X(-,\mathcal{G})$, it suffices to prove that $H^{d-i}(\mathbb{P}^d_k,\mathcal{O}_{\mathbb{P}^d_k}(a))^\vee=0$ for $i\geq 1$ and $a\ll 0$. This is true by an earlier result. $\square$

Remark. We now have a beautiful duality in $\mathbb{P}^d_k$ (over a field). We have not shown yet that the relative dualizing module exists for a projective scheme over a field in general, but later we will use the existence of $\omega_{X/k}$ to construct it.

Remark. Warning: taking duals of non-finitely presented objects often has unexpected consequences.

Lemma 2.11.5. *Following are some general facts about $\mathcal{E}xt$.*

1. *The formulation of $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},-)$ as the right derived functor of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-)$ commutes with restriction to opens.*
2. *(Local-to-global spectral sequence for Ext) There is a spectral sequence $E_2^{p,q}=H^p(X,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))\Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F},\mathcal{G})$.*
3. *If $\mathcal{F}$ is finite locally free, then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})=0$ for $i>0$ and $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})=\mathcal{F}^\vee\otimes_{\mathcal{O}_X}\mathcal{G}$ where $\mathcal{F}^\vee:=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$ is also finite locally free. Then the local-to-global spectral sequence gives $\mathrm{Ext}^n_X(\mathcal{F},\mathcal{G})=H^n(X,\mathcal{F}^\vee\otimes_{\mathcal{O}_X}\mathcal{G})$.*

Corollary 2.11.6 (Serre duality for $\mathbb{P}^d_k$)**.** *If $\mathcal{F}$ is finite locally free, then*

$$H^i(\mathbb{P}^d_k,\mathcal{F}^\vee(-d-1)) = H^i(\mathbb{P}^d_k,\mathcal{F}^\vee\otimes\omega_{\mathbb{P}^d/k}) = \mathrm{Ext}^i_{\mathbb{P}^d}(\mathcal{F},\omega_{\mathbb{P}^d/k}) = H^{d-i}(\mathbb{P}^d_k,\mathcal{F})^\vee.$$

Let $\mathrm{ev}\colon\mathcal{F}^\vee\otimes\mathcal{F}\to\mathcal{O}$ be the evaluation map. Since all cohomologies are finite-dimensional vector spaces over a field, there is a perfect pairing

$$H^i(\mathbb{P}^d_k,\mathcal{F}^\vee(-d-1))\times H^{d-i}(\mathbb{P}^d_k,\mathcal{F})\to H^d(\mathbb{P}^d_k,\mathcal{F}^\vee(-d-1)\otimes_{\mathcal{O}_X}\mathcal{F})\xrightarrow{\mathrm{ev}} H^d(\mathbb{P}^d_k,\mathcal{O}_{\mathbb{P}^d_k}(-d-1))=k.$$

Lemma 2.11.7. *If X is locally Noetherian, $\mathcal{F}$ is coherent, and $\mathcal{G}$ is quasi-coherent, then:*

1. *$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is also quasi-coherent, and coherent if $\mathcal{G}$ is coherent;*

2. *on $U = \operatorname{Spec} A \subset X$ affine open, $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U = \widetilde{\operatorname{Ext}^i_A(M,N)}$ where $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$.*

Lemma 2.11.8. *Let k be a field. Let $P \coloneqq k[x_1,\dots,x_n] \twoheadrightarrow A$ be a surjection. Assume $\dim A = d$.*

(A1) For any A-module M, we have $\operatorname{Ext}^i_P(M,P) = 0$ for $i < n-d$.

(A2) For $i = n-d$, there is a canonical isomorphism $\operatorname{Ext}^{n-d}_P(M,P) = \operatorname{Hom}_A(M, \operatorname{Ext}^{n-d}_P(A,P))$.

(A3) If A is Cohen–Macaulay and equidimensional of dimension d, then $\operatorname{Ext}^i_P(A,P) = 0$ if $i \neq n-d$. (Moreover, $\operatorname{Ext}^{n-d+i}_P(M,P) = \operatorname{Ext}^i_A(M, \operatorname{Ext}^{n-d}_P(A,P))$.)

Lemma 2.11.9. *If X is projective over k, then X has a dualizing module $\omega_{X/k}$ and in fact*

$$i_*\omega_{X/k} = \mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n_k/k})$$

where $i\colon X \to \mathbb{P}^n_k$ is a closed immersion.

Proof sketch (see Hartshorne for more details). By duality on $\mathbb{P}^n_k$, we have an isomorphism of functors (in $\mathcal{F}$)

$$H^d(X,\mathcal{F})^\vee = H^d(\mathbb{P}^n_k, i_*\mathcal{F})^\vee = \operatorname{Ext}^{n-d}_{\mathbb{P}^n_k}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k}).$$

Use the local-to-global spectral sequence of $\mathcal{E}xt$:

$$E_2^{a,b} = H^a(\mathbb{P}^n_k, \mathcal{E}xt^b_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k})) \Rightarrow \operatorname{Ext}^*_{\mathbb{P}^n_k}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k}).$$

Result (A1) of the lemma above implies $\mathcal{E}xt^b_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k}) = 0$ for $b < n-d$, so the first $n-d$ rows of the spectral sequence are zero. Hence

$$\operatorname{Ext}^{n-d}_{\mathbb{P}^n_k}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k}) = H^0(\mathbb{P}^n_k, \mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k})).$$

Result (A2) of the lemma above implies $\mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k/k}) \cong i_*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/k})$. Finally,

$$H^0(\mathbb{P}^n_k, i_*\,\mathcal{H}om(\mathcal{F}, \omega_{X/k})) = H^0(X, \mathcal{H}om(\mathcal{F}, \omega_{X/k})) = \operatorname{Hom}(\mathcal{F}, \omega_{X/k}).$$ □

Lemma 2.11.10. *If X is projective over k and is Cohen–Macaulay and equidimensional of dimension d, then all the maps*

$$t^i_{\mathcal{F}}\colon \operatorname{Ext}^i_X(\mathcal{F}, \omega_{X/k}) \to H^{d-i}(X,\mathcal{F})^\vee$$

are isomorphisms.

Proof. We have seen it is enough to show $H^{d-i}(X, \mathcal{O}_X(a))^\vee = 0$ for $i > 0$ and $a \ll 0$. To do this, we use

$$H^{d-i}(X, \mathcal{O}_X(a))^\vee = H^{d-i}(\mathbb{P}^n_k, i_*\mathcal{O}_X(a))^\vee = \operatorname{Ext}^{n-d+i}_{\mathbb{P}^n_k}(i_*\mathcal{O}_X(a), \omega_{\mathbb{P}^n_k/k}).$$

This is computed by the spectral sequence $E_2^{a,b} = H^a(\mathbb{P}^n_k, \mathcal{E}xt^b_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_X(a), \omega_{\mathbb{P}^n_k/k}))$. Result (A3) of the lemma above says there is only a single non-zero row, at $b = n-d$. But the terms of the single non-zero row vanish:

$$H^i(\mathbb{P}^n_k, \mathcal{E}xt^{d-n}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^n_k/k})(-a)) = 0.$$ □

2.12 Dualizing modules for smooth projective schemes

Our goal now is to show that if X/k is smooth projective and irreducible of dimension d, then

$$\omega_{X/k} \cong \Omega^d_{X/k} := \bigwedge^d \Omega^1_{X/k}.$$

Definition 2.12.1. We define the **sheaf of differentials** Ω in several cases, of increasing generality.

1. $\Omega_{k[x_1,\dots,x_n]/k} := \bigoplus_{i=1}^n k[x_1,\dots,x_n]dx_i$, with universal differential $d(f) := \sum_{i=1}^n (\partial f/\partial x_i)dx_i$.
2. If A/k is finite type, write $A = k[x_1,\dots,x_n]/J$. Then

   $$\Omega_{A/k} := \operatorname{coker}(J/J^2 \to \bigoplus_{i=1}^n A\,dx_i).$$

   Explicitly, $\Omega_{X/k}$ is the free A-module on $dx_1,\dots,dx_n$ mod the sub-module generated by the $d(f_j)$ if $J = (f_1,\dots,f_m)$. There is a universal differential $d\colon A \to \Omega_{A/k}$ such that the following diagram commutes:

   $$\begin{array}{ccc} k[x] & \xrightarrow{d} & \Omega_{k[x]/k} \\ \downarrow & & \downarrow \\ A & \xrightarrow{d} & \Omega_{A/k}. \end{array}$$

3. Fact: $\Omega_{A_f/k} = (\Omega_{A/k})_f$, so $\Omega_{A/k}$ sheafifies correctly. So if $X \to \operatorname{Spec} k$ is locally of finite type, we can define $\Omega_{X/k}$.

Definition 2.12.2 (Alternate definition using universal property)**.** A map $D\colon A \to M$ is a **k-derivation** if it is a k-linear map satisfying the Leibniz rule. Then $(\Omega_{A/k}, d)$ is the initial object among derivations $A \to M$, i.e.

$$\operatorname{Der}_k(A, M) = \operatorname{Hom}_A(\Omega_{A/k}, M).$$

Definition 2.12.3. We say A/k of finite type is **smooth** any of the following equivalent conditions hold:

1. $\Omega_{A/k}$ is finite locally free of rank $\dim A$;
2. when writing $A = k[x]/J$, we have J/J^2 is finite locally free and $J/J^2 \to \bigoplus_{i=1}^n A\,dx_i$ has maximal rank at every point;
3. $\Omega_{A/k}$ is finite locally free and $\ker(J/J^2 \to \bigoplus_{i=1}^n A\,dx_i) = 0$. (Johan likes this definition.)

Example 2.12.4. If $A = k[x_1, x_2]/(x_1x_2)$, then $x_1x_2 \mapsto (x_2\,dx_1, x_1\,dx_2)$. But then at the point $x_1 = x_2 = 0$, this map $J/J^2 \to A\,dx_1 \oplus A\,dx_2$ is not maximal rank (as a map from a free rank 1 module into a free rank 2 module). In general, we see that a hypersurface $A = k[x_1,\dots,x_n]/(f)$ is smooth iff

$$V\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = \emptyset.$$

Remark. Warning: the definition of smoothness will be different in other texts, but will all mean the same thing. The equivalence(s) will in general be hard to prove (e.g. formal smoothness).

Proposition 2.12.5. *If $A \to B$ is smooth, then we have some good properties:*

1. *$A \to B$ is flat (i.e. B is a flat A-module);*
2. *B is a local complete intersection over A;*

3. *for any $A \to A'$, the base change $A' \to A' \otimes_A B$ is smooth;*

4. *if $A = k$ is a field, then B smooth over k implies B is regular. (If B is a regular finite-type k-algebra and* $\operatorname{char} k = 0$*, then $k \to B$ is smooth; this is false when* $\operatorname{char} k \neq 0$*.)*

Example 2.12.6. Let $k = \mathbb{F}_p(t)$. The standard example of a regular variety which is not smooth is $B = k[x,y]/(x^2 + y^p - t)$ (which works for $p > 2$; for $p = 2$, take $k[x,y]/(x^3 + y^2 - t)$). Compute that $J/J^2 = B(x^2 + y^p - t) \to B\,dx \oplus B\,dy$ has image generated by $2x\,dx + 0$. This does not have a kernel, since $2x$ is not a zero-divisor in $B\,dx \oplus B\,dy$. The cokernel is $B/(2x)B \oplus B$ and therefore not projective (because $B/(2x)B$ is torsion). Alternatively, the rank jumps at $x = 0$. Hence B is not smooth at $x = 0$. However, the local ring there is still regular, because $\mathfrak{m}_x = (x, y^p - t)$, and $y^p - t$ is generated by x.

Example 2.12.7. Why do we not define "smooth" to mean "$\Omega_{B/A}$ is free"? Take $B = k[x]/(x^p)$ where $\operatorname{char} k = p$. Then $J/J^2 = Bx^p \to B\,dx$ is the zero map, so $\Omega_{B/k} = B\,dx$ and the kernel is non-zero. So having a free $\Omega_{B/A}$ is not good enough for smoothness.

Example 2.12.8. For what values of $\lambda \in k$ is the curve $x_0^3 + x_1^3 + x_2^3 - \lambda x_0 x_1 x_2 = 0$ in $\mathbb{P}^2_k$ not smooth?

1. Method 1: compute $V(F, \partial_0 F, \partial_1 F, \partial_2 F)$.

2. Method 2: affine locally, get $1 + x^3 + y^3 + \lambda xy$, and look at $3x^2 - \lambda y = 0$ and $3y^2 - \lambda x = 0$.

2.13 Koszul complex

Definition 2.13.1. Let R be a ring and $f_1, \ldots, f_r \in R$. Then $K_\bullet(R, f_1, \ldots, f_r)$ is the complex

$$0 \to \wedge^r R^{\oplus r} \to \cdots \to \wedge^2 R^{\oplus r} \to R^{\oplus r} \xrightarrow{f_1, \ldots, f_r} R$$

with maps $d(v_1 \wedge \cdots \wedge v_t) \coloneqq \sum_{i=1}^t (-1)^j \varphi(v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_t$. Then $(\wedge^* R^{\oplus r}, d)$ is a differential graded algebra.

Definition 2.13.2. We say $f_1, \ldots, f_r$ is a **Koszul-regular sequence** iff $H_*(K_\bullet(R, f_1, \ldots, f_r)) = 0$ for $i > 0$. (The homology in degree-0 is always $R/(f_1, \ldots, f_r)$.)

Theorem 2.13.3. *If R is Noetherian, then the following are equivalent:*

1. *$f_1, \ldots, f_r$ is a Koszul-regular sequence;*

2. *for all $\mathfrak{p} \in V(f_1, \ldots, f_r)$, the sequence $f_1, \ldots, f_r$ is a regular sequence in the local ring $R_\mathfrak{p}$.*

Definition 2.13.4. An ideal $I \subset R$ is **Koszul-regular** iff locally, I can be generated by Koszul-regular sequence. Explicitly, for all $\mathfrak{p} \in V(I)$, there exists $f \in R \setminus \mathfrak{p}$ such that $I_f \subset R_f$ can be generated by a Koszul-regular sequence.

Remark. If I is Koszul-regular, then it is **quasi-regular**, i.e. I/I^2 is a finite locally free R/I-module.

Definition 2.13.5. A ring map $A \to B$ is a **local complete intersection homomorphism** iff $A \to B$ is of finite presentation and for any presentation $B = A[x_1, \ldots, x_n]/J$, the ideal J is Koszul-regular.

2.14 Closed immersions and (co)normal sheaves

Definition 2.14.1. Let $i\colon Z \hookrightarrow X$ be a closed immersion of schemes. Then

$$\mathcal{I} \coloneqq \ker(\mathcal{O}_X \to i_* \mathcal{O}_Z)$$

is a quasi-coherent sheaf of ideals. The **conormal sheaf** of Z in X is

$$\mathcal{C}_{Z/X} \coloneqq i^* \mathcal{I}.$$

Remark. There is an equivalence of categories between $\mathsf{Mod}(\mathcal{O}_Z)$ and the subcategory of $\mathsf{Mod}(\mathcal{O}_X)$ annihilated by $\mathcal{I}$. So the following lemma uniquely specifies $\mathcal{C}_{Z/X}$, and is the reason why some people define $\mathcal{C}_{Z/X}$ as a sheaf on X by $\mathcal{I}/\mathcal{I}^2$.

Lemma 2.14.2. $i_*\mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$.

Proof. Affine locally, we have $I \subset A$ an ideal where $\mathcal{I} = \tilde{I}$. Then $i^*\mathcal{I} = \widetilde{I \otimes_A A/I}$. Look at $0 \to I \to A \to A/I \to 0$ and apply $I \otimes_A -$, which is right exact, to get $I \otimes_A I \xrightarrow{m} I \to I \otimes_A A/I \to 0$. Hence $I \otimes_A A/I$ is just I/I^2. ☐

Definition 2.14.3. The **normal sheaf** is $\mathcal{C}_{Z/X}^\vee := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{X/Z}, \mathcal{O}_Z)$.

Definition 2.14.4. We say $i\colon Z \hookrightarrow X$ is a **(Koszul-)regular closed immersion** iff $\mathcal{I} \subset \mathcal{O}_X$ is a Koszul-regular ideal, i.e. affine locally, it corresponds to Koszul-regular ideals.

Lemma 2.14.5. *If $i\colon Z \hookrightarrow X$ is a regular closed immersion, then the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{O}_X) = \begin{cases} 0 & i \neq c \\ \wedge^c \mathcal{C}_{Z/X}^\vee & i = c \end{cases}$$

where c is the rank of the normal sheaf $\mathcal{C}_{X/Z}^\vee$ (in an affine open where the rank is constant).

Proof. Affine locally, we have seen that $\mathcal{C}_{Z/X}$ is finite locally free. Compute affine locally with $Z = \operatorname{Spec}(A/I) \hookrightarrow X = \operatorname{Spec} A$ and $I = (f_1, \ldots, f_c)$ with $f_1, \ldots, f_c$ a Koszul-regular sequence. Then

$$K_\bullet(A, f_1, \ldots, f_r) \to A/I$$

is a finite free resolution. Hence $\operatorname{Ext}^*_A(A/I, A) = H^*(\operatorname{Hom}_A(K_\bullet, A))$. Note that the Koszul complex $K_\bullet$ is **self-dual** (up to sign), i.e. there is a pairing

$$\langle \cdot, \cdot \rangle \colon \wedge^* A^{\oplus c} \times \wedge^* A^{\oplus c} \to \wedge^c A^{\oplus c} \cong A, \quad (\omega, \eta) \mapsto \omega \wedge \eta$$

and $\langle d\alpha, \beta \rangle + \langle \alpha, d\beta \rangle = 0$. So $\operatorname{Hom}_A(K_\bullet, A) = K_\bullet[-c]$, where the shifting happens because of the Hom_A. This gives the vanishing we want, and gives $\operatorname{Ext}^c_A(A/I, A) \cong A/I = \wedge^c I/I^2$. Each of these isomorphisms is not canonical, but the composition is canonical. ☐

Corollary 2.14.6 (Adjunction formula)**.** *Suppose $X \subset \mathbb{P}^n_k$ is a local complete intersection of pure codimension c. Then*

$$\begin{aligned} i_*\omega_{X/k} &\cong \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n_k/k}) = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(i_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \otimes \omega_{\mathbb{P}^n_k/k} \\ &\cong \left(i_* \bigwedge^c (\mathcal{C}_{X/\mathbb{P}^n})^\vee \right) \otimes \omega_{\mathbb{P}^n_k/k} \\ &\cong i_*\left((\bigwedge^c \mathcal{N}_{X/\mathbb{P}^n}) \otimes_{\mathcal{O}_X} i^*\omega_{\mathbb{P}^n_k/k} \right). \end{aligned}$$

In this situation, $\omega_{X/k}$ is invertible.

2.15 Dualizing sheaf in the smooth case

Lemma 2.15.1. *If $i\colon X \to \mathbb{P}^n_k$ is a closed immersion and X is smooth over k, then there exists a canonical short exact sequence*

$$0 \to \mathcal{C}_{X/\mathbb{P}^n} \to i^*\Omega_{\mathbb{P}^n/k} \to \Omega_{X/k} \to 0.$$

Proof. Over $D_+(T_0)$ (set $x_i \coloneqq T_i/T_0$), we get a presentation

$$0 \to J \to k[x_1, \dots, x_n] \to \mathcal{O}_X(X \cap D_+(T_0)) \to 0$$

The definition of X being smooth gives a short exact sequence $0 \to J/J^2 \to \bigoplus B\, dx_i \to \Omega_{B/k} \to 0$ where $B \coloneqq \mathcal{O}_X(X \cap D_+(T_0))$. The same holds over $D_+(T_i)$ and everything glues. $\square$

Lemma 2.15.2. $\omega_{\mathbb{P}^n/k} \cong \mathcal{O}_{\mathbb{P}^n/k}(-n-1) = \bigwedge^n \Omega_{\mathbb{P}^n/k}$.

Proof 1. On $D_+(T_i)$, the sheaf $\bigwedge^n \Omega_{\mathbb{P}^n/k}$ is free with basis $\omega_i = d(T_0/T_i) \wedge \cdots \wedge \widehat{d(T_i/T_i)} \wedge \cdots \wedge d(T_n/T_i)$. Note that if $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^n$, then on $D_+(T_i)$ this is free with basis T_i, and $T_i|_{D_+(T_j T_i)} = (T_i/T_j)T_j|_{D_+(T_i T_j)}$. It suffices to check the transition functions of ω_i are $(T_i/T_j)^{-n-1}$. $\square$

Proof 2. Show that there is a canonical short exact sequence of finite locally free modules (called the **Euler sequence**)

$$0 \to \Omega_{\mathbb{P}^n_k/k} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

where we think of $\mathcal{O}(-1)^{\oplus(n+1)}$ as having a basis of "dT_i" (although they are not global sections). The first map sends on $D_+(T_i)$

$$d(T_i/T_j) \mapsto \frac{1}{T_j} dT_i - \frac{T_i}{T_j^2} dT_j$$

and the second map is multiplication by $(T_0, \dots, T_n)$. Then use the general fact that if $0 \to A \to B \to C \to 0$ is a short exact sequence of finite locally free $\mathcal{O}$-modules, then $\bigwedge^{\text{top}} B \cong \bigwedge^{\text{top}} A \otimes_{\mathcal{O}} \bigwedge^{\text{top}} C$. $\square$

Corollary 2.15.3. *If $i\colon X \to \mathbb{P}^n_k$ is a closed immersion with X smooth over k, then $\omega_{X/k} \cong \Omega^{\dim X}_{X/k} \coloneqq \bigwedge^{\dim X} \Omega_{X/k}$.*

Proof. We already know $\omega_{X/k} \cong \bigwedge^c (\mathcal{C}_{X/k})^\vee \otimes i^*\omega_{\mathbb{P}^n/k}$ where $c \coloneqq \operatorname{codim}_{\mathbb{P}^n} X$. (This uses that i is a regular embedding, because X is smooth over k.) But this is just $\bigwedge^c (\mathcal{C}_{X/k})^\vee \otimes \bigwedge^n (i^*\Omega_{\mathbb{P}^n/k})$ by the previous lemma and that i^* commutes with $\wedge$. By another previous lemma,

$$0 \to \mathcal{C}_{X/k} \to i^*\Omega_{\mathbb{P}^n/k} \to \Omega_{X/k} \to 0$$

is a short exact sequence, so $\bigwedge^n i^*\Omega_{\mathbb{P}^n/k} \cong \bigwedge^c \mathcal{C}_{X/k} \otimes_{\mathcal{O}_X} \bigwedge^{n-c} \Omega_{X/k}$. $\square$

Remark. If $X \to Y$ is a closed immersion of projective schemes smooth over k, then the previous arguments show that $\bigwedge^{\text{top}} \Omega_{X/k} \cong \bigwedge^{\text{top}} (\mathcal{N}_{X/Y}) \otimes_{\mathcal{O}_X} i^*\Omega^{\text{top}}_{Y/K}$. This is also called the **adjunction formula**.

Example 2.15.4. If $X \subset \mathbb{P}^2_k$ is a curve of degree d, then $\omega_{X/k} \cong \mathcal{N}_{X/\mathbb{P}^2} \otimes \mathcal{O}_X(-3)$. But we have

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X \to 0$$

where $F \in k[X_0, X_1, X_2]_d$ is the defining polynomial. So the ideal sheaf of X is $\mathcal{I} = \mathcal{O}_{\mathbb{P}^2}(-d)$, and by definition $\mathcal{C}_{X/\mathbb{P}^2 2} \cong i^*\mathcal{I} = \mathcal{O}_X(-d)$. The normal bundle is therefore $\mathcal{N}_{X/\mathbb{P}^2} = \mathcal{O}_X(d)$. Hence $\omega_{X/k} = \mathcal{O}_X(d-3)$. By Serre duality,

$$\dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_{X/k}) = \dim H^0(X, \mathcal{O}_X(d-3)).$$

Twist the short exact sequence above by $\mathcal{O}_{\mathbb{P}^2}(d-3)$ to get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2}(d-3) \to i_*\mathcal{O}_X(d-3) \to 0.$$

The cohomology sequence gives

$$H^0(i_*\mathcal{O}_X(d-3)) = H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) = k[T_0, T_1, T_2]_{d-3}, \quad H^1(i_*\mathcal{O}_X(d-3)) = H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) = k.$$

Hence $\dim H^1(X, \mathcal{O}_X) = \dim k[T_0, T_1, T_2]_{d-3} = \binom{d-3+2}{2} = \binom{d-1}{2}$. (Note that we also get $H^0(X, \mathcal{O}_X) \cong H^1(i_*\mathcal{O}_X(d-3)) = k$ by Serre duality.)

Definition 2.15.5. Let X be a projective scheme of dimension 1 over k with $H^0(X, \mathcal{O}_X) = k$. The **genus** of X is

$$g \coloneqq \dim_k H^1(X, \mathcal{O}_X).$$

The example above shows **plane curves** of degree d have genus $\binom{d-1}{2}$. If X is smooth, then we have $g = \dim_k H^0(X, \Omega_{X/k})$ and $1 = \dim_k H^1(X, \Omega_{X/k})$.

Chapter 3

Curves

3.1 Degree on curves

Definition 3.1.1. Let X be projective over k a field, and $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$. The **Euler characteristic** of $\mathcal{F}$ is

$$\chi(\mathcal{F}) \coloneqq \chi(X, \mathcal{F}) \coloneqq \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Remark. Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of coherent sheaves on X, we have additivity, i.e. $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$. Hence we can interpret χ as a homomorphism $\chi\colon K_0(\mathsf{Coh}(\mathcal{O}_X)) \to \mathbb{Z}$. More generally, if $f\colon X \to \operatorname{Spec} A$ is projective and A is a Noetherian ring, then we get a similar map

$$K_0(\mathsf{Coh}(\mathcal{O}_X)) \to K_0(\mathsf{Coh}(\mathcal{O}_{\operatorname{Spec}(A)})) = K_0'(A), \quad [\mathcal{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathcal{F}].$$

Example 3.1.2. Let $X = \mathbb{P}^1_k$. Then $\chi(\mathcal{O}_X(n)) = n + 1$.

Definition 3.1.3. Let X be a projective scheme of dimension ≤ 1 over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the **degree of** $\mathcal{L}$ is

$$\deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

Remark. Riemann–Roch is built into this definition, because for $H^0(X, \mathcal{O}_X) = k$ (i.e. the case where we define the genus $g \coloneqq h^1(\mathcal{O}_X)$),

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\mathcal{L}) + h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = \deg(\mathcal{L}) + 1 - g.$$

Corollary 3.1.4. *If* $\deg(\mathcal{L}) > 2g - 2$*, then* $h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$ *and* $h^1(\mathcal{L}) = 0$.

Definition 3.1.5. Let k be a field. A **variety** is a scheme over k of finite type (locally of finite type and quasi-compact), separated (diagonal is a closed immersion), reduced (no non-zero nilpotents in local rings), and irreducible. A **curve** is a variety of dimension 1.

Remark. Warning: the definition of variety is not the same in every text. For example, some people want *geometrically* reduced and irreducible, i.e. stable under base changes. For example, $\operatorname{Spec}(\mathbb{Q}(i))$ is a variety over $\operatorname{Spec}(\mathbb{Q})$, but the base change to $\mathbb{C}$ gives $\operatorname{Spec}(\mathbb{C}) \times_{\operatorname{Spec}(\mathbb{Q})} \operatorname{Spec}(\mathbb{Q}(i)) = \operatorname{Spec}(\mathbb{C}) \sqcup \operatorname{Spec}(\mathbb{C})$ which is reducible.

Lemma 3.1.6. *Let X be a projective curve. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let $s \in \Gamma(X, \mathcal{L})$ be non-zero. Then:*

1. *the zeros scheme $Z = Z(s)$ is nice closed subscheme of X (an effective Cartier divisor of X);*

2. $Z \to \operatorname{Spec}(k)$ *is finite, i.e.* $Z = \operatorname{Spec}(A)$ *and* $\dim_k(A) < \infty$*;*

3. $\dim_k(A) = \sum_{x \in X} \operatorname{ord}_x(s)[\kappa(x) : k]$ *where* $\operatorname{ord}_x(s)$ *is the order of vanishing of* s *at* x*.*

Definition 3.1.7. In the setting of the lemma, we define the **degree** $\deg(Z) \coloneqq \deg(Z \to \operatorname{Spec}(A)) \coloneqq \dim_k(A)$. The **degree of** $\mathcal{L}$ is defined to be $\deg(\mathcal{L}) \coloneqq \deg(Z)$.

Proof. For any $x \in X$, find an affine open $U = \operatorname{Spec}(B) \subset X$ containing x, and get an isomorphism $\varphi \colon \mathcal{L}|_U \to \mathcal{O}_U$. Set $f \coloneqq \varphi(s|_U) \in \mathcal{O}_U(U) = B$. Since $s \neq 0$, we know $f \neq 0$. So f is a non-zerodivisor. An effective Cartier divisor is a regular immersion of codimension 1, so we have obtained an effective Cartier divisor. We know $Z \cap U \coloneqq \operatorname{Spec}(B/fB)$, and these pieces glue because $\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X$ gives a quasi-coherent ideal sheaf. This ideal sheaf does not contain the generic point, so $\dim Z = 0$ and $Z = \{x_1, \ldots, x_n\}$ is a set of closed points. If $x = x_i$, then by shrinking U we may assume $U \cap Z = \{x_i\}$. Suppose x corresponds to $\mathfrak{m} \subset B$ the maximal ideal. If we set

$$\operatorname{ord}_x(s) \coloneqq \operatorname{length}_{B_{\mathfrak{m}}}(B_{\mathfrak{m}}/fB_{\mathfrak{m}})$$

then $\dim_k B/fB = \dim_k B_{\mathfrak{m}}/fB_{\mathfrak{m}} = \operatorname{ord}_x(s)[\kappa(x) : k]$, because B is supported only at x. Now as schemes, $Z = Z_1 \sqcup \cdots \sqcup Z_n$, where on the algebra side we have $A = B_1/f_1B_1 \times \cdots \times B_n/f_nB_n$, and $\dim_k A$ is the sum of the individual dimensions. Finally, take

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to 0$$

(where really $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ means $i_*i^*\mathcal{L}$). Since $\operatorname{Pic}(Z) = \{\mathcal{O}_Z\}$, this last term really is just $\mathcal{O}_Z$. So $\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_Z)$ and it follows that $\deg(\mathcal{L}) = \deg(Z)$. ☐

Lemma 3.1.8. *Let* X *be projective over* k *with* $\dim X \leq 1$*. Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be invertible sheaves on* X*. Then*

$$\deg(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2).$$

Proof. We prove this in the case of X a projective curve. Suppose $\mathcal{L}_1$ has a non-zero section $s_1 \in \Gamma(X, \mathcal{L}_1)$. Then we get $0 \to \mathcal{O}_X \xrightarrow{s_1} \mathcal{L}_1 \to \mathcal{O}_{Z_1} \to 0$ where $Z_1 \coloneqq Z(s_1)$. Tensoring with $\mathcal{L}_2$, an invertible sheaf, is an exact functor, so we get

$$0 \to \mathcal{L}_2 \xrightarrow{s_1} \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2 \to \mathcal{O}_{Z_1} \to 0$$

where the last term remains $\mathcal{O}_{Z_1}$ because $\operatorname{Pic}(Z_1)$ is trivial, by the discussion in the proof of the previous lemma. Hence

$$\chi(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2) = \chi(\mathcal{L}_2) + \deg(Z_1) = \chi(\mathcal{L}_2) + \deg(\mathcal{L}_1)$$

so by the definition of $\deg(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)$ we are done.

Now suppose $\mathcal{L}_1$ has no non-zero sections. Choose a closed immersion $i \colon X \hookrightarrow \mathbb{P}^n_k$. Then $\mathcal{O}_X(d) = i^*\mathcal{O}_{\mathbb{P}^n_k}(d)$. Since $\mathcal{O}_X(d)$ for $d \geq 0$ has a non-zero section, by the preceding case, $\deg(\mathcal{L}_1(d)) = \deg(\mathcal{L}_1) + \deg(\mathcal{O}_X(d))$ and similarly for $(\mathcal{L}_1 \otimes \mathcal{L}_2)(d)$. But now for $d \gg 0$, we see $\mathcal{L}_1(d)$ is globally generated and therefore has a non-zero section. Hence

$$\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) + \deg(\mathcal{O}_X(d)) = \deg(\mathcal{L}_1(d) \otimes \mathcal{L}_2) = \deg(\mathcal{L}_1(d)) + \deg(\mathcal{L}_2) = \deg(\mathcal{L}_1) + \deg(\mathcal{O}_X(d)) + \deg(\mathcal{L}_2). \quad ☐$$

Corollary 3.1.9. *The degree of an invertible sheaf* $\mathcal{L}$ *on a projective curve is the leading coefficient of the linear polynomial* $n \mapsto \chi(X, \mathcal{L}^{\otimes n})$*.*

Proof. By the lemma, this polynomial is just $n \deg(\mathcal{L}) + \chi(\mathcal{O}_X)$. ☐

Remark. This is a pretty good alternative for defining the degree of an invertible sheaf. Also, on a curve, if the degree is positive, then we can also look at the growth rate of $h^0(\mathcal{L}^{\otimes n})$. If $h^0(\mathcal{L}) > 0$, then $\deg(\mathcal{L}) \geq 0$. If in addition $\deg(\mathcal{L}) = 0$ (equivalently, $h^0(\mathcal{L}^{-1}) > 0$), then $\mathcal{L} \cong \mathcal{O}_X$. If $\deg(\mathcal{L}) > 0$, then $h^0(\mathcal{L}^{\otimes n}) > 0$ for some $n > 0$.

3.2 Linear series

Definition 3.2.1. Let X be a projective curve. Let $d \geq 0$ and $r \geq 0$. A **linear series of degree d and dimension r** is a pair $(\mathcal{L}, V)$ where $\mathcal{L} \in \operatorname{Pic}(X)$ is an invertible $\mathcal{O}_X$-module of degree d, and $V \subset H^0(X, \mathcal{L})$ is a vector subspace of dimension $r+1$. We abbreviate this as "$(\mathcal{L}, V)$ is a g^r_d." If $V = H^0(X, \mathcal{L})$, then we say $(\mathcal{L}, V)$ is a **complete linear system**.

Definition 3.2.2. Given $(\mathcal{L}, V)$, we get an $\mathcal{O}_X$-module map $V \otimes_k \mathcal{O}_X \to \mathcal{L}$ given by $\sum s_i \otimes f_i \mapsto \sum f_i s_i$. If this map is surjective, we get a morphism

$$\varphi_{(\mathcal{L},V)} \colon X \to \mathbb{P}^r_k = \mathbb{P}(V) \coloneqq \operatorname{Proj}(\operatorname{Sym}^* V)$$

such that $\varphi^*\mathcal{O}(1) = \mathcal{L}$. The general construction is as follows. Given a scheme X over a base S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ where $\mathcal{L} \in \operatorname{Pic}(X)$ is an invertible $\mathcal{O}_X$-module and $s_i \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$, there is a canonical map

$$\varphi = \varphi_{(\mathcal{L},V)} \colon X \to \mathbb{P}^n_S$$

with $\varphi^*\mathcal{O}(1) = \mathcal{L}$ such that $T_i \in \Gamma(\mathcal{O}(1))$ pulls back to s_i. Namely, $X = \bigcup_i X_{s_i}$ where $X_{s_i} \coloneqq \{x \in X : s_i \text{ does not vanish at } x\}$. Then use s_i to get an isomorphism

$$\mathcal{L}|_{X_{s_i}} \xrightarrow{s_i^{-1}} \mathcal{O}_{X_{s_i}}, \quad s_j \mapsto f_{i,j} \coloneqq s_j s_i \in \mathcal{O}_{X_{s_i}}(X_{s_i}).$$

These glue. (Conversely, a morphism $X \to \mathbb{P}^n_S$ gives $(s_0, \ldots, s_n)$ by pulling back $T_0, \ldots, T_n$.)

Remark. This construction shows that

$$\operatorname{Mor}_S(X, \mathbb{P}^n_S) = \{(\mathcal{L}, (s_0, \ldots, s_n))\}/\cong$$

where $(\mathcal{L}, (s_0, \ldots, s_n)) \cong (\mathcal{L}', (s'_0, \ldots, s'_n))$ if there is an isomorphism $\mathcal{L} \to \mathcal{L}'$ sending s_i to s'_i. By Yoneda's lemma, this means we know $\mathbb{P}^n_S$ if we know all line bundles and their global sections. In particular, $\operatorname{Mor}_S(X, \mathbb{A}^n_S) = \Gamma(X, \mathcal{O}_X)^{\oplus n}$.

Lemma 3.2.3. *Given $(\mathcal{L}, V)$, then V generates $\mathcal{L}$ iff for any closed point $x \in X$, we have $V \to H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_x)$ is non-zero, i.e. there exists a section in V not vanishing at x. If it is a complete linear system, then this is equivalent to* $\dim_k H^0(X, \mathcal{I}\mathcal{L}) < \dim_k H^0(X, \mathcal{L})$ *where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of the point x.*

3.3 Normalization and normal varieties

Definition 3.3.1. A variety is called **normal** iff the following equivalent conditions hold:

1. for any non-empty $U \subset X$ affine open, $\mathcal{O}_X(U)$ is a normal domain;
2. for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

Remark. If X is a curve, then for $x \in X$ closed, $\mathcal{O}_{X,x}$ is normal iff $\mathcal{O}_{X,x}$ is regular iff $\mathcal{O}_{X,x}$ is a DVR. Hence X is normal iff X is regular. In characteristic 0, regular is equivalent to smooth, so X is normal iff $X \to \operatorname{Spec} k$ is smooth. (In any characteristic and any dimension, smooth implies normal.)

Proposition 3.3.2. *Given a variety X, there exists a finite birational morphism $\nu \colon X^\nu \to X$ such that X^ν is a normal variety and such that for $U = \operatorname{Spec}(A) \subset X$ affine open, we have $\nu^{-1}(U)$ is the Spec of the integral closure of A in its fraction field.*

Remark. The universal property of the normalization $\nu \colon X^\nu \to X$ is not for all schemes; it is for all dominant morphisms of normal varieties into X.

Example 3.3.3. Let $X = \operatorname{Spec} k[x,y]/(x^2 - y^3) = \operatorname{Spec} k[t^2, t^3]$. The integral closure is $k[t]$, so the normalization is $X^\nu \coloneqq \operatorname{Spec} k[t]$.

Remark. There is an often-used trick: if X is a normal curve, then any non-zero coherent sub-sheaf $\mathcal{L}' \subset \mathcal{L}$ is invertible. This is because the classification of modules over DVRs tells us the stalks are free. Therefore given $(\mathcal{L}, V)$ a g^r_d with $r \geq 1$ on a projective normal curve, then $\mathcal{L}' \coloneqq \operatorname{im}(V \otimes_k \mathcal{O}_X \to \mathcal{L})$ is invertible of degree $d' \leq d$ and $V \subset H^0(X, \mathcal{L}') \subset H^0(X, \mathcal{L})$ generates $\mathcal{L}'$. So we get $\varphi_{(\mathcal{L}', V)} \colon X \to \mathbb{P}(V)$ with $\varphi^*\mathcal{O}(1) = \mathcal{L}'$.

3.4 Genus zero projective curves

Situation: k is a field, X is a projective curve over k, and $H^0(X, \mathcal{O}_X) = k$. Under these conditions, we are allowed to talk about genus. Suppose $g = 0 = \dim_k H^1(X, \mathcal{O}_X)$. (Then $\chi(\mathcal{O}_X) = 1$.) Examples include $\mathbb{P}^1_k$, and any plane curve of degree 1 or 2.

Example 3.4.1. In $\mathbb{P}^2_{\mathbb{R}}$, the curve $t_0^2 + t_1^2 + t_2^2 = 0$ has no real points. The curve $t_0^2 + t_1^2$ is an irreducible non-smooth degree-2 curve. (It is singular at $[0:0:1]$. In fact it is also not geometrically irreducible.) The curve $aX_0^2 + bX_0^2 + X_2^2 = 0$ is geometrically non-reduced in $\mathbb{P}^2_{\mathbb{F}_2(a,b)}$.

Remark. Warning: the unique $\mathbb{R}$-rational closed point on $T_0^2 + T_1^2 = 0$ on $\mathbb{P}^2_{\mathbb{R}}$ does not give us a Cartier divisor.

Definition 3.4.2. A variety X over k is **geometrically** $\mathcal{P}$ if $X/\bar{k} \coloneqq X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ has property $\mathcal{P}$.

Lemma 3.4.3. *If $\mathcal{L}$ on X has degree 0, then $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Using Riemann–Roch, $\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X) = 0 + 1 > 0$. So $\mathcal{L}$ has a non-zero section. This section must be non-vanishing, so $\mathcal{L} \cong \mathcal{O}_X$. □

Corollary 3.4.4. $\operatorname{Pic}(X) \subset \mathbb{Z}$*, and it is non-empty.*

Lemma 3.4.5. *If $\mathcal{L}$ has degree $d > 0$, then $\dim_k H^0(X, \mathcal{L}) = d + 1$ and $H^1(X, \mathcal{L}) = 0$.*

Proof. By Riemann–Roch, $\chi(\mathcal{L}) = d+1 > 0$. So there exists a non-zero section, giving a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_Z \to 0$ where $Z = Z(s)$. Then $H^0(Z, \mathcal{O}_Z) \cong k^{\oplus d}$. Since $\mathcal{O}_Z$ is a sum of skyscrapers, $H^1(X, \mathcal{O}_Z) = 0$. Since $H^1(X, \mathcal{O}_X) = 0$, we get $H^1(X, \mathcal{L}) = 0$ and $\dim H^0(X, \mathcal{L}) = d + 1$. □

Make the assumption that ω_X is invertible. For this, it suffices for X to be Gorenstein. Then we get $-1 = \chi(\omega_X) = \deg(\omega_X) + \chi(\mathcal{O}_X)$. This shows $\deg(\omega_X) = -2$. Taking the dual $\mathcal{L} \coloneqq \omega_X^\vee$, we get $\deg(\mathcal{L}) = 2$, i.e. $2\mathbb{Z} \subset \operatorname{Pic}(X)$. Hence $\dim_k H^0(X, \mathcal{L}) = 3$. (So we have a g^2_2.) Consider $0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to \mathcal{O}_Z \to 0$. By applying global sections, we get $0 \to \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{L}) \to \Gamma(\mathcal{O}_Z) \to 0$. For every $x \in X$ closed, there exists $t \in \Gamma(X, \mathcal{L})$ such that t does not vanish at x.

1. Case 1: if $x \notin Z$, then s does not vanish at x.
2. Case 2: if $x \in Z$, then there exists $f \in \Gamma(\mathcal{O}_Z)$ that does not vanish at x. Because $\Gamma(X, \mathcal{L}) \twoheadrightarrow \Gamma(X, \mathcal{O}_Z)$, let $t \in \Gamma(X, \mathcal{L})$ be a lift of f.

Apply the construction of maps to get $\varphi \colon X \to \mathbb{P}^2_k$ with $\varphi^*\mathcal{O}(1) = \mathcal{L} = \omega_X^{\otimes -1}$, and $\varphi^*T_0, \varphi^*T_1, \varphi^*T_2$ form a basis for $H^0(X, \mathcal{L})$. Since X is projective, $X \to \mathbb{P}^2_k$ is proper, and therefore closed. So $\varphi(X) \subset \mathbb{P}^2_k$ is closed. We know $\dim \varphi(X) \leq 1$ and $\varphi(X)$ is irreducible. However $\dim \varphi(X) = 0$ is impossible, otherwise $\varphi^*\mathcal{O}(1)$ is trivial instead of $\mathcal{L}$. Hence $\varphi(X) = V_+(F)$ for some homogeneous polynomial $F \in k[T_0, T_1, T_2]$ irreducible. Let $d = \deg(F) \geq 1$. Set $Y = V_+(F)$ as a scheme. Thus we obtain a factorization $X \xrightarrow{\psi} Y \xrightarrow{f} \mathbb{P}^2_k$.

Remark. Given $\varphi \colon X \to \mathbb{P}^2_k$, how do we determine what F is? View $F(T_0, T_1, T_2) \in \Gamma(\mathbb{P}^2, \mathcal{O}(d))$. Then $F(s_0, s_1, s_2) \coloneqq \varphi^*F(T_0, T_1, T_2) \in \Gamma(X, \varphi(\mathcal{O}(d))) = \Gamma(X, \mathcal{L}^{\otimes d})$, and we want to know if this F is zero or not. We know $\dim_k \Gamma(X, \mathcal{L}^{\otimes 2}) = \deg(\mathcal{L}^{\otimes 2}) + 1 = 4 + 1 = 5$. But we have monomials $s_0^2, s_0s_1, s_1^2, s_1s_2, s_2^2, s_0s_2$. Hence there is a linear relation.

Summary: assume k is a field and X is a projective curve over k with $H^0(X, \mathcal{O}_X) = k$ and $H^1(X, \mathcal{O}_X) = 0$, i.e. it has genus 0. Assume $\omega_{X/k}$ is an invertible $\mathcal{O}_X$-module (Gorenstein condition). Then $\mathcal{L} = \omega_{X/k}^{\otimes -1}$ is a degree 2 invertible $\mathcal{O}_X$-module, $h^0(\mathcal{L}) = 3$, and choosing a basis s_0, s_1, s_2 of $H^0(X, \mathcal{L})$, we get a morphism $\varphi_{\mathcal{L},(s_0,s_1,s_2)} \colon X \to \mathbb{P}^2_{\mathcal{L}}$ which factors through $Y = V_+(F)$ where $F \in k[T_0, T_1, T_2]$ is homogeneous of degree 2 and irreducible.

Theorem 3.4.6. *Actually, $X \cong Y$. In other words, any Gorenstein genus 0 curve is a conic.*

Proof. Let $\varphi \colon X \to Y$ be the morphism of schemes. We know $\mathcal{L} = \varphi^*\mathcal{O}_Y(1)$. Then $\mathcal{L}^{\otimes n} = \varphi^*\mathcal{O}_Y(n)$. Compute

$$\varphi_*(\mathcal{L}^{\otimes n}) = \varphi_*\varphi^*\mathcal{O}_Y(n) = (\varphi_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$$

by the projection formula (see below). Fact: if $f \colon X \to Y$ is either a morphism of projective varieties or a locally projective morphism of schemes and all fibers have dimension 0, then f is affine and even finite. (This comes from Zariski's main theorem.) Another fact: if $f \colon X \to Y$ is an affine morphism of schemes, then f is an isomorphism iff $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Last fact: if $i \colon Y \hookrightarrow \mathbb{P}^n_k$ and $\mathcal{O}_Y(1) = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent $\mathcal{O}_Y$-modules, then α is an isomorphism iff $\Gamma(Y, \mathcal{F}(d)) \xrightarrow{\sim} \Gamma(Y, \mathcal{G}(d))$ for $d \gg 0$. Applying all of the above to $\mathcal{O}_Y \xrightarrow{\varphi^\#} \varphi_*\mathcal{O}_X$, it suffices to show

$$\Gamma(Y, \mathcal{O}_Y(d)) \xrightarrow{\sim} \Gamma(Y, \varphi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(d)) = \Gamma(X, \mathcal{L}^{\otimes d}).$$

Our choice of F was exactly such that this map is injective for all $d \geq 0$. By definition,

$$\dim \Gamma(Y, \mathcal{O}_Y(d)) = 2d + 1, \quad \dim \Gamma(X, \mathcal{L}^{\otimes d}) = \deg(\mathcal{L}^{\otimes d}) + 1 = 2d + 1$$

where for the last equality we used crucially that X is genus 0. Hence we have our isomorphism. ☐

Lemma 3.4.7 (Projection formula)**.** *Let $f \colon X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a finite locally free $\mathcal{O}_Y$-module and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = (f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G}$$

and in fact,

$$R^if_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = (R^if_*\mathcal{F}) \otimes_Y \mathcal{G}.$$

Proof. This is clear if $\mathcal{G} = \mathcal{O}_Y^{\oplus n}$. In general, localize. (To make this cleaner, by adjunction, construct a global map

$$f^*f_*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = f^*(f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \to \mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}.$$

This saves us from having to show the proof is independent of choice of basis.) ☐

Remark. Picture of all curves:

$$\text{smooth} \subset \text{non-singular} = \text{regular} = \text{normal} \subset \text{Gorenstein} \subset \text{all curves}.$$

Note that all curves are necessarily CM, so for curves there is always a dualizing module and a perfect duality, but the module is not always invertible.

Theorem 3.4.8. *If X/k is a singular projective curve with $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$, then*

1. *X has a k-rational point x which is the unique singular point,*
2. *the normalization X_ν of X is isomorphic to $\mathbb{P}^1_{k'}$ with k'/k a non-trivial finite extension, and*

3. *there is a k'-rational point $x' \in \mathbb{P}^1_{k'}$ mapping to x such that*

$$\begin{array}{ccc} \operatorname{Spec} k' = x' & \longrightarrow & \mathbb{P}^1_{k'} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k = x & \longrightarrow & X \end{array}$$

   is a pushout diagram.

Example 3.4.9. If $k = \bar{k}$ is algebraically closed, $\mathbb{P}^1_{\bar{k}}$ is the only genus zero curve. If $k = \mathbb{R}$, then $\mathbb{P}^1_{\mathbb{R}}$ and $T_0^2 + T_1^2 + T_2^2 = 0$ and $T_0^2 + T_1^2 = 0$ are the only genus zero curves.

Example 3.4.10. Over $k = \mathbb{F}_2$ or $k = \overline{\mathbb{F}}_2$, consider $Z \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ given by $X_0^2 Y_0 + X_1^2 Y_1 + X_2^2 Y_2 = 0$, where the homogeneous coordinates are $[X_0 : X_1 : X_2]$ and $[Y_0 : Y_1 : Y_2]$. (View Z as defined by a global section of $\mathcal{O}(2,1)$.) Claim: Z is a smooth projective 3-fold over k. We give two proofs of this.

1. Look in the affine piece $(X_0 \neq 0) \times (Y_0 \neq 0)$. The equation for Z becomes $1 + x_1^2 y_1 + x_2^2 y_2 = 0$. This is smooth by the Jacobian criterion. By symmetry, we only have to check on one more affine open. Let's check on $(X_0 \neq 0) \times (Y_1 \neq 0)$. The equation for Z becomes $y_0 + \cdots = 0$, and again it is smooth by the Jacobian criterion.

2. Consider $\mathrm{pr}_1 \colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ and let $\pi \colon \mathrm{pr}_1|_Z \colon Z \to \mathbb{P}^2$. The fibers of π are smooth because they are lines. This "implies" π is smooth after we show π is flat. Now note that $Z \to \mathbb{P}^2 \to \operatorname{Spec} k$ is smooth as the composition of smooth morphisms.

Let $\gamma \colon \mathrm{pr}_2|_Z \colon Z \to \mathbb{P}^2$. Let $\eta \in \mathbb{P}^2$ be the generic point. Then the residue field $\kappa(\eta) = k(s,t)$ is the function field, where $s = Y_1/Y_0$ and $t = Y_2/Y_0$. Observe that the generic fiber of γ, i.e. the scheme $X = Z \times_\gamma \eta$ over η, is given by $X_0^2 + sX_1^2 + tX_2^2 = 0$. We know X is regular, because Z is. However it is not geometrically reduced, by base changing to any extension where s and t are both squares. Upshot: in positive characteristic, generic fibers of morphisms between smooth projective varieties (geometrically good) can be geometrically non-reduced even though they are always regular. In characteristic 0, such a generic fiber is actually always smooth, but is not necessarily geometrically irreducible.

3.5 Varieties and rational maps

We first state some facts about properties of curves. These facts will be important later on.

Proposition 3.5.1. *Over an algebraically closed field, every regular variety is smooth.*

Corollary 3.5.2. *Normal curves are smooth. More generally, given a curve X, the normalization X^ν is smooth.*

Proposition 3.5.3. *If $f \colon X \to Y$ is a finite morphism and Y is projective over k, then X is projective over k.*

Corollary 3.5.4. *Given a projective curve X, the normalization X^ν is a projective curve.*

Proposition 3.5.5. *Every curve is either affine or projective.*

Proposition 3.5.6. *If the curve X is affine, then there exists an open immersion $X \hookrightarrow \bar{X}$ with $\bar{X}$ a projective curve. If X smooth, then we may pick $\bar{X}$ smooth.*

Definition 3.5.7. Let X and Y be varieties. A **rational map from X to Y** is an equivalence class of pairs (U, f) where $U \subset X$ is non-empty open and $f \colon U \to Y$ is a morphism of varieties, and $(U, f) \sim (U', f')$ iff f and f' agree on some non-empty open $U'' \subset U \cap U'$. Notation: $f \colon X \dashrightarrow Y$.

Definition 3.5.8. A rational map $f\colon X \dashrightarrow Y$ is **dominant** iff $f(U)$ is dense for any $f = (U, f)$. This is iff f sends the generic point of X to the generic point of Y. (This equivalence follows from Chevalley's theorem.)

Remark. Note that the composition of rational maps does not make sense: the image of the first map may completely miss the domain of definition of the second map. Luckily, we can compose dominant rational maps $f\colon U \to Y$ and $g\colon V \to Z$, by taking $g \circ f\colon f^{-1}(V) \to Z$.

Definition 3.5.9. Two varieties X and Y are **birational** iff they are isomorphic in the category of varieties and dominant rational maps. If $f\colon X \dashrightarrow Y$ is a dominant rational map, then we get a map

$$k(X) \coloneqq \kappa(\text{generic point of } X) \xleftarrow{f^\#} \kappa(\text{generic point of } Y) = k(Y)$$

where $k(X)$ is the **function field** of X, and is equal to the fraction field of $\mathcal{O}_X(U)$ for any $U \subset X$ non-empty affine open, or $\mathcal{O}_{X,\eta} = \kappa(\eta)$ where $\eta \in X$ is the generic point. (From last semester, we know $\dim X = \operatorname{trdeg}_k k(X)$.)

Theorem 3.5.10. *The functor*

$$\begin{pmatrix}\text{varieties with}\\ \text{dominant rational maps}\end{pmatrix} \to \begin{pmatrix}\text{finitely generated field extensions } K/k\\ \text{and maps of fields over } k\end{pmatrix}$$

is an equivalence

Proof. Step 1: check the functor is essentially surjective. Pick K/k in the rhs. Pick generators $\theta_1, \dots, \theta_n \in K$, and let A be the k-algebra generated by $\theta_1, \dots, \theta_n$ inside K. Then A is a finitely generated k-algebra, and $K = \operatorname{Frac} A$ by construction. So $X = \operatorname{Spec} A$.

Step 2: check the functor is full. Let X and Y be varieties, and $k(X) \xleftarrow{\varphi} k(Y)$ be a morphism in the rhs. Pick $U \subset X$ and $V \subset Y$ non-empty affine open. Then we have the situation

$$\begin{tikzcd} k(X) & k(Y) \arrow[l, "\varphi"'] \\ \mathcal{O}_X(U) \arrow[u] & \mathcal{O}_Y(V). \arrow[u] \end{tikzcd}$$

Pick generators $\theta_1, \dots, \theta_n$ for $\mathcal{O}_Y(V)$ as a k-algebra. Suppose $\varphi(\theta_i) = a_i/b_i$ where $a_i, b_i \in \mathcal{O}_X(U)$, with $b_i \neq 0$. Then let U' be the principal open obtained from U by localizing at $b_1, \dots, b_n$, so that

$$\mathcal{O}_X(U)[1/b_1 \cdots b_n] = \mathcal{O}_X(U').$$

Then $\mathcal{O}_X(U') \xleftarrow{\varphi} \mathcal{O}_Y(V)$ induces $\operatorname{Spec}(\varphi)\colon U' \to V$, and we are done. $\square$

Corollary 3.5.11. *A variety X is* **rational** *iff X is birational to $\mathbb{P}^n_k$. Equivalently, its function field is a purely transcendental extension of the base field.*

Theorem 3.5.12. *The construction of the theorem above defines an equivalence of categories*

$$\begin{pmatrix}\text{smooth projective curves}\\ \text{and non-constant (so finite) morphisms}\end{pmatrix} \to \begin{pmatrix}\text{finitely generated field extensions } K/k\\ \text{with } \operatorname{trdeg}_k(K) = 1 \text{ and maps of fields over } k\end{pmatrix}.$$

Proof. The functor is essentially surjective: use the facts stated above for curves. The only thing left to show is that given smooth projective curves X, Y, any morphism $X \supset U \xrightarrow{f} Y$ extends to all of X. Put $Y \hookrightarrow \mathbb{P}^n$. Then it is enough to show that $X \supset U \to \mathbb{P}^n$ lifts to $X \to \mathbb{P}^n$. Since $\dim X = 1$, $X - U = \{x_1, \dots, x_n\}$ is a finite set of points. So we can extend the morphism one point at a time. Main point: $\mathcal{O}_{X,x_i}$ is a DVR. It is enough to show that given a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k, \end{tikzcd}$$

where R is a DVR over k and $K = \mathrm{Frac}(R)$, the arrow $\mathrm{Spec}(R) \to \mathbb{P}^n$ exists. Then apply this to $R = \mathcal{O}_{X,x_i}$ and $K = \mathrm{Frac}(\mathcal{O}_{X,x_i}) = k(X)$.

The morphism $\mathrm{Spec}\, K \to \mathbb{P}^n$ is given by $(a_0 : \cdots : a_n)$ where $a_0, \ldots, a_n \in K$ are not all zero. Pick a uniformizer $\pi \in R$ and write $a_i = u_i \pi^{e_i}$ where $u_i \in R$ are units and $e_i \in \mathbb{Z}$. Wlog, e_0 is the smallest. Then

$$(a_0 : \cdots : a_n) = (u_0 : u_1 \pi^{e_1 - e_0} : \cdots : u_n \pi^{e_n - e_0}),$$

so that we get the desired map $\mathrm{Spec}\, R \to \mathbb{P}^n$. $\square$

Lemma 3.5.13. *Given X, Y varieties, points $x \in X$ and $y \in Y$, and a local k-algebra map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, there exists an open neighborhood $U \ni x$ and $f \colon U \to Y$ such that $y = f(x)$ and $\varphi = f_x^{\#} \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.*

Proof. Exactly as in the proof of fullness for the functor from varieties above. $\square$

3.6 Weil divisors

Definition 3.6.1. A **Weil divisor** D on a smooth curve X is $D = \sum_{i=1}^{N} n_i x_i$ with $n_i \in \mathbb{Z}$, i.e. a formal linear combination of closed points x_i. The **support of** D, denoted $\mathrm{supp}\, D$, is the union of all x_i with non-zero n_i. We say D is **effective** if $n_i \geq 0$ for every i, and write $D \geq 0$. If X is projective, the **degree** is $\deg(D) \coloneqq \sum n_i$.

Definition 3.6.2. Given $f \in k(X)^*$ (the units in $k(X)$), define the **divisor associated to** f as $\mathrm{div}_X(f) \coloneqq \mathrm{div}(f) \coloneqq \sum_{x \in X \text{ closed}} \mathrm{ord}_x(f) x$, where $\mathrm{ord}_x(f)$ is the order of vanishing of f at x. (This sum is finite because $f \in k(X)$ invertible means f is invertible in a Zariski open, and the complement of a Zariski open in a curve is dimension 0.) It is determined by the valuation on the DVR $\mathcal{O}_{X,x}$.

Example 3.6.3. Let $X = \mathbb{A}^1_{\mathbb{C}}$ and $f = (t - \pi)/(t - e)^{10}$. Then $\mathrm{div}_X(f) = 1 \cdot \pi - 10 \cdot e$.

Definition 3.6.4. Given a Weil divisor D, define $\mathcal{O}_X(D)$ to be the sheaf of $\mathcal{O}_X$-modules defined by

$$U \mapsto \{0\} \cup \{f \in k(X)^* : \mathrm{div}_U(f) + D|_U \geq 0\}.$$

(If $f = 0$, the order of vanishing $\mathrm{ord}_x(f) = \infty$ for any $x \in X$.) Note that $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-submodule of the constant sheaf with value $k(X)$.

Example 3.6.5. In the preceding example of $\mathrm{div}_X(f)$, we have $f \in \mathcal{O}_X(10 \cdot e)(X)$.

Example 3.6.6. $\mathcal{O}_X(-x)$ is the ideal sheaf of x: it consists of regular functions which vanish to at least order 1 on x. On the other hand, $\mathcal{O}_X(x)$ is the sheaf of functions regular everywhere except x, where we are allowed a simple pole.

Lemma 3.6.7. *$\mathcal{O}_X(D)$ is an invertible $\mathcal{O}_X$-module.*

Proof idea. Pick $x \in X$. Let n be the coefficient of x in D. Pick $f \in k(X)^*$ with $\mathrm{ord}_x(f) = -n$. Then there exists $U \ni x$ open such that:

1. for all $u \in U$ with $u \neq x$, we have $\mathrm{ord}_u(f) = 0$;
2. $U \cap \mathrm{supp}(D) = \{x\}$.

Then we see immediately that $\mathcal{O}_U \cdot f = \mathcal{O}_U(D)$. $\square$

Lemma 3.6.8. *Given D and $f \in k(X)^*$, multiplication by f induces an isomorphism $\mathcal{O}_X(D) \xrightarrow{\sim} \mathcal{O}_X(D - \mathrm{div}_X(f))$.*

Proof. Immediate. $\square$

Remark. We are consistently using the fact that if $U \subset X$ is affine, then $\mathcal{O}_U(U) = \mathcal{O}_X(U) = \bigcap_{u \in U} \mathcal{O}_{U,u}$, where the intersection happens in $k(X)$. This is true for varieties in general, and just says that a function is regular in U iff it is regular at each point $u \in U$.

Lemma 3.6.9. *There is a canonical isomorphism $\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1 + D_2)$ given by multiplication on sections.*

Lemma 3.6.10. *If X is projective and smooth, then $\deg(\mathcal{O}_X(D)) = \deg(D)$.*

Proof. We know the degree of line bundles is additive, and degree of Weil divisors is additive. So it is enough to show this in the case $D = -x$. Then $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{F} \to 0$ where $\mathcal{F}$ is a skyscraper sheaf at x with value k. Hence the degree is 1. (Alternatively, write $0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to i_*\mathcal{O}_{X,x} \to 0$.) $\square$

Corollary 3.6.11. *If X is projective and smooth, then $\deg \operatorname{div}_X(f) = 0$.*

Proof. $\deg \operatorname{div}_X(f) = \deg \mathcal{O}_X(f) = \deg \mathcal{O}_X = 0$, by the isomorphism $\mathcal{O}_X(f) \xrightarrow{\sim} \mathcal{O}_X(\operatorname{div}_X(f) - \operatorname{div}_X(f))$. $\square$

Corollary 3.6.12. *(X need not be projective, but must still be smooth.) The Picard group $\operatorname{Pic}(X)$ is isomorphic to the group of Weil divisors mod the subgroup $\{\operatorname{div}_X(f) : f \in k(X)^*\}$ of principal divisors:*

Proof. By a previous lemma, there exists a well-defined map taking a Weil divisor to $\operatorname{Pic}(X)$ that descends to Weil/principal, and by another previous lemma it is additive. If $\mathcal{O}_X \to \mathcal{O}_X(D)$ is an isomorphism, then the image of 1 is $f \in k(X)^*$ with $\operatorname{div}_X(f) = -D$, so we have injectivity. For surjectivity, if $\mathcal{L} \in \operatorname{Pic}(X)$ has a non-zero global section, then we get

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to \mathcal{L}|_Z = \mathcal{O}_Z \to 0$$

where $Z \coloneqq Z(s)$ is the zero locus of the section s. Set-theoretically, $Z = \{x_1, \ldots, x_n\}$, say with multiplicities n_i. This means

$$\mathcal{O}_{Z,x_i} = \mathcal{O}_{X,x_i}/(\pi_i)^{n_i}$$

where π_i is the uniformizer for $\mathcal{O}_{X,x_i}$. So in terms of locally trivializing sections of $\mathcal{L}$, the section s vanishes to order n_i in $\mathcal{L}$, i.e. $\mathcal{O}_X(\sum n_i x_i) = \mathcal{L}$. By a previous argument, any $\mathcal{L}$ is of the form $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ where both $\mathcal{L}_1, \mathcal{L}_2$ have non-vanishing global sections. $\square$

3.7 Separating points and tangent vectors

Let X be a smooth projective curve.

Lemma 3.7.1. *$\mathcal{L}$ is globally generated iff $h^0(\mathcal{L}(-x)) = h^0(\mathcal{L}) - 1$ for all $x \in X$ closed.*

Lemma 3.7.2. *If for all $x, y \in X$ (not necessarily distinct) we have $h^0(\mathcal{L}(-x-y)) = h^0(\mathcal{L}) - 2$, then with $V = H^0(X, \mathcal{L})$, we get a closed immersion $\varphi_{\mathcal{L},V} \colon X \to \mathbb{P}(V)$.*

Proof. By a previous lemma, $\mathcal{L}$ is globally generated, so we get $\varphi \colon X \to \mathbb{P}(V)$. We know φ is injective because if $x \neq y$, then $\varphi(x) \neq \varphi(y)$; otherwise the set of hyperplanes passing through $\varphi(x)$ and $\varphi(y)$ is the same as the set of hyperplanes passing only through one of them, and therefore has codimension 1. But we assumed it has codimension 2. So $\varphi \colon X \to \varphi(X) \subset \mathbb{P}(V)$ is a homeomorphism. To finish, we need to show $\mathcal{O}_{\mathbb{P}(V)} \to \varphi_*\mathcal{O}_X$ is surjective. Pick $x \in X$. It suffices to check at stalks

$$\mathcal{O}_{\mathbb{P}(V),\varphi(x)} \to (\varphi_*\mathcal{O}_X)_{\varphi(x)} = \mathcal{O}_{X,x}$$

where the equality holds because φ is a homeomorphism onto its image. (In general this holds if φ is a closed map with a unique point in the fiber.) Since φ is finite, we have $(\varphi_*\mathcal{O}_X)_{\varphi(x)}$ is a finite module over $\mathcal{O}_{\mathbb{P}(V),\varphi(x)}$. So Nakayama applies. It is enough to show $\mathcal{O}_{X,x}/\mathfrak{m}_{\varphi(x)}\mathcal{O}_{X,x} = k$, the residue field at any closed point of $\mathbb{P}(V)$. Take $x = y \in X$. Then there exists an $s \in V$ such that s vanishes to order exactly 1 at x. So $\mathfrak{m}_{\varphi(x)}\mathcal{O}_{X,x}$ contains a uniformizer; if $s' \in V$ does not vanish at x, then $f \coloneqq s/s' \in \mathcal{O}_{\mathbb{P}(V)}(D_+(s'))$ maps to the uniformizer at x. $\square$

Definition 3.7.3. We call V the **complete linear system** associated to $\mathcal{L}$.

Corollary 3.7.4. *If* $\deg \mathcal{L} \geq 2g+1$ *with* $V := H^0(X, \mathcal{L})$, *then* $\varphi_{\mathcal{L},V} : X \to \mathbb{P}(V)$ *is a closed immersion.*

Corollary 3.7.5. *If* D *is effective and* $\deg D \geq 2g+1$, *then* $X - \operatorname{supp}(D)$ *is affine.*

Proof. $\mathcal{L} = \mathcal{O}_X(D)$ gives $\varphi\colon X \hookrightarrow \mathbb{P}(V)$. The section $s := 1 \in V$ vanishes exactly at $\operatorname{supp}(D)$. So $X - \operatorname{supp}(D) = \varphi^{-1}(D_+(s))$. $\square$

Corollary 3.7.6. *Any genus* 1 *(smooth projective) curve is a plane cubic.*

Definition 3.7.7. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible on X. Then we get a multiplication map

$$H^0(X, \mathcal{L}_1) \otimes_k H^0(X, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2), \quad s_1 \otimes s_2 \mapsto s_1 s_2.$$

If $s_1, s_2 \neq 0$, then $s_1 s_2 \neq 0$. This map does not kill pure tensors.

Lemma 3.7.8. *Let* $\mu\colon V_1 \otimes_k V_2 \to V$ *be a linear map which does not kill pure tensors. Then* $\dim V \geq \dim V_1 + \dim V_2 - 1$.

Proof. Let $n_i := \dim V_i$. Let $\mathbb{A}_k^{n_1 n_2}$ be the affine space whose points are $V_1 \otimes_k V_2$, and $C \subset \mathbb{A}_k^{n_1 n_2}$ be the cone of pure tensors. Then $\dim C = n_1 + n_2 - 1$. By dimension theory, C intersected with n hyperplanes in $\mathbb{A}^{n_1 n_2}$ passing through 0 has dimension at least $n_1 + n_2 - 1 - n$. Note that the intersection is never empty, because $0 \in C$. It will have another non-zero vector if $n < n_1 + n_2 - 1$. Hence apply this to the hyperplanes $\lambda_i = 0$ for $i = 1, \ldots, n$ where $\epsilon_1, \ldots, \epsilon_n$ is a basis for V^* and $\lambda_i := \epsilon_i \circ \mu$. $\square$

Corollary 3.7.9 (Clifford's theorem)**.** *If* $h^0(\mathcal{L}) > d + 1 - g$ *and* $d \geq 0$, *then* $2h^0(\mathcal{L}) \leq d + 2$.

Proof. In this case, $h^0(\omega_X \otimes \mathcal{L}^{-1}) > 0$. By the lemma applied to $H^0(\mathcal{L}) \otimes_k H^0(\omega_X \otimes \mathcal{L}^{-1}) \to H^0(\omega_X)$, we get

$$g \geq h^0(\mathcal{L}) + (h^0(\mathcal{L}) - d - 1 + g) - 1 = 2h^0(\mathcal{L}) - d - 1 + g - 1.$$ $\square$

Remark. If equality holds in Clifford's theorem, then X is hyperelliptic.

3.8 Degree of morphisms and ramification

Definition 3.8.1. Let $f\colon X \to Y$ be a dominant morphism of varieties of the same dimension. Then the **degree** of f is the $\deg f := [k(X) : k(Y)]$. (This is a finite extension because the transcendence degree of both function fields is the same.)

Lemma 3.8.2. *If* $f\colon X \to Y$ *is a non-constant (and therefore dominant) morphism of projective curves, then*

$$\deg(f^*\mathcal{L}) := \deg(f) \cdot \deg(\mathcal{L}).$$

Proof for smooth curves. Let $y \in Y$ be a closed point. We will show $\deg(f^*\mathcal{O}_Y(-y)) = -\deg(f)$, which is enough by additivity. Choose $V \subset Y$ affine open containing y. Then $f\colon U := f^{-1}(V) \to V$ is a finite (because its fibers are finite) morphism of affine schemes which are spectra of Dedekind domains (because X, Y are smooth). Then $f^*\mathcal{O}_Y(-y)$ is the ideal sheaf of the scheme-theoretic fiber $X_y \subset X$. So there is a short exact sequence

$$0 \to f^*\mathcal{O}_Y(-y) \to \mathcal{O}_X \to \mathcal{O}_{X_y} \to 0$$

Then $\deg(f^*\mathcal{O}_Y(-y)) = -\deg(\mathcal{O}_{X_y})$. So it suffices to show $\deg(X_y) = \deg(f)$. This follows from the next lemma. $\square$

Lemma 3.8.3. *Let* $A \subset B$ *be a finite extension of Dedekind domains. Then for* $\mathfrak{m} \subset A$ *maximal, we have* $\dim_{\kappa(x)} B/\mathfrak{m}B = [\operatorname{Frac}(B) : \operatorname{Frac}(A)]$.

Remark. In fact, if $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \subset B$ are the maximal ideals of B lying over $\mathfrak{m} \subset A$, and if e_i is the **ramification index** of $A_\mathfrak{m} \to B_{\mathfrak{m}_i}$, then

$$n \coloneqq [\mathrm{Frac}(B) : \mathrm{Frac}(A)] = \sum_{i=1}^{r} e_i f_i$$

where $f_i \coloneqq [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$. This comes from $B/\mathfrak{m}B$ being an Artinian ring and the structure theorem of Artinian rings.

Definition 3.8.4. Let $f\colon X \to Y$ be a non-constant morphism of smooth curves. Then the **ramification index** of f at a closed point $x \in X$ is the ramification index of $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Remark. If g is a local function on Y defined in a neighborhood of $f(x)$, then

$$(\text{vanishing order of } g \circ f \text{ at } x) = (\text{ramification index of } f \text{ at } x) \cdot (\text{vanishing order of } g \circ f \text{ at } f(x)).$$

Definition 3.8.5 (Pullback of divisors)**.** Let $f\colon X \to Y$ be a non-constant morphism of smooth curves. Then define the **pullback**

$$f^*\colon \mathrm{Div}(Y) \to \mathrm{Div}(X), \quad y \mapsto \sum_{x \in f^{-1}(y)} (\text{ramification index of } f \text{ at } x) \cdot x.$$

Then there is a commutative (by the remark above) diagram

$$\begin{tikzcd}
k(Y)^* \arrow[r, "\mathrm{div}_Y"] \arrow[d, "f^\#"'] & \mathrm{Div}(Y) \arrow[r] \arrow[d, "f^*"'] & \mathrm{Pic}(Y) \arrow[r] & 0 \\
k(X)^* \arrow[r, "\mathrm{div}_X"] & \mathrm{Div}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & 0.
\end{tikzcd}$$

Definition 3.8.6 (Pushforward of divisors)**.** Assume f is finite and X, Y smooth. Then define the **pushforward**

$$f_*\colon \mathrm{Div}(X) \to \mathrm{Div}(Y), \quad \sum n_i x_i \mapsto \sum n_i f(x_i).$$

Proposition 3.8.7. $f_* \operatorname{div}_X(g) = \operatorname{div}_Y(\operatorname{Nm} g)$ *where* $g \in k(X)^*$.

Remark. It follows that there is an induced map $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ given by the diagram

$$\begin{tikzcd}
k(X)^* \arrow[r] \arrow[d, "\mathrm{Nm}"'] & \mathrm{Div}(X) \arrow[r] \arrow[d, "f_*"'] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "\mathrm{Nm}"'] & 0 \\
k(Y)^* \arrow[r] & \mathrm{Div}(Y) \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & 0.
\end{tikzcd}$$

There is actually a direct way to construct $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$: define

$$\mathrm{Nm}(\mathcal{L}) \coloneqq \wedge^n (f_*\mathcal{L}) \otimes_{\mathcal{O}_Y} \wedge^n (f_*\mathcal{O}_X)^{\otimes -1}$$

where $n \coloneqq \deg(f)$. (View the second term as an "adjustment," so that if we plug in $\mathcal{L} = \mathcal{O}_X$, we get $\mathcal{O}_Y$.)

Lemma 3.8.8. *Let $A \subset B$ be a finite extension of Dedekind domains. Let $\mathfrak{m} \subset A$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \subset B$ lying over $\mathfrak{m}$. Then for $b \in B$ we have*

$$\mathrm{ord}_{A_\mathfrak{m}}(\mathrm{Nm}\, b) = \sum \mathrm{ord}_{B_{\mathfrak{m}_i}}(b) \cdot f_i.$$

(See tag 02MJ for a more general statement.)

Proof. The lhs is precisely $\mathrm{length}_{A_\mathfrak{m}}((B/bB)_\mathfrak{m})$ because

$$\mathrm{Nm}(b) = \det_{A_\mathfrak{m}}(B_\mathfrak{m} \xrightarrow{b} B_\mathfrak{m}).$$

Now use that $(B/bB)_\mathfrak{m} = \bigoplus_i (B/bB)_{\mathfrak{m}_i}$ by the structure theorem for Artinian local rings. $\square$

Remark. If $f\colon X \to Y$ is non-constant and X and Y are smooth projective curves, then the composition

$$\mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\mathrm{Nm}} \mathrm{Pic}(Y)$$

is just multiplication by $\deg(f)$.

3.9 Hyperelliptic curves

Definition 3.9.1. A curve is **hyperelliptic** if it has a degree-2 morphism to $\mathbb{P}^1$. The **gonality** of a smooth projective curve X is the smallest degree d of a non-constant morphism $X \to \mathbb{P}^1$.

Example 3.9.2. Gonality 1 curves are $\mathbb{P}^1$, and gonality 2 curves are hyperelliptic. Gonality 3 curves are called **trigonal** curves.

Example 3.9.3. Any genus 1 curve is hyperelliptic: pick any two points x and y, and then $\mathcal{L} \coloneqq \mathcal{O}_X(x+y)$ has $h^0(\mathcal{L}) = 2 + 1 - 1$ and must be globally generated, because otherwise we drop a degree and end up isomorphic to $\mathbb{P}^1$.

Example 3.9.4. Any genus 2 curve is hyperelliptic: the canonical bundle K_X has two sections, and every other line bundle has only one section (cf. Clifford's theorem).

Lemma 3.9.5. *The function field $k(X)$ of a hyperelliptic curve X is a degree 2 extension of $k(\mathbb{P}^1) = k(t)$. So if $\mathrm{char}\, k \neq 2$, then*

$$k(X) = k(t)[y]/(y^2 - (t-a_1)\cdots(t-a_r))$$

for pairwise distinct $a_1, \ldots, a_r \in k$. For $\mathrm{char}\, k = 2$,

$$k(X) = k(t)[y]/(y^2 - y + f(t)),\ f(t) \in k(t) \quad \text{or} \quad k(t)[y]/(y^2 - t) \cong k(y).$$

Remark. From this description, we see that whenever $t \neq a_i$, there will be two points, each with ramification index 1. When $t = a_i$ or ∞, then there will be one point with ramification index 2. We see that

$$g = \begin{cases} (r-2)/2 & r \text{ even} \\ (r-1)/2 & r \text{ odd.} \end{cases}$$

3.10 Riemann–Hurwitz

Note that in characteristic $p > 0$, it can happen that all ramification indices are > 1. For example, take $k = \bar{k}$ with characteristic $p > 0$. Then the map $t \mapsto t^p$ from $\mathbb{P}^1_k \to \mathbb{P}^1_k$ is such a map. Keep this in mind while we prove Riemann–Hurwitz.

Lemma 3.10.1. *Let $f\colon X \to Y$ be a non-constant morphism of projective curves. The following are equivalent:*

1. *$k(X)/k(Y)$ is separable;*
2. *for all but finitely many (closed) points $x \in X$, the ramification index e_x is 1;*
3. *the module of relative differentials $\Omega_{X/Y}$ is supported in finitely many points;*

4. $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is non-zero.

Proof. Pick affine opens $\mathrm{Spec}(A) \subset Y$, and let $\mathrm{Spec}(B) \subset X$ be the inverse image of f. This gives

$$\begin{array}{ccc} B & \longrightarrow & k(X) \\ \uparrow & & \uparrow \\ A & \longrightarrow & k(Y). \end{array}$$

In this situation, commutative algebra gives an exact sequence $\Omega_{A/k} \otimes_A B \to \Omega_{B/k} \to \Omega_{B/A} \to 0$. Scheme-theoretically, the sequence is $f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$. Taking stalks at the generic point (i.e. localizing at $(0) \subset B$) gives

$$\Omega_{k(Y)/k} \otimes_{k(Y)} k(X) \to \Omega_{k(X)/k} \to \Omega_{k(X)/k(Y)} \to 0.$$

Because A and B are finite-type k-algebras, and moreover these Ω's are finite, in fact of rank 1, the map $f^*\Omega_{Y/K} \to \Omega_{X/k}$ is non-zero iff $\Omega_{X/Y}$ is supported at finitely many points. Field theory fact: $k(X)/k(Y)$ is separable iff $\Omega_{k(X)/k(Y)} = 0$ (if you have a double root, you have a non-zero differential). Pick $x \in X$ closed and set $y := f(x)$ and consider $\mathcal{O}_{Y,y} \xrightarrow{f^\#} \mathcal{O}_{X,x}$, and take uniformizers π_y and π_z.

Schemes fact: $(\Omega_{X/k})_x = \Omega_{\mathcal{O}_{X,x}/k}$. Algebra fact: if A/k is a DVR, essentially of finite type, with residue field k and uniformizer π (i.e. local ring of a curve at a smooth point), then $\Omega_{A/k} = A\,d\pi$. This is because $\Omega_{A/k}$ is a finite A-module, and for $a \in A$ we can write $a = \lambda + \pi a'$ where $\lambda \in k$ and $a' \in A$. Then $da = d\lambda + \pi(da') + a'\,d\pi$. By Nakayama's lemma, we conclude $d\pi$ is a generator of $\Omega_{A/k}$. We finish by noting $\Omega_{A/k}$ cannot be torsion, because the localization at the generic point is $\Omega_{k(X)/k}$, which has transcendence degree 1. Then by the classification of modules over a DVR, $\Omega_{A/k}$ is free.

By taking stalks at x, we get $\Omega_{\mathcal{O}_{Y,y}/k} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \Omega_{\mathcal{O}_{X,x}/k} \to (\Omega_{X/Y})_x \to 0$. The generator of the first term is $d\pi_y$, and the generator of the second term is $d\pi_x$. We know $\pi_y \mapsto u\pi_x^{e_x}$ where $u \in \mathcal{O}_{X,x}$ is a unit. So

$$d\pi_y = d(u\pi_x^{e_x}) = \pi_x^{e_x}\,du + e_x\pi_x^{e_x-1}u\,d\pi_x = (\pi_x^{e_x-1}\,d\pi_x)(e_xu\, + \pi_x\,du/d\pi_x).$$

This whole thing iff $e_x = 1$, and $e_xu \neq 0$. But $u \neq 0$, and $e_xu = 0$ iff $p \mid e_x$. Set $d_x := \mathrm{length}_{\mathcal{O}_{X,x}}((\Omega_{X/Y})_x)$. We have actually shown

$$d_x = \mathrm{ord}_{\mathcal{O}_{X,x}}((e_xu + \pi_x du/d\pi_x)\pi_x^{e_x-1}). \qquad \square$$

Definition 3.10.2. If any of the equivalent conditions of the lemma hold, then we say f is **separable**. In characteristic 0, everything is separable. In general (when f is not necessarily separable), there exists $n \geq 0$ and a factorization

$$f\colon X \xrightarrow{F^n_{X/k}} X^{(p^n)} \xrightarrow{g} Y.$$

such that g is separable. The morphism $F^n_{X/k}$ arises as follows. Take the Cartesian diagram

$$\begin{array}{ccc} X^{(p^n)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{F^n_{\mathrm{Spec}\,k}} & \mathrm{Spec}(k) \end{array}$$

where $F^n_{\mathrm{Spec}\,k}$ is the (p^n)-th power map, and note that the resulting map $X^{(p^n)} \to X$ is not a morphism over $\mathrm{Spec}\,k$. The **absolute Frobenius** $F_X\colon (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ is id_X on topological spaces, and is $a \mapsto a^p$ on $\mathcal{O}_X$. It gives a map $X \to X$, so because the diagram above is Cartesian, there is a factorization via $F^n_{X/k}\colon X \to X^{(p^n)}$, which is the **relative Frobenius**.

Example 3.10.3. Suppose $X \subset \mathbb{P}^n_k$ is cut out by $\sum_{i=1}^r a_{i,I}T^I = 0$, with $a_{i,I} \in k$. Then $X^{(p^n)}$ is cut out by $\sum a_{i,I}^{p^n}T^I = 0$ and $F^n_{X/k}\colon X \to X^{(p^n)}$ is given by $T_i \mapsto (T_i)^{p^n}$.

Corollary 3.10.4 (Riemann–Hurwitz). *In the situation of the lemma, assume f is separable.*

1. *There is a short exact sequence $0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$.*
2. *Let $d_x := \mathrm{length}_{\mathcal{O}_{X,x}}((\Omega_{X/Y})_x)$ be the multiplicity of $\Omega_{X/Y}$ at x. Then $2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum d_x$ by taking degrees.*
3. *The relationship between d_x and the ramification index e_x is that $d_x \geq e_x - 1$, with equality iff $p \nmid e_x$.*

Definition 3.10.5. We say f is **tamely ramified** if $p \nmid e_x$ for all $x \in X$ (and the residue field extensions are separable).

Corollary 3.10.6. *Let K_X denote the canonical divisor on X, i.e. $\Omega_{X/k} = \mathcal{O}_X(K_X)$, and similarly for K_Y. Then $K_X \sim f^*K_Y + R$, where $R = \sum d_x \cdot x$ is the ramification divisor.*

Example 3.10.7. Some applications of Riemann–Hurwitz:

1. $g(X) \geq g(Y)$ (actually this is also true in the inseparable case, by the factorization via the relative Frobenius);
2. if $g(X) = g(Y)$ and $g(Y) \geq 2$, then $\deg f = 1$, i.e. $X \cong Y$;
3. suppose $Y = \mathbb{P}^1$ and $f(\operatorname{supp} R) = \{\infty\}$, so that if we assume only tame ramification,

$$2g(X) - 2 = -2\deg(f) + \sum d_x = -2\deg(f) + \sum_{x \mapsto \infty} (e_x - 1) = -\deg(f) - \#(\text{points above } \infty),$$

   which is impossible when $\deg f \geq 2$;

Example 3.10.8. Consider a hyperelliptic curve $y^2 = (t - a_1) \cdots (t - a_r)$. Clearly we have ramification with $e_x = 2$ at $a_1, \ldots, a_r$. So Riemann–Hurwitz gives

$$2g(X) - 2 = 2(2g(Y) - 2) + \sum (e_x - 1).$$

Since the first two terms are even, we see that there is ramification iff r is odd.

Chapter 4

Extra stuff

4.1 Picard scheme and Jacobian variety

Let X be a smooth projective curve over $k = \bar{k}$. Let $x \in X(k)$ be a point.

Definition 4.1.1. The **Picard functor** $\mathrm{Pic}_{X/k,x}$ is the functor $(\mathsf{Sch}/k)^{\mathrm{op}} \to \mathsf{Set}$ given by

$$S/k \mapsto \left(\begin{array}{c} \text{isomorphism classes of pairs } (\mathcal{L}, \alpha) \\ \text{where } \mathcal{L} \text{ is an invertible } \mathcal{O}_{X \times S} \text{ module} \\ \text{and } \alpha \colon \mathcal{O}_S \xrightarrow{\sim} x_S^* \mathcal{L} \text{ where } x_S \coloneqq x \times \mathrm{id}_S \colon S \to X \times S. \end{array} \right).$$

The way to think about his is "invertible modules on $X \times S$ trivialized along the section x." Given $h \colon S' \to S$ a morphism of Sch/k, define

$$\begin{aligned} \mathrm{Pic}_{X/k,x}(S) &\xrightarrow{\mathrm{Pic}_{X/S,x}(h)} \mathrm{Pic}_{X/k,x}(S') \\ (\mathcal{L}, \alpha) &\mapsto ((X \times S' \to X \times S)^* \mathcal{L}, \text{suitable pullback of } \alpha). \end{aligned}$$

Theorem 4.1.2. *The functor* $\mathrm{Pic}_{X/k,x}$ *is representable by a scheme* $\underline{\mathrm{Pic}}_{X/k}$*, and there is a universal pair* $(\mathcal{L}_{univ}, \alpha_{univ})$ *(where* $\mathcal{L}_{univ}$ *is on* $X \times \underline{\mathrm{Pic}}_{X/k}$*, and* $\alpha_{univ} \colon \mathcal{O}_{\underline{\mathrm{Pic}}_{X/k}} \xrightarrow{\sim} x^* \mathcal{L}_{univ}$*) such that*

$$\begin{aligned} \mathrm{Mor}_k(S, \underline{\mathrm{Pic}}_{X/k}) &= \mathrm{Pic}_{X/k,x}(S) \\ h &\mapsto \mathrm{Pic}_{X/k,x}(h)(\mathcal{L}_{univ}, \alpha_{univ}). \end{aligned}$$

Remark. The k-points of $\underline{\mathrm{Pic}}_{X/k}$ are $\mathrm{Pic}_{X/k,x}(\operatorname{Spec} k) = \mathrm{Pic}(X)$.

Lemma 4.1.3. $\underline{\mathrm{Pic}}_{X/k}$ *is an abelian group object in* Sch/k*.*

Proof. By the Yoneda lemma, the group law on $\underline{\mathrm{Pic}}_{X/k}$ comes from the fact that $\mathrm{Pic}_{X/k,x}$ is really a functor $(\mathsf{Sch}/k)^{\mathrm{op}} \to \mathsf{Ab}$ via the rule $(\mathcal{L}, \alpha) \cdot (\mathcal{L}', \alpha') \coloneqq (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$. ☐

Lemma 4.1.4. *There is a morphism of schemes*

$$\underline{\deg} \colon \underline{\mathrm{Pic}}_{X/k} \to \coprod_{d \in \mathbb{Z}} \mathrm{Spec}(k)$$

such that $\underline{\mathrm{Pic}}^d_{X/k} = \underline{\deg}^{-1}(d)$ *parametrizes degree* d *invertible modules on* X*.*

Remark. If $\mathcal{L}_0$ on X/k has degree d, then there is an isomorphism $\underline{\mathrm{Pic}}^0_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$ given by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{L}_0$. So these components are all isomorphic k-schemes.

Definition 4.1.5. The **symmetric powers of** X are given by $\mathrm{Sym}^d(X) \coloneqq (X \times \cdots \times X)/S_d$, where the symmetric group acts by permutations.

Theorem 4.1.6. *$\mathrm{Sym}^d(X)$ is a smooth projective variety over k.*

Theorem 4.1.7. *There is a universal effective Cartier divisor $D_{univ} \subset X \times \mathrm{Sym}^d(X)$ whose fiber over the closed point "$x_1 + \cdots + x_d$" (a multi-set of closed points of X) is the divisor $\sum x_i$ on X.*

Definition 4.1.8. Now we have $\mathcal{L}_d \coloneqq \mathcal{O}_{X\times \mathrm{Sym}^d(X)}(D) \otimes \mathrm{pr}_2^*(\text{something})$ where the "something" is to make it trivialized along x. By the universal property, we get a map $\mathrm{Sym}^d(X) \to \underline{\mathrm{Pic}}^d_{X/k}$ such that the point "$x_1 + \cdots + x_d$" maps to $\mathcal{O}_X(x_1 + \cdots + x_d)$.

Theorem 4.1.9. *For $d = g$, this morphism is an isomorphism over a non-empty open of $\underline{\mathrm{Pic}}^g_{X/k}$.*

Corollary 4.1.10. *$\underline{\mathrm{Pic}}^d_{X/k}$ is a smooth projective variety for all d.*

Theorem 4.1.11. *If $n \in \mathbb{Z}$ is coprime to $\mathrm{char}\, k$, then $\mathrm{Pic}(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$.*

Definition 4.1.12. An **abelian variety** is a group object in the category of varieties over k which is projective over k. (Automatically, it is commutative.)

Example 4.1.13. $\underline{\mathrm{Pic}}^0_{X/k}$ is an abelian variety of dimension g.

Theorem 4.1.14. *If A is an abelian variety over k $(= \bar{k})$, then $A(k)[n] = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2\dim A}$.*

Proof sketch. Look at the "multiplication by n" map $[n]\colon A \to A$. The group law is a morphism, so this is also a morphism. First look at the induced map $d[n]\colon T_0A \to T_0A$, which really is multiplication by n. Since n is coprime to $\mathrm{char}\, k$, this is an isomorphism of tangent spaces. Hence $[n]$ is a dominant morphism, and $\deg([n])$ is well-defined. Because of the group structure, $d[n]$ has maximal rank everywhere, so $[n]$ is étale. Then $A(k)[n] = [n]^{-1}(\{0\})$ has $\deg([n])$ points. So it suffices to show $\deg([n]) = n^{2\dim A}$. To do this, we can try to prove

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n^2}$$

for some $\mathcal{L}$ ample on A, because if so, then $[n]^*c_1(\mathcal{L}) = n^2 c_1(\mathcal{L})$ and taking powers gives $[n]^*c_1(\mathcal{L})^{\dim A} = n^{2\dim A}c_1(\mathcal{L})^{\dim A}$ in the Chow ring. But $c_1(\mathcal{L})^{\dim A}$ are the same zero cycle, with positive degree because $\mathcal{L}$ is ample. Hence $\deg([n])\deg(c_1(\mathcal{L})^{\dim A}) = n^{2\dim A}\deg(c_1(\mathcal{L})^{\dim A})$. Finally, to get an ample $\mathcal{L}$ with this property, use the theorem of the cube. □

4.2 Some open problems

Conjecture 4.2.1 (Hartshorne)**.** *Let k be a field. Any codimension 2 closed sub-variety of $\mathbb{P}^n_k$ is a complete intersection for $n \geq 37$.*

Remark. This is known for codimension 1. It is probably better to assume that it is a local complete intersection. (This holds in P^n_k for high n.) There is an analogous conjecture for higher codimension. Also, 37 is way too big; usually it is 6 or 7.

Theorem 4.2.2 (Quillen–Suslin)**.** *Any vector bundle on affine space $\mathbb{A}^n_k$ is trivial.*

Conjecture 4.2.3 (Hartshorne)**.** *Any finite locally free $\mathcal{O}_{\mathbb{P}^n}$-module (vector bundle) of rank 2 splits, i.e. is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(a) \oplus \mathcal{O}_{\mathbb{P}^n}(b)$ for $n \geq 37$.*

Conjecture 4.2.4 (Hartshorne)**.** *Let $(R, \mathfrak{m})$ be a regular local ring. Let $U = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of R. Every rank 2 vector bundle on U is trivial, i.e. isomorphic to $\mathcal{O}_U^{\oplus 2}$, when $\dim R \leq 38$.*

Remark. This conjecture implies the previous one: take $R = k[X_0, \ldots, X_n]_{(X_0,\ldots,X_n)}$ so that there are morphisms $U \to \mathbb{A}^{n+1}_k - \{0\} \to \mathbb{P}^n_k$. Then we can pullback vector bundles from $\mathbb{P}^n_k$ to U.

Definition 4.2.5. Given a Noetherian scheme X, we say X **has the resolution property** iff every coherent $\mathcal{O}_X$-module is the quotient of a vector bundle.

Example 4.2.6. The resolution property holds for X quasi-projective over a field, because then there is an ample line bundle we can twist by to get more global sections.

Problem 4.2.7. *Find a separated finite-type scheme (or algebraic space) X over a field which does not have the resolution property.*

Example 4.2.8. Every smooth variety has the resolution property. (This comes from every divisor on a smooth variety being an effective Cartier divisor.) However we don't know this for algebraic spaces.

Remark. The real problem underneath these conjectures seems to be: find X which have "few" vector bundles.

Theorem 4.2.9 (Totaro)**.** *If X is separated of finite type over a field and has the resolution property, then*

$$X \cong (\textit{affine finite type } k\textit{-scheme})/(\textit{linear algebraic group}).$$

Problem 4.2.10 ("Jason's Mathoverflow question")**.** *Does there exist a smooth projective variety X over $\mathbb{C}$ and a smooth morphism $f\colon X \to \mathbb{P}^1_{\mathbb{C}}$ which does not have a section?*

Remark. The general question here is: can fibers have "interesting" moduli?

Problem 4.2.11. *Does there exist an $F \in \mathbb{Z}[x_0, \ldots, x_n]$ homogeneous of degree 3 such that $X \coloneqq V_+(F) \subset \mathbb{P}^n_{\mathbb{Z}}$ is smooth over $\mathbb{Z}$?*

Remark. In degree 2, we can just take $x_0x_1 + x_2x_3 + \cdots + x_{2n}x_{2n+1}$. We also have objects which are smooth proper over $\operatorname{Spec} \mathbb{Z}$, e.g. flag varieties, $\overline{\mathcal{M}}_g$.