Notes for Lie Groups & Representations
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Abstract

These are my live-texed notes for the Spring 2017 offering of MATH GR6344 Lie Groups & Representations. There are known omissions. Let me know when you find errors or typos. I'm sure there are plenty.

Chapter 1

Kac–Moody Lie Algebras

Given a semisimple Lie algebra, we can construct an associated root system, and from the root system we can construct a discrete group W generated by reflections (called the Weyl group).

1.1 Root systems and Weyl groups

Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a **Cartan subalgebra**. Recall that $\mathfrak{g}$ has a non-degenerate bilinear form $(\cdot,\cdot)$ which is preserved by the adjoint action, i.e.

$$([x,y],z) + (y,[x,z]) = 0 \quad \forall x,y,z \in \mathfrak{g}.$$

By picking a Cartan subalgebra, we get a **weight decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where $\Phi \subset \mathfrak{h}^* \setminus \{0\}$. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ and $\alpha \neq -\beta$, then $(x,y) = 0$. Hence $\mathfrak{h}$ is orthogonal to $\bigoplus_{\alpha\in\Phi} g_\alpha$. Since $(\cdot,\cdot)$ is non-degenerate, $(\cdot,\cdot)|_\mathfrak{h}$ is non-degenerate. Pick $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ a real vector subspace, such that $\mathfrak{h} = (\mathfrak{h}_\mathbb{R})_\mathbb{C}$ and $\Phi \subset \mathfrak{h}_\mathbb{R}$. Then $(\cdot,\cdot)|_{\mathfrak{h}_\mathbb{R}}$ is a positive definite real form.

Example 1.1.1. If $\mathfrak{g}$ is the complex Lie algebra of a compact Lie group, then $\mathfrak{t} \otimes \mathbb{C} = \mathfrak{h}$, and $(\cdot,\cdot)|_\mathfrak{t}$ is a negative definite bilinear form. So the correct choice here is $\mathfrak{h}_\mathbb{R} = i\mathfrak{t}$.

For each root $\alpha \in \Phi$ we can associate a Lie subalgebra $\mathfrak{g}^{(\alpha)} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. This Lie subalgebra lies in $\mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \mathfrak{g}_{-\alpha}$. So by the adjoint action, $\mathfrak{g}$ becomes a $\mathfrak{g}^{(\alpha)}$-representation. This is nice because we know a lot about the representation theory of $\mathfrak{sl}_2$.

To construct $\mathfrak{g}^{(\alpha)}$, pick $e_\alpha \in \mathfrak{g}_\alpha$ such that $e_\alpha \neq 0$. Since $(\cdot,\cdot)$ is non-degenerate on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, we can pick $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}.$$

Now let $\alpha^\vee := 2(\alpha,\cdot)/(\alpha,\alpha) \in \mathfrak{h}$, called a **co-root**. The triple $(e_\alpha, f_\alpha, \alpha^\vee)$ is a $\mathfrak{sl}_2$**-triple**, i.e. they satisfy the usual relations for generators of $\mathfrak{sl}_2$:

$$[\alpha^\vee, e_\alpha] = 2e_\alpha, \quad [\alpha^\vee, f_\alpha] = -2f_\alpha, \quad [e_\alpha, f_\alpha] = 2\alpha^\vee.$$

It follows that

$$\operatorname{ad}_{e_\alpha} g_\beta \subset \mathfrak{g}_{\beta+\alpha}, \quad \operatorname{ad}_{f_\alpha} g_\beta \subset \mathfrak{g}_{\beta-\alpha},$$

so $\mathfrak{g}$ decomposes as a $\mathfrak{g}^{(\alpha)}$-representation into chains of the form $\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$. The weight of $\mathfrak{g}_\beta$ is

$$\beta(\alpha^\vee) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} =: \langle\beta|\alpha\rangle.$$

This is linear only in β. Finally, define the reflection map

$$r_\alpha \colon \mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^*, \quad \beta \mapsto \beta - \langle \beta | \alpha \rangle \alpha$$

which reflects across the hyperplane $H_\alpha \coloneqq \{\beta \in \mathfrak{h}_\mathbb{R} : \langle \beta | \alpha \rangle = 0\}$. This map preserves Φ, i.e. $r_\alpha(\Phi) = \Phi$.

For $x \in \mathfrak{g}_\alpha$, we have $[e_\alpha, x] = c_1 \alpha^\vee$. For $y \in \mathfrak{g}_\alpha$, we have $[f_\alpha, y] = c_2 \alpha^\vee$. So a non-trivial irreducible subrep of $V = \bigoplus_{c \in \mathbb{R}} \mathfrak{g}_{c\alpha}$ intersects with $\mathfrak{h}$ only by $\operatorname{span}\{\alpha^\vee\}$. Hence there is at most one irreducible subrep of $\bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{n\alpha}$. It follows that V has only one non-trivial $\mathfrak{g}^{(\alpha)}$-irrep $\operatorname{span}\{e_\alpha, f_\alpha, \alpha^\vee\}$. All the rest $(\alpha^\vee)^\perp \subset \mathfrak{h}_\mathbb{R}$ are trivial reps.

Corollary 1.1.2. *If $\alpha \in \Phi$, then $c\alpha \in \Phi$ iff $c \in \{\pm 1\}$.*

Corollary 1.1.3. $\dim \mathfrak{g}_\alpha = 1$ *for $\alpha \neq 0$.*

Note that for all $\alpha, \beta \in \Phi$, we have $\langle \alpha | \beta \rangle \in \mathbb{Z}$ as an $\mathfrak{sl}_2$-weight.

Definition 1.1.4. A subset $\Phi \subset E$ with the following properties is a **root system**:

1. Φ is finite and $0 \neq \Phi$;
2. for all $\alpha \in \Phi$, we have $r_\alpha \Phi = \Phi$;
3. for all $\alpha, \beta \in \Phi$, we have $\langle \alpha | \beta \rangle \in \mathbb{Z}$;

If in addition it satisfies

4. for all $\alpha \in \Phi$, if $c\alpha \in \Phi$ for $c \in \mathbb{R}$, then $c \in \{\pm 1\}$

then it is a **reduced root system**. To a root system Φ we can associate the group $W \coloneqq \{r_\alpha : \alpha \in \Phi\}$, called the **Weyl group**. .

1.2 Reflection groups

Let X be a Riemannian manifold with constant curvature. Such an X has universal cover which is one of:

1. S^n for $n \geq 2$;
2. (Euclidean space) E^n;
3. (Lobachevsky space) L^n.

The classification of reflection groups in the first two is known; the classification for L^n is combinatorially messy and still not complete.

Definition 1.2.1. A subgroup $W \subset \operatorname{Isom}(X)$ is called a **discrete group generated by reflections** if:

1. there exists a family of reflections $\{r_i\}_{i \in I} \subset \operatorname{Isom}(X)$ such that $W = \langle r_i : i \in I \rangle$;
2. for all $x \in X$, the image Wx has only isolated points.

Note that the second condition is automatic if W is finite, but we will have a slightly more general situation here.

Definition 1.2.2. Let $\pi(W)$ be the set of all hyperplanes such that W has a reflection with respect to it.

Since W is a group, if $H \in \pi(W)$, then for all $g \in W$, we have $gH \in \pi(W)$. If the angle between two such hyperplanes is α, then we can obtain any angle $n\alpha$ by repeatedly reflecting.

Lemma 1.2.3. *Let $U \coloneqq X \setminus \bigcup_{H \in \pi(W)} H$. Then $U \neq \emptyset$ and is open. Its connected components are called* **chambers***, and have the following properties:*

1. *they are finite convex polytopes, i.e. intersections of finitely many half-spaces;*

2. *angles between faces of a fixed chamber are of the form π/m for $m \geq 2$ an integer.*

Definition 1.2.4. All polytopes satisfying (1) and (2) are called **Coxeter polytopes**.

Proposition 1.2.5. *The reflections with respect to faces of a fixed chamber generate W.*

Proposition 1.2.6. *Any Coxeter polytope generates a reflection group by reflections with respect to faces. In particular, there are no mirrors which intersect the polytope.*

Remark. It follows that the classification of reflection groups is equivalent to the classification of Coxeter polytopes.

Definition 1.2.7. We say $W \subset \mathrm{Isom}(E)$ is:

1. **reducible** if there exists E_1, E_2 and $W_1 \subset \mathrm{Isom}(E_1)$ and $W_2 \subset \mathrm{Isom}(E_2)$ such that $E = E_1 \oplus E_2$ and $W = W_1 \times W_2$;

2. **degenerate** if there exists $E' \subset E$ such that $W \subset \mathrm{Isom}(E')$.

Lemma 1.2.8. *An irreducible non-degenerate Coxeter polytope in E^n is a simplex or a cone over a simplex.*

Definition 1.2.9. We say an irreducible non-degenerate Coxeter polytope is

1. **parabolic** if it is a simplex, and

2. **elliptic** if it is a cone over a simplex.

Definition 1.2.10. A Coxeter polytope can be encoded by a graph, called a **Coxeter diagram**, as follows:

1. for each facet, assign a vertex;

2. for an angle π/m, assign an edge with multiplicity $m-2$ (if the angle is 0, the multiplicity is ∞).

From a Coxeter diagram we can construct a group generated by symbols $s_1, \ldots, s_n$ corresponding to vertices, with relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is the multiplicity of the edge plus two.

Lemma 1.2.11. *This group constructed from the Coxeter diagram is isomorphic to W.*

Definition 1.2.12. For each facet, we assign a unit vector e_i normal to this facet pointing outward, so that $(e_i, e_i) = 1$ and $(e_i, e_j) \leq 0$. Construct the **Gram matrix** G whose entries are given by $G_{ij} \coloneqq (e_i, e_j)$.

Lemma 1.2.13 (Elliptic case)**.** *Let G be a Gram matrix.*

1. *(Elliptic case) If all principal minors are positive, then there exist $e_1, \ldots, e_n$ which have the necessary scalar products, and the Coxeter polytope is the corresponding cone.*

2. *(Parabolic case) If all proper principal minors are positive and $\det G = 0$, then a similar statement holds.*

Definition 1.2.14. For a diagram D, define $\det D$ to be the determinant of its Gram matrix. So an equivalent statement of the lemma is: if D satisfies either condition

1. (elliptic) the determinants of subdiagrams are positive, or

2. (parabolic) the determinants of strict subdiagrams are positive, and the determinant of D is 0,

then there is a Coxeter polytope associated to D.

Example 1.2.15. The determinant of $\tilde{A}_n$, whose diagram is a cycle, is 0. For a tree S with an edge e of multiplicity m between v_1, v_2, we can apply the recursive formula

$$\det S = \det(S - \{v_1, e\}) - \cos^2(\pi/m)\det(S - \{v_1, e, v_2\}).$$

Definition 1.2.16. A **simplification** of a diagram involves reducing the multiplicities on edges.

Theorem 1.2.17. *Simplifications of parabolic or elliptic diagrams are elliptic. So if after simplification we get a parabolic diagram, the original cannot have been parabolic or elliptic.*

1.3 Regular polytopes and Coxeter groups

For dimension 3, there are five regular polytopes: the tetrahedron, the cube, octahedron, icosahedron, dodecahedron. There is a duality which pairs the cube and octahedron, and icosahedron and dodecahedron. In dimensions ≥ 5, there are three: the simplex, cube, and cube$^\vee$. In dimension 4, there are six: simplex, cube, cube$^\vee$, icosahedron, dodecahedron, and the 120-cell. So the number of regular polytopes in dimension n forms the sequence

$$1, \infty, 5, 6, 3, 3, 3, \ldots.$$

Definition 1.3.1. An n-dimensional polytope $\Delta \subset \mathbb{R}^n$ is **regular** if all facets of Δ are congruent to a regular $(n-1)$-dimensional polytope and the dihedral angles between any two facets are equal.

Remark. This implies that the group $\mathrm{Aut}(\Delta) \subset \mathrm{Isom}(\mathbb{R}^n)$ acts transitively on the set of **flags** of faces $F_0 \subset F_1 \subset \cdots \subset F_n$ (where $\dim F_i = i$).

Lemma 1.3.2. *A reflection across the hyperplane that bisects two $(n-1)$-dimensional faces F and F' takes F to F' and preserves Δ.*

Lemma 1.3.3. *Assume $w \in \mathrm{Aut}(F_{n-1}) \subset \mathrm{Isom}(\mathrm{span}\, F_{n-1}) \hookrightarrow \mathrm{Isom}(\mathbb{R}^n)$ acts on F_{n-1}. Then $w \in \mathrm{Aut}(\Delta)$.*

Lemma 1.3.4. *The group $\mathrm{Aut}(\Delta)$ is generated by $\{\mathrm{Aut}(F_{n-1}), r\}$ for reflections r.*

Corollary 1.3.5. *$\mathrm{Aut}(\Delta)$ is finitely generated by reflections.*

Remark. In fact, we can choose a specific set of reflections generating $\mathrm{Aut}(\Delta)$, corresponding to flags $F_0 \subset F_1 \subset \cdots \subset F_n$. Take the centers $c_0, c_1, \ldots, c_n$ of the corresponding faces, and let r be a reflection in $\mathrm{span}\{c_0, c_1, \ldots, \hat{c}_{n-1}, c_n\}$.

Remark. Note that $r = s_n$ commutes with $s_{n-2}, s_{n-3}, \ldots$, because those hyperplanes are perpendicular to F_1. Hence in the Coxeter diagram, there are only edges connecting s_i to s_{i-1}. The multiplicity of this edge is the number of facets that meet at F_{n-3}.

Theorem 1.3.6 (Schläfli)**.** *Regular polytopes are classified by (Schläfli symbols) Coxeter diagrams that are linear and connected and with a choice of orientation.*

Let $W \subset \mathrm{Isom}(\mathbb{R}^n)$ be a discrete group generated by reflections. Let Δ be a connected component, and $s_1, \ldots, s_{n+1}$ be reflections in hyperplanes bounding Δ. Then $W = \langle s_1, \ldots, s_{n+1} \rangle$.

Proposition 1.3.7. *Δ is a fundamental domain for W and $|Wx \cap \Delta| = 1$ for every generic x.*

Proof. Consider the length function on W, i.e. let $\ell(w)$ be the number of reflection hyperplanes that separate Δ from w. We will show that $\ell(w)$ is the minimal length k such that $w = s_{i_1} \cdots s_{i_k}$. This will imply $\ell(w) = 0$ iff $w\Delta = \Delta$ iff $w = \mathrm{id}$. Clearly $\ell(ws_i) = \ell(w) + 1$ or $\ell(ws_i) = \ell(w) - 1$, since Δ either lies on one side of s_i or the other. So as long as we keep reflecting away from Δ, we get a minimal path. $\square$

Definition 1.3.8. This is a special case of a more general notion. Let $\Gamma \subset \mathrm{Isom}(S)$ where $S = S^n, \mathbb{R}^n, \mathbb{H}^n$ is a constant curvature geometry. Let

$$\mathcal{D} = \{y : d(y, x) < d(y, \gamma_i x)\ \forall \gamma_i \in \Gamma\},$$

called the **Dirichlet domain**.

Theorem 1.3.9. $W = \langle s_1, s_2, \ldots \rangle / (s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1)$.

Proof. Let $\tilde{W}$ be the given group, so that $1 \to \ker \to \tilde{W} \to W \to 1$ and we must prove ker is trivial. Let $\Delta_3 = \Delta - \{\text{strata of codim } \geq 3\}$ and M_3 be the same for M. Take $\{\tilde{w}\Delta_3\}_{\tilde{w} \in \tilde{W}}$ and make a manifold $\tilde{M}_3$ out of it by gluing together the pieces along codim 1 and codim 2 strata. Then $\tilde{M}_3 \to M_3$ is a covering. But clearly M_3 is simply connected. Hence $\tilde{M}_3 \cong M_3$. $\square$

Definition 1.3.10. Given such a group W, define the **braid group**

$$B_W \coloneqq \langle \sigma_1, \sigma_2, \ldots \rangle / (\sigma_i \sigma_j \sigma_i \cdots = \sigma_j \sigma_i \sigma_j \cdots)$$

where the relations may have m_{ij} terms on both sides. It clearly comes with a surjective homomorphism to W. The kernel of $B_W \to W$ is called the **pure braid group**.

Remark. There is a nice geometric interpretation of B_W. Take $\mathbb{C}^n \setminus \{H_\alpha \otimes \mathbb{C}\}$, which still has a W-action. Each point has trivial stabilizer now, so there is a quotient

$$\mathbb{C}^n_{\text{reg}} \coloneqq (\mathbb{C}^n \setminus \{H_\alpha \otimes \mathbb{C}\} \to (\mathbb{C}^n \setminus \{H_\alpha \otimes \mathbb{C}\})/W$$

which is a normal covering. Hence there is a sequence of fundamental groups

$$1 \to \pi_1(\mathbb{C}^n_{\text{reg}}) \to \pi_1(\mathbb{C}^n_{\text{reg}}/W) \to W \to 1.$$

The claim is that $\pi_1(\mathbb{C}^n_{\text{reg}}/W) \cong B_W$. The generators are the paths γ that connects x to $s_i x$ with $\operatorname{im} H_i|_\gamma > 0$ generate. The relations come from homotopy.

Example 1.3.11. The most famous example is when $W = S(n)$, the symmetric group on n letters. Then $\mathbb{C}^n_{\text{reg}}$ consists of n-tuples of distinct complex numbers modulo permutation. This is the same as polynomials with only simple roots. The hyperplanes are of the form $\{x_i = x_j : i \neq j\}$. An element of the fundamental group can be viewed as paths (strands) connecting these roots, which may intertwine. The generators are $\{H_i = \{x_i = x_{i+1}\}$, which in terms of these strands, are just the elements which twist two adjacent strands. The relations are given by Reidemeister moves. As a group, this is $\langle \sigma_1, \sigma_2, \ldots \rangle / (\sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j)$.

1.4 Kac–Moody Lie algebras

Recall that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$. The root system $\{\alpha \in \mathfrak{h}^*\}$ gives rise to a finite reflection group where the multiplicities m_{ij} are restricted to $m_{ij} = 2, 3, 4, 6$. We will now classify Cartan matrices, and specify how they give rise to Kac–Moody Lie algebras.

Definition 1.4.1. Given roots $\{\alpha\} \subset \mathfrak{h}^*$, let the reflection group $W = \langle s_\alpha \rangle$ consist of reflections s_α around hyperplanes defined by α. Pick a fundamental domain C of W, i.e. a **Weyl chamber**. The closure of C partitions roots into positive and negative roots. The **simple roots** α_i are those such that $\{\alpha_i = 0\}$ is a boundary of C. So s_{α_i} for simple roots α_i are generators of W. The **Cartan matrix** has entries $\alpha_j(h_i)$ for $h_i \coloneqq h_{\alpha_i}$ the co-root corresponding to α_i.

Remark. Observe that $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ and $[e_i, f_i] = h_i$ form a copy of $\mathfrak{sl}_2$. The Cartan matrix tells us which angle these $\mathfrak{sl}_2$ sit inside our Lie algebra $\mathfrak{g}$, since $[h_i, e_j] = \alpha_j(h_i) e_j$, and these scalars $\alpha_j(h_i)$ are precisely the entries of the Cartan matrix. Note that $[e_i, f_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$, but this root is positive near α_i and negative near α_j, and therefore must be 0. By finite-dimensionality, we also have the **Serre relations** $\operatorname{ad}(e_i)^{1 - a_{ij}} e_j = 0$, where $[h_i, e_j] = a_{ij} e_j$.

Definition 1.4.2. Given a Cartan matrix (a_{ij}), create the Lie algebra $\tilde{\mathfrak{g}} \coloneqq \langle e_i, f_i, h_i \rangle / \sim$ where $\sim$ encodes the relations in the remark above, excluding the Serre relations. This has representations, called **Verma modules**, generated by $|\lambda\rangle$ for $\lambda \in \mathfrak{h}^*$. Fact: $\dim \mathfrak{h} = n + \operatorname{corank}$ of C, and there exists a unique (up to isometry) $\mathfrak{h}$ of this dimension with linearly independent $h_1, \ldots, h_n \in \mathfrak{h}$ and $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ such that $\alpha_j(h_i) = a_{ij}$. The number of simple roots is n, and the center is $\{h \in \mathfrak{h} : \alpha_i(h) = 0\}$. There is a **highest weight vector** $|\lambda\rangle$ satisfying $e_i|\lambda\rangle = 0$ and $h|\lambda\rangle = \lambda(h)|\lambda\rangle$, and the Verma module is the span of $f_{i_1} f_{i_2} \cdots |\lambda\rangle$. The weight of such an element is $h(\cdots) = \lambda(h) - \sum_k \alpha_k(h)$.

Theorem 1.4.3. *1. The sub-algebra $\tilde{\mathfrak{n}}_- \coloneqq \langle f_1, \ldots, f_n \rangle$ is a free Lie algebra, i.e. $\mathcal{U}(\tilde{\mathfrak{n}}_-)$ is isomorphic to a free associative algebra.*

2. $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ *and the decomposition is unique.*

3. *The map* $e_i \mapsto -f_i$ *and* $f_i \mapsto -e_i$ *and* $h_i \mapsto -h_i$ *is an involution, called the* **Chevalley involution**.

4. $\tilde{\mathfrak{g}} = \mathfrak{h} \oplus \bigoplus_{\alpha=\sum n_i\alpha_i,\, n_i>0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha=\sum n_i\alpha_i,\, n_i<0} \tilde{\mathfrak{g}}_\alpha$ *where* $\dim \tilde{\mathfrak{g}}_\alpha < \infty$ *and is given by an explicit formula.*

Theorem 1.4.4. *Consider ideals* $I \subset \tilde{\mathfrak{g}}$*. Then* I *decomposes as* $(I \cap \mathfrak{h}) \oplus \bigoplus (I \cap \tilde{\mathfrak{g}}_\alpha)$*. Among all ideals* $I \subset \tilde{\mathfrak{g}}$ *with* $I \cap \mathfrak{h} = 0$*, there exists a unique maximal ideal* $\mathfrak{r}$*. Moreover:*

1. $\mathfrak{r}$ *is stable under the Chevalley involution;*

2. $\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-)$.

Definition 1.4.5. A **Kac–Moody Lie algebra** is $\mathfrak{g} \coloneqq \tilde{\mathfrak{g}}/\mathfrak{r}$. This object is "simple" in the sense that it has an ideal, but the ideal is central.

Remark. Eventually we will show $\dim \mathfrak{g} < \infty$ iff C is the Cartan matrix of a root system.

Remark. The center of $\mathfrak{g}$ must lie in $\mathfrak{h}$, and must commute with generators, i.e. $[h, e] = 0$, so $\alpha_i(h) = 0$. Hence the center is the set $\{c \in \mathfrak{h} : \alpha_i(c) = 0\}$. In particular, if the Cartan matrix is non-degenerate, the center is trivial. This center in general does not split off.

Proposition 1.4.6. *Suppose* C *is indecomposable, i.e.* $C \neq C_1 \oplus C_2$*. Then any ideal* $I \subset \mathfrak{g}$ *is either the center or contains the ideal generated by the* e_i *and* f_i*.*

Proof. Suppose $I \neq 0$. Then $I \cap \mathfrak{h} \neq 0$ by the construction of $\mathfrak{g}$. Let $h \in I \cap \mathfrak{h}$. Either

1. $\alpha_i(h) = 0$ for all i, i.e. h is in the center, or

2. there exists i such that $\alpha_i(h) \neq 0$, so that $e_i \in I$, and similarly $f_i \in I$, which implies $h_i \in I$.

In the second case, there exists $j \neq i$ such that $a_{ji} \neq 0$, i.e. $\alpha_j(h_i) \neq 0$. Then $e_j, f_j, h_j \in I$ by the same argument. Since C is indecomposable, we can find a chain of elements such that everything turns out to be in I. □

Definition 1.4.7. Consider the $\mathfrak{sl}(2)_i$-module generated by e_j with $j \neq i$. We know $[f_i, e_j] = 0$, so e_j is of lowest weight a_{ij} because $[h_i, e_j] = a_{ij} e_j$. Assume $a_{ij} \in \mathbb{Z}_{\leq 0}$. If we keep applying e_i, we get $(\operatorname{ad} e_i)^{-a_{ij}} e_j$, which has weight $-a_{ij}$. In finite-dimensional modules, this would be the highest weight vector. But for Verma modules, we can keep going. However, the vector $(\operatorname{ad} e_i)^{-a_{ij}+1}$ is singular, i.e. killed by f_i. Hence $[f_i, (\operatorname{ad} e_i)^{-a_{ij}+1} e_j] = 0$. In fact, this is true for any f_k. These are the **Serre relations**.

Definition 1.4.8. Recall that if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, then $\mathfrak{g}$ has a Cartan–Killing invariant bilinear form $(a, b) = \operatorname{tr}(\operatorname{ad}(a) \operatorname{ad}(b))$, and we can write

$$(\alpha_j(h_i)) = (2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}) = 2 \operatorname{diag}(1/(\alpha_i, \alpha_i))((\alpha_i, \alpha_j)) =: 2DB$$

as matrices. Note that D is non-degenerate, and B is symmetric. If the Cartan matrix has this form, the Kac–Moody Lie algebra is **symmetrizable**.

Example 1.4.9. The Cartan matrix of a finite root system in $\mathbb{R}^n$ is always symmetrizable.

Proposition 1.4.10. *A Kac–Moody Lie algebra is symmetrizable iff it has a non-degenerate invariant bilinear form.*

Proof. If there exists a non-degenerate invariant bilinear form,

$$(h_i, [e_j, f_j]) = ([h_i, e_j], f_j) = \alpha_j(h_i)(e_i, f_j).$$

Clearly (h_i, h_j) is symmetric, and (e_i, f_j) is diagonal. Hence this is the matrix we want. Conversely, if $\mathfrak{g}$ is symmetrizable, we construct the matrix inductively via a pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$ for each root α. Specifically, if $\alpha = \sum_i m_i\alpha_i$, we induct on the **height** $\sum_i m_i$. If $\mathrm{ht}(\alpha) > 1$, then given $x \in \tilde{\mathfrak{g}}_\alpha$ and $y \in \tilde{\mathfrak{g}}_{-\alpha}$, we have $x = \sum[e_i, x_i]$ for $x_i \in \tilde{\mathfrak{g}}_{\alpha-\alpha_i}$ and therefore $(x, y) = -\sum(x_i, [e_i, y])$. This is defined already by induction, since $[e_i, y] \in \tilde{\mathfrak{g}}_{-\alpha+\alpha_i}$, which has smaller height. If $v \in \mathfrak{g}_\alpha$ for $\alpha > 0$ is singular, i.e.$f_i v = 0$ for all i, then $(v, -) = 0$. Observe that $\ker(\cdot, \cdot) \coloneqq \{v : (v, -) = 0\}$ is an ideal, and does not intersect $\mathfrak{h}$. Hence it is inside $\mathfrak{r}$. So this construction descends to a non-degenerate form on $\mathfrak{g}$. □

Remark. We can figure out an explicit formula for the pairing: it is $(e, f) = [e, f]/v$ where v is the generator for the weight space that $[e, f]$ lies in.

1.5 Examples of Kac–Moody algebras

Example 1.5.1. Let $\mathfrak{g}$ be a simple Lie algebra. Take the polynomial **loop algebra** $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, which has a Lie bracket

$$[p(t) \otimes a, q(t) \otimes b] \coloneqq p(t)q(t) \otimes [a, b].$$

(This comes from loop groups.) Given a short exact sequence $0 \to k \to M \to \mathfrak{g} \to 0$, if the image of K is central in M, we say M is a **central extension** of $\mathfrak{g}$. An **affine Lie algebra** is

$$\hat{\mathfrak{g}} \coloneqq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

as a vector space, with Lie bracket

$$[a \otimes t^n + \alpha c, b \otimes t^m + \beta c] = [a, b] \otimes t^{m+n} + (a, b)n\delta_{n+m,0}c$$

where (a, b) is the standard bilinear form on $\mathfrak{g}$. Note that c is central, because $[c, \cdot] = 0$ by definition. The Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ extends to $\tilde{\mathfrak{h}} \coloneqq 1 \otimes \mathfrak{h} \oplus \mathbb{C}c$. Given $\tilde{h} \in \tilde{\mathfrak{h}}$,

$$[\tilde{h}, t^n \otimes x_\alpha] = \langle \alpha, h\rangle t^n \otimes x_\alpha, \quad [\tilde{h}, t^n \otimes \mathfrak{h}] = 0.$$

Example 1.5.2. Define the **affine Kac–Moody algebra** $\hat{\mathfrak{g}} \coloneqq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, with

$$[d, p(t) \otimes x] \coloneqq t\frac{dp}{dt} \otimes x, \quad [d, c] \coloneqq 0.$$

Define $\hat{\mathfrak{h}} \coloneqq \tilde{\mathfrak{h}} \oplus \mathbb{C}d$. Then

$$[1 \otimes h + \mu c + \nu d,^m \otimes x_\alpha] = \langle \alpha, h\rangle t^n \otimes x_\alpha + \nu n t^n \otimes x_\alpha = (\langle \alpha, h\rangle + \nu n)t^n \otimes x_\alpha.$$

In $\tilde{\mathfrak{h}}^*$, extend functions from $\tilde{h}^*$ by $\lambda(d) = 0$. There is a root space decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha+n\delta\in\hat{\Delta}} \mathfrak{g}_{\alpha+n\delta} \oplus \bigoplus_{n\delta\in\hat{\Delta}} \mathfrak{g}_{n\delta}$$

where $\hat{\Delta} \coloneqq \{\alpha + n\delta : \alpha \in \Delta,\ n \in \mathbb{Z}_{\neq 0}\}$. Clearly $\dim \mathfrak{g}_{\alpha+n\delta} = 1$ as before, and $\dim \mathfrak{g}_{n\delta} = \dim \mathfrak{h}$, the original Cartan. Define a standard bilinear form by

$$(p(t)\otimes a, q(t)\otimes b) \coloneqq \mathrm{Res}(t^{-1}p(t)q(t))(a, b), \quad (c, p(t)\otimes a) = (d, p(t)\otimes a) \coloneqq 0, \quad (c, c) = (d, d) = 0, \quad (c, d) = 1,$$

and we make this symmetric and bilinear.

Definition 1.5.3. Let $\{E_1, \ldots, E_k, F_1, \ldots, F_k\}$ be generators for $\mathfrak{g}$, where

$$[E_i, F_j] = \delta_{ij} H_j, \quad [H_i, E_j] = \langle \alpha_j, \alpha_i^\vee \rangle E_j = A_{ij} E_j, \quad [H_i, F_j] = \langle -\alpha_j, \alpha_i^\vee \rangle F_j = -A_{ij} F_j.$$

Define F_0 such that $F_0 \in \mathfrak{g}_\theta$ where $\theta \coloneqq \sum_i \alpha_i$ is the sum of positive roots, and $(F_0, w(F_0)) = -1$ where w is the Chevalley involution. Define $E_0 \coloneqq -w(F_0)$ so that $E_0, \ldots, E_k, w$ generate. Write

$$e_0 \coloneqq t \otimes E_0, \quad e_i \coloneqq 1 \otimes E_i, \quad f_0 \coloneqq t^{-1} \otimes F_0, \quad f_i \coloneqq 1 \otimes F_i$$

and $\alpha_0 \coloneqq \delta - \theta$. Then these are the **Chevalley generators** for the affine Kac–Moody algebra.

Example 1.5.4. For $\widehat{\mathfrak{sl}}(2)$, start with $\mathfrak{sl}(2) = \operatorname{span}\{e, f, h\}$. Then the roots are $\Delta = \{-\alpha, \alpha\}$, with $\theta = \alpha$ and $E_0 = F$. The Chevalley generators are therefore

$$e_0 = t \otimes f, \quad e_1 = 1 \otimes 1 \otimes e, \quad f_0 = t^{-1} \otimes e, \quad f_1 = 1 \otimes f, \quad h_0 = 2c - 1 \otimes h, \quad h_1 \coloneqq 1 \otimes h, \quad d.$$

1.6 Category O

Definition 1.6.1. We define a full subcategory of the category of $\mathfrak{g}$-modules. Recall Verma modules $M(\lambda)$ where $\lambda \in \mathfrak{h}^*$. These are defined by $M(\lambda) = \mathcal{U}(\mathfrak{n}_-) |\lambda\rangle$ where $|\lambda\rangle$ is highest weight λ, i.e.

$$h |\lambda\rangle = \lambda |\lambda\rangle, \quad \mathfrak{n}_+ |\lambda\rangle = 0.$$

Then $M(\lambda) = \bigoplus_{\mu = \lambda - \sum \alpha_i} M(\lambda)_\mu$ where the μ are all valid weights. Note that $\dim M(\lambda)_\mu < \infty$, and there is a "cone" of weights.

A $\mathfrak{g}$-module M is in **category** $\mathcal{O}$ if $\dim M_\mu < \infty$ and $\{\mu : M_\mu = 0\}$ is a finite union of cones of weights (like for Verma modules). Let $R(\lambda)$ be the maximal submodule not containing $|\lambda\rangle$. Then the quotient $L(\lambda) \coloneqq M(\lambda)/R(\lambda)$ is irreducible.

Remark. It is not true that every $M \in \mathcal{O}$ has an irreducible quotient. This is because in category $\mathcal{O}$, there is a duality: define $M^\vee$ by $(M^\vee)_\mu \coloneqq (M_\mu)^*$, the dual space, and $\langle x \cdot \xi, m \rangle \coloneqq -\langle \xi, w(x) m \rangle$ where $w(x)$ is the Chevalley involution. (The involution is necessary to change the otherwise lowest-weight module back into a highest-weight module.) Now it could be that $M \supset M' \supset M'' \supset \cdots$ has an infinite composition series. For example, $M(0)$ has this property where $\dim \mathfrak{g} = \infty$. Then M has no irreducible submodule L, i.e. $0 \to L \to M$ does not exist, and hence by duality $M^\vee$ has no irreducible quotient, i.e. $M^\vee \to L \to 0$ does not exist.

Definition 1.6.2. The **Casimir operator** Ω splits $\mathcal{O}$ into blocks. If $\dim \mathfrak{g} < \infty$ and $\{e_i\}$ and $\{e^i\}$ are dual bases of $\mathfrak{g}$ with respect to the invariant bilinear form, then $e_i \otimes e^i \in S^2\mathfrak{g}$ is $\mathfrak{g}$-invariant. The map $\mathfrak{g} \otimes \mathfrak{g} \to \mathcal{U}\mathfrak{g}$ given by $a \otimes b \mapsto ab$ is an invariant map, since $[x, ab] = [x, a]b + a[x, b]$. Hence $\Omega \coloneqq \sum e_i e^i \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$, the center of $\mathcal{U}\mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$, then in general $\Omega \coloneqq \sum_\alpha e_{-\alpha}^{(i)} e_\alpha^{(i)}$. This is an infinite expression, but we can rewrite it so that it makes sense in $\mathcal{O}$ using **normal ordering**:

$$\Omega = \sum_{\alpha \geq 0} e_{-\alpha}^{(i)} e_\alpha^{(i)} + \sum_{\alpha < 0} \left(e_\alpha^{(i)} e_{-\alpha}^{(i)} + [e_{-\alpha}^{(i)}, e_\alpha^{(i)}] \right).$$

The last term $[e_{-\alpha}^{(i)}, e_\alpha^{(i)}]$ makes sense because it is just some multiple of the vector h_α. Hence

$$\Omega = 2\rho + \sum_{\alpha = 0} e_{-\alpha}^{(i)} e_\alpha^{(i)} + 2 \sum_{\alpha > 0} e_{-\alpha}^{(i)} e_\alpha^{(i)}$$

where ρ is the analogue of the half-sum of the positive roots in the finite dimensional case, i.e. it satisfies $(\rho, \alpha_i) = \alpha_i$. Now Ω is well-defined on modules in $\mathcal{O}$ and commutes with $\mathfrak{g}$. (It is convenient to think of ρ as an element in both $\mathfrak{h}$ and $\mathfrak{h}^*$.)

Example 1.6.3. Let $|\lambda\rangle$ be a highest weight vector. Then

$$\Omega\,|\lambda\rangle = 2(\rho,\lambda) + (\lambda,\lambda) + 0 = (\lambda+\rho,\lambda+\rho) - (\rho,\rho)$$

because every $e_\alpha^{(i)}$ kills $|\lambda\rangle$. Let this constant be c. Since Ω commutes with everything, i.e.

$$\Omega\bigg(\prod_k f_{i_k}\bigg)|\lambda\rangle = \bigg(\prod_k f_{i_k}\bigg)\Omega\,|\lambda\rangle\,,$$

we have $\Omega|_{M(\lambda)} = c\cdot\mathrm{id}$. Hence given any non-zero homomorphism $M(\mu)\to M(\lambda)$, the operator Ω acts the same on both $M(\mu)$ and $M(\lambda)$, i.e. $\|\lambda+\rho\|^2 = \|\mu+\rho\|^2$. More generally, this holds for $M(\nu)\to M(\lambda)/U$ given by $|\nu\rangle \to$ a primitive vector$v$ (i.e. $v\notin U$ but $e_i v \in U$). We have shown the following.

Proposition 1.6.4. *If μ is the weight of a primitive vector in $M(\lambda)$, then $\|\lambda+\rho\|^2 = \|\mu+\rho\|^2$ (and $\mu = \lambda - \sum_j \alpha_j$).*

Definition 1.6.5. Take $\mathcal{U}(\tilde{\mathfrak{n}}_-)_{\mathrm{aug}}$ to be the **augmentation ideal**, i.e. $\mathcal{U}(\tilde{\mathfrak{n}}_-)$ without the constant elements. Then $\mathfrak{r}_- \subset \mathcal{U}(\tilde{\mathfrak{n}}_-)_{\mathrm{aug}}$. Take the map

$$\begin{aligned}\tilde{\varphi}\colon \mathfrak{r}_- \subset \mathcal{U}(\tilde{\mathfrak{n}}_-)_{\mathrm{aug}} &\to \bigoplus_{i=1}^n M(-\alpha_i)\\ \sum u_i f_i &\mapsto \sum u_i\,|-\alpha_i\rangle\end{aligned}$$

where $M(-\alpha_i)$ are Verma modules for $\mathfrak{g}$, not $\tilde{\mathfrak{g}}$, i.e. $\mathfrak{r}$ acts by 0.

Proposition 1.6.6. *1. This is a homomorphism of $\mathfrak{g}$-modules.*

2. It factors through $\mathfrak{r}_-/[\mathfrak{r}_-,\mathfrak{r}_-]$.

3. The induced map $\varphi\colon \mathfrak{r}_-/[\mathfrak{r}_-,\mathfrak{r}_-] \to \bigoplus_{i=1}^n M(-\alpha_i)$ is injective.

Proof. Take $R = \sum u_i f_i \in \mathfrak{r}_-$. Then $[x,R] = xR - Rx \mapsto \sum x u_i\,|-\alpha_i\rangle + 0$ since $\mathfrak{r}$ acts by 0 on the Verma modules. Similarly, if $R\in[\mathfrak{r}_-,\mathfrak{r}_-]$, then $R = R_1R_2 - R_2R_1 \mapsto 0$. Now note that $\varphi(R) = 0$ iff each u_i vanishes in $\mathcal{U}(\mathfrak{n}_-)$, i.e. $u_i \in \mathfrak{r}\mathcal{U}(\mathfrak{n}_-)$. So $R\in \mathfrak{r}\cap\mathfrak{r}\mathcal{U}(\mathfrak{n}_-)_{\mathrm{aug}}$. But then in fact $R\in\mathfrak{r}\cap\mathfrak{r}\mathcal{U}(\mathfrak{r})_{\mathrm{aug}}$, because by PBW, higher (quadratic or above) terms in $\mathfrak{r}\mathcal{U}(\mathfrak{n}_-)_{\mathrm{aug}}$ belonging to $\mathfrak{r}$ actually live in $\mathcal{U}(\mathfrak{r})_{\mathrm{aug}}$. By the following lemma, we are done. □

Lemma 1.6.7. *For any Lie algebra $\mathfrak{g}$, we have $\mathfrak{g}\cap\mathfrak{g}\mathcal{U}(\mathfrak{g})_{aug} = [\mathfrak{g},\mathfrak{g}]$.*

Proof. Consider the quotient $\mathfrak{g}_{\mathrm{ab}} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Then

$$\mathfrak{g}_{\mathrm{ab}} \cap \mathfrak{g}_{\mathrm{ab}} U(\mathfrak{g}_{\mathrm{ab}})_{\mathrm{aug}} = 0,$$

because the first term is linear polynomials and the second term is all quadratic and higher-order polynomials. □

Corollary 1.6.8. *If $-\alpha$ is the weight of a generator (hence primitive vector), in $\mathfrak{r}_-$, then $\|-\alpha+\rho\|^2 = \|-\alpha_i+\rho\|^2$ for some i. Since $2(\rho,\alpha_i) = (\alpha_i,\alpha_i)$, this gives*

$$(\alpha,\alpha) = 2(\alpha,\rho).$$

1.7 Gabber–Kac theorem

Definition 1.7.1. Take a Cartan matrix $C = (a_{ij})$, i.e. C satisfies:

1. $a_{ii} = 2$,
2. $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
3. $C = DB$ where D is diagonal invertible, and B is symmetric.

Note that (3) implies $a_{ij} = 0$ iff $a_{ji} = 0$. From C we produce $\tilde{\mathfrak{g}}$ which has a maximal ideal $\mathfrak{r}$, and the **Kac–Moody Lie algebra** obtained is $\mathfrak{g} := \tilde{\mathfrak{g}}/\mathfrak{r}$. We know some relations in $\mathfrak{r}$: the Serre relations $\operatorname{ad}(e_i)^{1-a_{ij}} e_j = 0$.

Theorem 1.7.2 (Gabber–Kac)**.** *If C is symmetrizable, then the Serre relations generate the ideal $\mathfrak{r}$.*

Remark. Here is the big picture. There are three levels of generality. Given a matrix $C = (a_{ij})$, we can require, successively, that

1. (**Borcherds–KM algebra**) a_{ii} does not necessarily have to be 2,
2. (**symmetrizable BKM algebra**) $C = DB$ is symmetrizable,
3. (**symmetrizable KM algebra**) $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}_{\leq 0}$.

At the level of (1), there is an exact sequence (see homework)

$$0 \to \mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-] \to \bigoplus_{i=1}^{n} M(-\alpha_i) \to M(0) \to \Bbbk \to 0.$$

At the level of (2), we have an invariant bilinear form, and any generator appears in the weight $-\alpha$ where $2(\rho, \alpha) = (\alpha, \alpha)$. This equality gives, at the level of (3), that $\mathfrak{r}$ is generated by Serre relations (Gabber–Kac).

Also, if we suppose every (including diagonals) $a_{ij} > 0$ (or every $a_{ij} < 0$), then $\mathfrak{r} = 0$, i.e. $\mathfrak{n}_-$ is freely generated by $f_1, \ldots, f_n$, and similarly $\mathfrak{n}_+$ is freely generated by $e_1, \ldots, e_n$. Suppose $\alpha = \sum m_i \alpha_i$. Then we can compute $(\alpha, \alpha) - 2(\rho, \alpha)$ is either positive or negative, but not zero.

Definition 1.7.3. Note that $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$ is in category $\mathcal{O}$ and also the category of integrable modules. A $\mathfrak{g}$-module M is **integrable** if $\mathfrak{h}$ acts semisimply and e_i, f_i act locally nilpotently, i.e. for every $m \in M$, there exists N such that $e_i^N m = f_i^N m = 0$.

Example 1.7.4. If $\mathfrak{g}$ is a KM Lie algebra, then the adjoint representation of $\mathfrak{g}$ (on $\mathcal{U}\mathfrak{g}$) is integrable. This is because:

1. (from the construction of $\tilde{\mathfrak{g}}$) $\operatorname{ad}(e_i) f_j = \delta_{ij} h_i$, so $(\operatorname{ad} e_i)^3 f_j = 0$;
2. (from the construction of $\mathfrak{g}$) $\operatorname{ad}(e_i)^{1-a_{ij}} e_j = 0$, the Serre relations.

Hence $\mathfrak{r}_-$ is integrable as a submodule of an integrable module, and $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$ is also integrable.

Example 1.7.5 (Main property of integrable modules)**.** Take the weight space M_μ. We can act on it by $\mathfrak{sl}_2$, giving a chain $M_{\mu + m\alpha_i}$ for $m \in \mathbb{Z}$. By integrability, this is a direct sum of finite $\mathfrak{sl}_2$-modules. But finite-dimensional $\mathfrak{sl}_2$-modules are symmetric with respect to reflection, so $M_\mu = M_{r_i \mu}$, where $r_i(\mu) := \mu - \mu(h_i)\alpha_i$.

Proof of Gabber–Kac. Let $\mathfrak{r}' \subset \mathfrak{r}$ be the ideal generated by the Serre relations and suppose $\mathfrak{r}' \neq \mathfrak{r}$. Define $\mathfrak{g}' := \tilde{\mathfrak{g}}/\mathfrak{r}'$. The adjoint representation of $\mathfrak{g}'$ is already integrable. So the weights of $\mathfrak{r}_-/\mathfrak{r}'_- + [\mathfrak{r}_-, \mathfrak{r}_-]$ are W-invariant (where W is the Weyl group). Let $-\alpha \in \mathfrak{r}_-/\mathfrak{r}'_- + [\mathfrak{r}_-, \mathfrak{r}_-]$ such that $\alpha = \sum m_i \alpha_i$ has height $\sum m_i$ minimal in its W-orbit. Since

$$r_k \alpha = \alpha - \alpha(h_k)\alpha_k,$$

we must have $\alpha(h_k) = (\alpha, \alpha_k) \leq 0$ for all k. On the other hand, $(\alpha, \alpha) = \sum m_i (\alpha, \alpha_i) \leq 0$. But $2(\alpha, \rho) = \sum m_i (\alpha_i, \alpha_i) > 0$, which is supposed to be equal to (α, α). ∎

1.8 Weyl–Kac character formula

Fix $\lambda \in \mathfrak{h}^*$ and let $M(\lambda)$ be the Verma module of highest weight λ. It contains a maximal submodule, and the quotient $L(\lambda)$ is irreducible of highest weight λ.

Proposition 1.8.1. *$L(\lambda)$ is integrable iff $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$, i.e. λ is* **dominant integral**.

Proof. If $f_i^N |\lambda\rangle = 0$, then $\mathfrak{sl}(2)_i$-representation theory says $\lambda(h_i)$ must be dominant integral. Conversely, if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. then write $v := f_i^{\lambda_i+1} |\lambda\rangle$. Then $e_j v = 0$, since if $v \neq 0$ in $L(\lambda)$, then v generates an infinite-dimensional $\mathfrak{sl}(2)_i$-module. □

Proposition 1.8.2. *Any $L(\lambda)$, integrable or not, has a resolution*

$$\cdots \to \bigoplus M(\lambda_i^{(2)}) \to \bigoplus M(\lambda_i^{(1)}) \to M(\lambda) \to L(\lambda) \to 0$$

which is finite in any given weight space.

Remark. This is in general because if we try to construct a resolution by picking a minimal set of generators at each step and forming the kernel consisting of relations between them, these relations will live in lower and lower weights.

Definition 1.8.3. The **character** of a module M is $\operatorname{char} M := \sum z^\mu \dim M_\mu$, where μ here is a multi-index. (Equivalently, $\mu \in \mathbb{Z}[\mathfrak{h}^*]$.)

Example 1.8.4. The existence of the above resolution of $L(\lambda)$ shows

$$\operatorname{char} L(\lambda) = \sum_{k,i} (-1)^k \operatorname{char} M(\lambda_i^{(k)}).$$

But the character of Verma modules is very simple, since $M(\lambda) = \mathcal{U}(\mathfrak{n}_-) |\lambda\rangle$:

$$\operatorname{char} M(\lambda) = z^\lambda \operatorname{char} \mathcal{U}(\mathfrak{n}_-) = z^\lambda \operatorname{char} \operatorname{Sym}(\mathfrak{n}_-) = \frac{z^\lambda}{\prod_{\alpha>0}(1 - z^{-\alpha})^{\dim \mathfrak{g}_{-\alpha}}}.$$

We call $\dim \mathfrak{g}_\alpha$ the **multiplicity** $\operatorname{mult}(\alpha)$ of α.

Theorem 1.8.5 (Weyl–Kac character formula)**.** *If $L(\lambda)$ is an integrable highest weight module over a symmetrizable KM Lie algebra $\mathfrak{g}$, then*

$$\operatorname{char} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \frac{z^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1 - z^{-\alpha})^{\operatorname{mult} \alpha}}.$$

Proof. From the resolution of $L(\lambda)$, we get $\operatorname{char} L(\lambda) = \sum_{\mu \in \lambda - Q_+} c_{\lambda\mu} \operatorname{char} M(\mu)$, and we have a formula for $\operatorname{char} M(\mu)$. So it suffices to show $c_{\lambda\mu} = (-1)^{\ell(w)}$ if $\mu + \rho = w(\lambda + \rho)$, and $c_{\lambda\mu} = 0$ otherwise.

Observe that $\Delta := z^\rho \prod_{\alpha>0}(1 - z^{-\alpha})^{\operatorname{mult} \alpha}$ is antisymmetric under r_i, i.e. $r_i(\Delta) = -\Delta$. Take $\mathfrak{sl}(2)_i$ acting on $\mathfrak{n}_- \ominus \mathfrak{g}_{-\alpha}$. Note that $r_i(-\alpha_i) = +\alpha_i$, and $r_i(-\alpha)$ is another negative root for $\alpha \neq \alpha_i$. Then clearly $\prod_{\alpha \neq \alpha_i, \alpha>0}(1 - z^{-\alpha})^{\operatorname{mult} \alpha}$ in Δ is invariant under r_i. What happens to the remaining part $z^\rho(1 - z^{-\alpha_i})$? We have $r_i(\mu) = \mu - 2(\mu, \alpha)/(\alpha, \alpha)\alpha_i$, so for ρ we get

$$r_i \colon z^\rho(1 - z^{-\alpha_i}) \mapsto z^{\rho-\alpha_i}(1 - z^{\alpha_i}) = -z^\rho(1 - z^{-\alpha_i}).$$

Hence Δ is anti-symmetric under r_i. It follows that $c_{\lambda\mu} = (-1)^{\ell(w)}$.

Now we must show $c_{\lambda\mu} = 0$ for $\mu + \rho \notin W(\lambda + \rho)$. (Note that this is special to infinite dimensions.) Pick an element $\nu := \mu' + \rho \in W(\mu + \rho)$ of maximal height (so that ν is in the dominant cone). Then $r_i(\nu) = \nu - 2(\nu, \alpha_i)/(\alpha_i, \alpha_i)\alpha_i$ must have smaller height, so $(\nu, \alpha_i) \geq 0$. **Key inequality**:

$$(\nu, \alpha_i) > 0,\ (\xi, \alpha_i) \geq 0,\ \nu \geq \xi,\ (\nu, \nu) = (\xi, \xi) \implies \nu = \xi.$$

Let $\nu = \lambda + \rho$ and $\xi = \mu' + \rho$. Then all the conditions of this inequality are satisfied, so $\lambda + \rho = \mu' + \rho$ and we are done.

To prove the key inequality, denote $\eta \coloneqq \nu - \xi$. Then

$$(\nu, \nu) = (\xi, \xi) = (\nu - \eta, \nu - \eta) = (\nu, \nu) - 2(\nu, \eta) + (\eta, \eta).$$

If we write $\eta = \sum m_i \alpha_i$, then $(\eta, \eta) = 2(\nu, \eta) = 2 \sum m_i (\alpha_i, \nu) > 0$. On the other hand,

$$(\xi, \xi) = (\nu, \nu) = (\xi + \eta, \xi + \eta) = (\xi, \xi) + 2(\xi, \eta) + (\eta, \eta).$$

Then $(\eta, \eta) = -2(\xi, \eta) = -2 \sum_i m_i (\alpha_i, \xi) \leq 0$. □

Remark. This argument actually shows something stronger: integrable modules in category $\mathcal{O}$ are semisimple. This arises from showing there are no non-trivial extensions.

Example 1.8.6. Consider $L(0)$ the 1-dimensional trivial representation. Clearly $\operatorname{char} L(0) = 1$. The Weyl–Kac formula becomes

$$1 = \sum_w (-1)^{\ell(w)} \frac{z^{w(\rho) - \rho}}{\prod_{\alpha > 0} (1 - z^{-\alpha})^{\operatorname{mult} \alpha}}.$$

This rearranges to give $\Delta = \sum_w (-1)^{\ell(w)} z^{w(\rho)}$, called the **Weyl denominator formula**. For example, for $\mathfrak{g} = \mathfrak{sl}(n)$, this is the expression for a Vandermonde determinant.

1.9 Weyl character formula

We will derive the Weyl character formula for compact groups. Let G be a compact Lie group. Fix $T \subset G$ a maximal torus. We want to study G-reps, but representations of compact Lie groups are the same as representations of their complexifications, so equivalently we study $G_{\mathbb{C}}$-reps.

Definition 1.9.1. Let V be a representation of G. The **character** of V is the function $\chi_V : G \to \mathbb{C}$ given by $g \mapsto \operatorname{tr}_V(g)$.

Remark. Note that these functions are **class functions**, i.e. invariant under Ad_G. In other words, $\chi_V(hgh^{-1}) = \chi_V(g)$ for any $h \in G$. So $\chi_V \in L^2(G)^G$.

Proposition 1.9.2. *Let $f \in L^2(G)^G$. Then f can be reconstructed from $f|_T$.*

Proof. Every element $g \in G$ is conjugate to some element $t \in T$. So let $f(g) = f|_T(t)$. Because f is a class function, this is well-defined. □

Remark. Since $N_G(T)$ acts on T by conjugation, $f|_T$ is W-invariant. (Recall $W \coloneqq N_G(T)/T$ is the Weyl group of G.) So there is a map

$$L^2(G)^G \to L^2(T)^W.$$

By Fourier, we know $L^2(S^1)$ has a dense subset $\mathbb{C}[z^{\pm 1}]$. So $L^2(T)^W$ has a dense subset $\mathbb{C}[z^{\pm w_1}, \ldots, z^{\pm w_n}]$ where $w_1, \ldots, w_n$ are generators of the weight lattice.

Example 1.9.3. Take $G = \mathrm{SU}(2)$ and let $f(g) = 1/\operatorname{tr} g$. Then $f \in L^2(T)^W$ but is not in $L^2(G)^G$ because of singularities.

Proposition 1.9.4. *If V is a G-rep, $(\chi_V)|_T \in \mathbb{Z}_{\geq 0}[z^{\pm w_1}, \ldots, z^{\pm w_n}]$.*

Proof. Make all $t \in T$ simultaneously upper triangular. Then $\chi_V(t) = \operatorname{tr}_V(t)$, but on the diagonal of t we have only elements of the form z^μ where μ is some weight. This proves the claim. □

Proposition 1.9.5. *If V_λ is a G-rep with highest weight λ, then $(\chi_V)|_T$ has leading term z^λ.*

Remark. Recall that on $L^2(G)$ we have $(f_1, f_2) := \int_G f_1(g)\overline{f_2(g)}\,dg$ where dg is the normalized Haar measure. If V_1, V_2 are irreps of G, then $(\chi_{V_1}, \chi_{V_2}) = \delta_{V_1V_2}$. Summary: $\{\chi_V : V \text{irrep of } G\}$ form an orthonormal basis in $\mathbb{Z}[z^{\pm w_1}, \dots, z^{\pm w_n}]^W$, because $\sum_{w\in W} z^{w(\lambda)}$ is a $\mathbb{Z}$-basis and $\chi|_{V_\lambda}$ are lower-triangular with respect to this basis with 1s on the diagonal.

Remark (Outline of proof of Weyl character formula). The logic that we will follow is as follows.

1. We can find an orthonormal basis in $\mathbb{Z}[z^{\pm w_1}, \dots, z^{\pm w_n}]$ with respect to $(\cdot,\cdot)$ of $L^2(G)^G$. Denote this basis by s_λ, where z^λ is the leading term of s_λ.
2. For every irrep V, we have $(\chi_V)|_T = s_\lambda$ for some λ. This is because if we write $(\chi_V)|_T = \sum_\lambda m_\lambda s_\lambda$, then $|\chi|_T|^2 = \sum_\lambda |m_\lambda|^2$ so only for one λ do we have $|m_\lambda| = 1$.
3. $\chi_{V_\lambda}|_T = \pm s_\lambda$, by comparing highest weights. If s_λ has a positive coefficient, then the sign is actually $+$.

The key thing is to compute $(\cdot,\cdot)_G$ in $\mathbb{Z}[z^{\pm w_1}, \dots, z^{\pm w_n}]^W$. We will relate $(\cdot,\cdot)_G$ with $(\cdot,\cdot)_T$.

Remark. Recall that the normalized Haar measure on T is $dt = \prod_{k=1}^n ds_k$ where $z_k := z^{w_k} = e^{2\pi i s_k}$. We define $L^2(T)$ with respect to this measure. Note that

$$(z^\lambda, z^\mu)_T = \int_T z^\lambda z^{-\mu}\,dt = \delta_{\lambda,\mu}.$$

So $\{z^\lambda\}$ is an orthonormal basis.

Theorem 1.9.6 (Weyl denominator formula)**.** $\Delta := \prod_{\alpha>0}(1 - z^{-\alpha}) = \sum_{w\in W}(-1)^{\ell(w)} z^{w(\rho)-\rho}$.

Proof. Take $X := \sum_{w\in W}(-1)^{\ell(w)} z^{w(\rho)}$. Since $2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 2$, we know ρ lies in the weight lattice. In fact it is a dominant weight. Clearly we have $s_\alpha X = -X$, i.e. $X = (1 - s_\alpha)X/2$. Note that for any μ in the weight lattice, $(1 - z^\alpha) \mid (1 - s_\alpha)z^\mu$. (This is essentially the statement $(1 - z^2) \mid (z^k - z^{-k})$.) Since X has only terms of the form z^μ where μ is in the weight lattice and the terms $(1 - z^{-\alpha})$ for $\alpha > 0$ are independent, we get $\prod_{\alpha>0}(1 - z^{-\alpha}) \mid X$. By comparing leading terms, we get $X = z^\rho \prod_{\alpha>0}(1 - z^{-\alpha})$. ☐

Theorem 1.9.7 (Weyl integration formula)**.** *If $f \in L^2(G)^G$, then*

$$\int_G f\,dg = \frac{1}{|W|}\int_T f|_T |\Delta(t)|^2\,dt$$

where $\Delta(t) := \prod_{\alpha>0}(z^\alpha - 1)$, or equivalently $\Delta(t) := \prod_{\alpha>0}(z^{\alpha/2} - z^{-\alpha/2})$.

Corollary 1.9.8. *Given $f_1, f_2 \in L^2(G)^G$, we get $(f_1, f_2)_G = (1/|W|)(f_1|_T\Delta, f_2|_T\Delta)_T$.*

Proof of Weyl integration formula. Consider the map $\varphi\colon T \times G/T \to G$ given by $(t, gT) \mapsto gtg^{-1}$. This map has $|W|$ pre-images for all points except points in a measure zero set (the points where W acts non-freely). Hence

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_{T\times G/T} f(gtg^{-1}) J(t)\,dt\,d(gT)$$

where $J(t)$ is the Jacobian of the map φ and $d(gT)$ is the invariant measure on G/T. If $f \in L^2(G)^G$, we get $f(gtg^{-1}) = f(t)$, and we can integrate out the constant factor $K = \int_{G/T} d(gT)$. Now we need to compute the Jacobian $J(t)$. We can use the point $(t, 1\cdot T)$ because J is independent of g. Let $\xi \in T_tT$ and $\eta \in T_{1\cdot T}(G/T) = \mathfrak{g}/\mathfrak{t}$. Then $t \mapsto t(1 + \xi)$ and $1 \mapsto 1 + \eta$, i.e. up to first-order, we get

$$\rho = (1 + \eta)t(1 + \xi)(1 - \eta) - t = t\xi + \eta t - t\eta.$$

Hence $t^{-1}\rho = \xi + (t^{-1}\eta t - \eta) \in \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t} = \mathfrak{g}$. It follows that $d\varphi\colon (\xi, \eta) \mapsto (\xi, (\mathrm{Ad}_{t^{-1}} - 1)\eta)$. So

$$J(t) = \det(d\varphi) = \det(\mathrm{Ad}_{t^{-1}} - 1) = \prod_{\alpha\neq 0}(z^\alpha - 1)$$

because we have the decomposition $(\mathfrak{g}/\mathfrak{t}) \otimes \mathbb{C} = \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$. Now note that $J(t) = \Delta(t)\overline{\Delta(t)}$. So we are done up to the constant K. Take $f_1 = f_2 = 1$, so that

$$1 = (1,1)_G = \frac{K}{|W|}(1 \cdot \Delta, 1 \cdot \Delta)_T.$$

To compute $(\Delta, \Delta)_T$, we use the Weyl denominator formula

$$\Delta = \prod_{\alpha>0}(1 - z^{-\alpha}) = \sum_{w \in W} (-1)^{\ell(w)} z^{w(\rho)-\rho}.$$

Using orthonormality,

$$(\Delta, \Delta)_T = \sum_{w,w' \in W} (-1)^{\ell(w)+\ell(w')} (z^{w(\rho)-\rho}, z^{w'(\rho)-\rho})_T = \sum_{w,w' \in W} (-1)^{\ell(w)+\ell(w')} \delta_{w(\rho),w'(\rho)}.$$

But w is uniquely determined by $w(\rho)$, since ρ is strictly dominant. Hence $\delta_{w(\rho),w'(\rho)} = 1$ iff $w = w'$, so $(\Delta, \Delta)_T = \sum_{w \in W}(-1)^{2\ell(w)} = |W|$. It follows that the constant we want is $K = 1$. □

Remark. Note now that it is easy to compute scalar products of things of the form $P(z)/\Delta$ where $P(z)$ is some polynomial and $\Delta \mid P(z)$.

Remark. Now define the deformed $\Delta_\lambda \coloneqq \sum_{w \in W}(-1)^{\ell(w)} z^{w(\lambda+\rho)-\rho}$ where λ is a dominant integral weight. In particular, $\Delta = \Delta_0$. By an argument analogous to the one for the Weyl denominator formula, $\Delta \mid \Delta_\lambda$. Define $s_\lambda \coloneqq \Delta_\lambda / \Delta$.

Theorem 1.9.9 (Weyl character formula)**.** $(\operatorname{char} V(\lambda))|_T = s_\lambda$.

Proof. By the same argument as for the Weyl denominator formula, we see that $(\operatorname{char} V(\lambda))|_T = \pm s_\mu$ for some μ. By comparing leading terms, we see $\lambda = \mu$ and we need the positive sign. (Refer to outline of proof earlier.) □

Example 1.9.10. For $G = U(n)$ we have $G_\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$. The roots are $e_i - e_j$, so $\Delta = \prod_{i<j}(z_i - z_j)$ is a Vandermonde determinant. The torus T consists of diagonal matrices, and $z_1, \ldots, z_n$ are the eigenvalues. The right hand side is $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i z_i^{\sigma(i)}$, which is just the determinantal expansion of the Vandermonde determinant. The formula for Δ_λ is similar.

Theorem 1.9.11 (Weyl dimension formula)**.** *For any dominant integral λ, let $V(\lambda)$ denote the irrep of highest weight λ. Then*

$$\dim V(\lambda) = \prod_{\alpha>0} \frac{(\lambda+\rho, \alpha^\vee)}{(\rho, \alpha^\vee)}.$$

Proof. Clearly $\dim V(\lambda) = (\operatorname{char} V(\lambda))(1)$. Given an element of $\mathbb{Z}[z^{\pm w_1}, \ldots, z^{\pm w_n}]$ and μ a real linear combination of dominant integral weights, we can send $z^{\pm w_i}$ to $e^{\pm(\mu, w_i)}$. This is called **evaluation at e^μ**. Hence $\dim V(\lambda) = \lim_{\hbar \to 0} s_\lambda(e^{\hbar\rho})$. We know $s_\lambda = \sum_{w \in W}(-1)^{\ell(w)} z^{w(\lambda+\rho)} / \prod_{\alpha>0}(z^{\alpha/2} - z^{-\alpha/2})$. Notation: $J_\lambda(\mu) \coloneqq \sum_{w \in W}(-1)^{\ell(w)} e^{(w(\lambda),\mu)} = z^\rho \Delta_{\lambda+\rho}(e^\mu)$. So write $s_\lambda(e^{\hbar\rho}) = J_{\lambda+\rho}(\hbar\rho)/J_\rho(\hbar\rho)$. Note that for $t \in \mathbb{R}$,

$$J_\lambda(t\mu) = \sum_{w \in W}(-1)^{\ell(w)} e^{t(w(\mu),\lambda)} = \sum_{w \in W}(-1)^{\ell(w)} e^{t(\mu, w(\lambda))} = J_\mu(t\lambda)$$

by the W-invariance of $(\cdot,\cdot)$. Hence

$$s_\lambda(e^{\hbar\rho}) = \frac{J_{\lambda+\rho}(\hbar\rho)}{J_\rho(\hbar\rho)} = \frac{J_\rho(\hbar(\lambda+\rho))}{J_\rho(\hbar\rho)} = \prod_{\alpha>0} \frac{e^{\hbar(\lambda+\rho,\alpha^\vee)} - e^{-\hbar(\lambda+\rho,\alpha^\vee)}}{e^{\hbar(\rho,\alpha^\vee)} - e^{-\hbar(\rho,\alpha^\vee)}}.$$

Taking the limit as $\hbar \to 0$, we are done by l'Hôpital's rule. □

1.10 Affine Kac–Moody Lie algebras

Theorem 1.10.1 (Weyl denominator identity)**.** $\prod_{\alpha>0}(1-z^{-\alpha}) = \sum_{w\in W}(-1)^{\ell(w)}z^{w(\rho)-\rho}$.

Definition 1.10.2. From the Weyl denominator identity, we see that if $|W| < \infty$, then $\dim \mathfrak{g} < \infty$. **Affine Kac–Moody Lie algebras** are the case where the Cartan matrix C corresponds to a discrete group $W = \langle v_i \rangle \subset \mathrm{Isom}(\mathbb{R}^n)$. In particular, $\operatorname{rank} C = n-1$ and we pick a positive vector (with coprime entries) δ such that $C\delta = 0$, and we pick K such that $C^T K = 0$.

Example 1.10.3. Take $\mathfrak{sl}_2 \otimes \mathbb{C}[z^{\pm 1}]$, with basis elements

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad h_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\delta = (1,1)$ and $K = (1,1)$. To make a central extension, we add an element d which we view as $z\partial_z$. If now we want a **twisted affine Kac–Moody Lie algebra**, let ω be an outer automorphism of $\mathfrak{p}$ a finite dimensional simple Lie algebra. In types A, D, and E_6, we have $\omega^2 = 1$. The only interesting case is D_4, where there is an element $\omega^3 = 1$. Then

$$\mathfrak{p} \otimes \mathbb{C}[z^{\pm 1}] \supset \mathfrak{g}/\text{center} = \{g(z) : g(\zeta z) = g(z)^\omega,\ \zeta^{\operatorname{ord}(\omega)} = 1\}.$$

Concretely, in the case of A_2, the automorphism is $X \mapsto -X^T$ where here $(-)^T$ denotes transpose with respect to the other diagonal. For A_2, we are looking at $g(z) \in A_2 \otimes \mathbb{C}[z^{\pm 1}]$ such that $g(z) = -g(z)^T$. If we expand $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, we get $g_{\text{even}} \in \mathfrak{p}_0 = \mathfrak{so}(3)$, and $g_{\text{odd}} \in \mathfrak{p}_- = \mathrm{Sym}$ (the -1 eigenvalues of ω). So the analogue of what we wrote before is

$$e_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & -1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix},$$
$$e_0 = \begin{pmatrix} 0 & & \\ & 0 & \\ z & & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & & z^{-1} \\ & 0 & \\ & & 0 \end{pmatrix}$$

and now we have commutators like $[h_1, e_0] = -4e_0$. This gives a twisted $\hat{A}_2^{(2)}$, as opposed to $\hat{A}_1^{(1)}$:

$$\hat{A}_1^{(1)} \sim C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad \hat{A}_2^{(2)} \sim C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Write $\hat{A}_1^{(1)} = \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$. The root spaces all sit in $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}]$, and are spanned by elements of the form

$$\begin{pmatrix} 0 & 0 \\ t^k & 0 \end{pmatrix} \implies h_1 = -2,\ d = k$$
$$\begin{pmatrix} 0 & t^k \\ 0 & 0 \end{pmatrix} \implies h_1 = 2,\ d = k$$
$$\begin{pmatrix} t^k & 0 \\ 0 & -t^k \end{pmatrix} \implies h_1 = 0,\ d = k.$$

The positive ones are when $k > 0$, $k \geq 0$, and $k > 0$ respectively. Hence the lhs of the denominator identity becomes

$$\prod_{\alpha>0}(1-z^{-\alpha}) = (1-z^{-1})\prod_{k>0}(1-q^k)(1-q^k w)(1-q^k w^{-1})$$

where q is another indeterminate we use for the level k. We get the same type of expression for $\hat{A}_2^{(2)}$, except we get 5 terms. In general, the lhs of the denominator identity will always be of the form $\prod 1/\Gamma_q(\text{monomial in } w)$ where

$$\frac{1}{\Gamma_q(y)} := (1-y)(1-qy)(1-q^2y)\cdots.$$

Definition 1.10.4. We have $\mathfrak{h} = \mathfrak{h}_{\text{finite}} \oplus (\mathbb{C}K + \mathbb{C}d)$. The notation for the dual is $\mathfrak{h}^* = \mathfrak{h}^*_{\text{finite}} \oplus (\mathbb{C}\Lambda_0 + \mathbb{C}\delta)$. View the action of W on $\mathfrak{h}^*$ as an affine linear transformation which preserves Λ_0. The action of W on δ, the remaining subspace, is uniquely specified by the fact that $w \in \operatorname{Isom}(\mathfrak{h}^*)$:

$$w \cdot \lambda = \lambda(K) \cdot \Lambda_0 + \text{how } w \text{ acts on } \mathfrak{h}^*_{\text{finite}} + ? \cdot \delta$$

and $\|w \cdot \lambda\| = \|\lambda\|$. Call the middle term $\bar{w} \cdot \bar{\lambda}$. Explicitly,

$$\|\lambda\|^2 = \|\bar{w} \cdot \bar{\lambda}\|^2 + 2\lambda(K) \cdot ?.$$

Corollary 1.10.5. *$w \mapsto \bar{w}$ is an isomorphism. Hence there is a short exact sequence*

$$0 \to M \to W \to W_{\mathit{finite}} \to 1$$

where M is a lattice and $W \subset \operatorname{Isom}(\mathfrak{h}^*_{\mathit{finite}})$.

Remark. In particular, elements $t_\mu \in M$ act as $t_\mu(\bar{\lambda}) = \bar{\lambda} + \mu$. If $\lambda = \lambda(K)\Lambda_0 + \bar{\lambda}$, then

$$t_\mu(\lambda) = \lambda(K)\Lambda_0 + \bar{\lambda} + \mu + \frac{\|\lambda\|^2 - \|\lambda + \mu\|^2}{2\lambda(K)}\delta.$$

Remark. Write $w \in W$ as σT, where $\sigma \in W_{\text{finite}}$ and T is a translation. If we split the formal variable z as (x, q), we can split the sum:

$$\sum_{w \in W} (-1)^{\ell(w)} z^{w\rho - \rho} = \sum_{\sigma \in W_{\text{finite}}} (-1)^\sigma \sum_{T \in M} q^{\text{quadratic}} x^{\sigma(\bar{\rho} + \mu) - \bar{\rho}}$$

where μ is the corresponding entry in the matrix of T. Interchanging the order of summation, we get some sort of Weyl character formula.

Example 1.10.6. Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. Then $\omega = \text{id}$ and $\mathfrak{p} = \mathfrak{sl}_2$, and explicitly we get

$$(1 - x^{-1}) \prod_n (1 - q^n)(1 - xq^n)(1 - x^{-1}q^n) = \sum_k q^{k + 2k^2}(x^{2k} - x^{-2k-1}) = \sum_n (-1)^n x^n q^{n(n+1)/2}.$$

These are two solutions of the q-difference equation: $f(qx) = -f(x)/qx$. In modern language, this says f is a section of a degree-1 line bundle over the elliptic curve $E \coloneqq \mathbb{C}/q^{\mathbb{Z}}$. By Riemann–Roch, a section of such a line bundle is uniquely determined up to constant.

Definition 1.10.7. Let C be a Cartan matrix, and suppose $(k_1, \ldots, k_n)C = 0$ normalized with positive and coprime entries. Then $K \coloneqq \sum k_i h_i$ is central. Let λ be a highest weight of an integrable module. Then $\lambda(K) \geq 0$ is the **level**. Levels add in tensor products. Level-1 representations are **basic representations**.

Example 1.10.8. Take $\mathfrak{g} = \widehat{\mathfrak{sl}_{n-1}}$. Then $K = h_1 + \cdots + h_n$. Embed $\widehat{\mathfrak{sl}_{n-1}} \hookrightarrow \mathfrak{gl}_\infty \oplus \text{center}$. The central extension in $\mathfrak{gl}_\infty \oplus \text{center}$ pulls back to the central extension in $\widehat{\mathfrak{sl}_{n-1}}$. The **Fock representations** Fock_c are level-1 representations in $\mathfrak{gl}_\infty \oplus \text{center}$, and pull back to the n basic representations π_i of $\widehat{\mathfrak{sl}_{n-1}}$ by $\text{Fock}_c|_{\widehat{\mathfrak{sl}_{n-1}}} = \pi_{c \bmod n}$. Explicitly, $\mathbb{C}^\infty = \bigoplus_{k \in \mathbb{Z}+1/2} \mathbb{C}\underline{k}$. Then

$$\text{Fock}_c = \bigoplus_{S = \{k_i\}} \underline{k_1} \wedge \underline{k_2} \wedge \cdots$$

where only finitely many negatives do not occur in S, i.e. $|S \triangle (\mathbb{Z} + 1/2)_{<0}| < \infty$. This number is the **charge**. There is a special element $\underline{-1/2} \wedge \underline{-3/2} \wedge \cdots$ called the **Dirac vacuum**. Any other vector arises from the Dirac vacuum by removing some finite number of negative elements, and adding some finite number of positive elements. (The charge is the number of additions minus the number of removals.) Consider the subset of $\mathfrak{gl}(\mathbb{C}^\infty)$ with finitely many diagonals.

1. Off-diagonal elements in $\mathfrak{gl}_\infty$ act fine on Fock modules. For example, $\sum_{k\in\mathbb{Z}+1/2} a_k E_{k+3,k}$ will act by the identity far enough into the negative range, and acts non-trivially only on the finite number of modifications.

2. Diagonal elements $\sum a_k E_{k,k}$ "act by scalar multiplication by $\sum_{k\in S} a_k$", which does not really make sense. Instead, we get rid of the infinitely many a_k for k negative: define the action as multiplication by $\sum_{k\in S} a_k - \sum_{k\in(\mathbb{Z}+1/2)_{<0}} a_k$. So the representation π satisfies $\pi([x,y]) = [\pi(x),\pi(y)] + \text{constant}$, and the constant may be non-zero. The way to deal with this is to introduce a central extension, and declare the constant to be constant $\cdot\, \pi(K)$, and declare $\pi(K) = 1$.

Now consider $\mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}]$. This acts on $\mathbb{C}^n \otimes \mathbb{C}[t^{\pm 1}] \cong \mathbb{C}^\infty$, where the identification is with basis

$$\ldots, t^{-1}e_1, \ldots, t^{-1}e_n, e_1, \ldots, e_n, te_1, \ldots, te_n, \ldots.$$

Declare e_1 to be $\underline{1/2}$. For example, $\underline{5/2} \wedge \underline{3/2} \wedge \underline{1/2} \wedge \cdots$ is annihilated by all lower-triangular matrices. Denote it $|0,3\rangle$. Then $(\sum a_k E_{k,k})\,|0,3\rangle = a_{5/2} + a_{3/2} + a_{1/2}$. The vector $|0,0\rangle$ has trivial action by the Cartan matrix, and the lowering operators, i.e. $(\mathfrak{sl}_n \otimes \mathbb{C}[t^{-1}])\,|0,0\rangle = 0$. The finite Cartan acts as

$$\operatorname{diag}(a_1,\ldots,a_n)\,|0,k\rangle = \sum_{i=1}^{k} a_k\,|0,k\rangle$$

for all $k = 0,\ldots,n-1$. This is an explicit description of level-k integrable modules for $\widehat{\mathfrak{sl}_n}$.

Observe that the **charge** is the action of the matrix $\operatorname{diag}(1,1,\ldots,1)$, the identity element $\sum E_{k,k} \in \widehat{\mathfrak{gl}_\infty}$. The **energy** is the action of the matrix $\sum kE_{k,k}$. Take the big Fock space $\mathcal{F} \coloneqq \bigoplus_c \mathrm{Fock}_c$, and compute $\operatorname{tr}_{\mathcal{F}}(q^{\text{energy}}x^{\text{charge}})$. Adding an element $\underline{k}$ introduces a factor xq^k, and removing an element $\underline{k}$ introduces a factor $x^{-1}q^k$. Hence

$$\operatorname{tr}_{\mathcal{F}}(q^{\text{energy}}x^{\text{charge}}) = \prod_{n\in\mathbb{Z}+1/2} (1+xq^n)(1+x^{-1}q^n).$$

But we can also write this as $\sum_{c\in\mathbb{Z}} x^c K$ where K is the generating function in q for the number of ways to get charge c. By pairing the additions with the removals in order, this is just $\sum_\lambda q^{|\lambda|}$ where λ ranges over all partitions. Hence

$$\operatorname{tr}_{\mathcal{F}}(q^{\text{energy}}x^{\text{charge}}) = \sum_{c\in\mathbb{Z}} x^c q^{\text{quadratic}} \sum_\lambda q^{|\lambda|} = \sum_{c\in\mathbb{Z}} x^c \frac{q^{\text{quadratic}}}{\prod(1-q^n)}.$$

This is precisely the triple product formula (once we compute $q^{\text{quadratic}} = q^{c^2/2}$).

Chapter 2

Equivariant K-theory

2.1 Equivariant sheaves

Let G be a linear algebraic group, and X be a G-variety, i.e. a variety with an action of G. This means there are two maps $a, p\colon G \times X \to X$ given by the action $a(g,x) \coloneqq g \cdot x$ and the projection $p(g,x) \coloneqq x$.

Definition 2.1.1. A function $f\colon X \to \mathbb{C}$ is **invariant** if $f(gx) = f(x)$ for any $x \in X$ and $g \in G$. Equivalently, $a^* f = p^* f$. Equivalently,

$$(\mathrm{id}_G \times a)^* a^* f = (m \times \mathrm{id}_X)^* p^* f$$

where $m\colon G \times G \to G$ is multiplication.

Remark. Recall that sheaf pullback $u^*\mathcal{F}$ is not the same as $\mathcal{O}_X$-module pullback $u^*\mathcal{F} \coloneqq \mathcal{O}_Y \otimes_{u^*\mathcal{O}_X} u^*\mathcal{F}$.

Definition 2.1.2. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on a G-variety X. We say $\mathcal{F}$ is G**-equivariant** if:

1. there is a *given* isomorphism of sheaves $I\colon a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$;
2. $p_{23}^* I \circ (\mathrm{id}_G \times a)^* I = (m \times \mathrm{id}_X)^* I$ where $p_{23}\colon G \times G \times X \to G \times X$ is projection onto the second and third factors;
3. $I|_{e\times X} = \mathrm{id}\colon \mathcal{F} = a^*\mathcal{F}|_{e\times X} \xrightarrow{\sim} p^*\mathcal{F}|_{e\times X}$ where $e \in G$ is the identity.

Remark. Note that condition 3 in the definition follows from 1 and 2. Also, equivariance is a *structure*, i.e. the same sheaf can carry different equivariant structures.

Remark. The structure sheaf $\mathcal{O}_X$ has a canonical G-equivariant structure given by $p^*\mathcal{O}_X \cong \mathcal{O}_{G\times X} \cong a^*\mathcal{O}_X$.

Example 2.1.3. A sheaf $\mathcal{F}$ over a point is a vector space V, and the group G acts on the sheaf by acting on the vector space. In fact we can check that giving an equivariant structure on $\mathcal{F}$ is equivalent to giving a representation $G \to \mathrm{End}(V)$.

Example 2.1.4. Let $\pi\colon F \to X$ be a vector bundle over X (with associated sheaf $\mathcal{F}$). Then the equivariant structure is given by a map $\Phi\colon G \times F \to F$, and conditions 2 and 3 impose the conditions

$$\Phi(h, \Phi(g, f)) = \Phi(gh, f), \quad \Phi(e, f) = f.$$

In addition, $\pi\colon F \to X$ should be compatible with Φ, with $F_x \mapsto F_{gx}$, and these maps should be linear.

Theorem 2.1.5. *Let G be a linear algebraic group and X a smooth (or normal) G-variety. Let $L \to X$ be a line bundle. Then there exists a positive integer n such that $L^{\otimes n}$ an equivariant structure.*

Remark. We will assume some results from algebraic geometry:

1. if X is a smooth algebraic variety, then the abelian group of isomorphism classes of line bundles is the class group $\mathrm{Cl}(X)$;
2. for any X, the pullback by $p\colon \mathbb{C}\times X \to X$ is an isomorphism $p^*\colon \mathrm{Cl}(X)\to \mathrm{Cl}(\mathbb{C}\times X)$ on class groups;
3. if $U\subset X$ is (Zariski) open dense and $C_1,\dots,C_n$ are irreducible components of $X-U$, then

$$\mathbb{Z}^n \xrightarrow{(k_1,\dots,k_n)\mapsto \sum k_i C_i} \mathrm{Cl}(X)\to \mathrm{Cl}(U)\to 0$$

   is exact. So $\mathrm{Cl}(X)\cong \mathrm{Cl}(\mathbb{C}^r\times(\mathbb{C}^\times)^s\times X)$ for any r,s.

Let M and X be smooth varieties and $p_M\colon M\times X\to M$ and $p_X\colon M\times X\to X$ be the projections. Assume there is an (Zariski) open dense subset $\mathbb{C}^r\times(\mathbb{C}^*)^s\subset M$. By applying result 3, we see that any line bundle on $M\times X$ is isomorphic to an external tensor product of line bundles on fibers, i.e. $L\cong p_M^*(L|_M)\otimes p_X^*(L|_X)$. (This is still true for normal varieties, but the proof is a bit more annoying.)

Remark. Recall also that given a linear algebraic group G, we can look at the unipotent radical G_u. Then G/G_u is reductive, and $G_u\cong\mathbb{C}^r$ as a variety. Furthermore, $G/G_u\times G_u$ as varieties.

Proof of theorem. We will show that any linear algebraic group contains $\mathbb{C}^r\times(\mathbb{C}^*)^s$ as an open dense subset. Then by the previous remark, it suffices to prove the theorem for reductive groups. Let B^+ and B^- be opposite Borels in a reductive G (i.e. $B^+\cap B^-=T$, the maximal torus). Then $B^+\cdot B^-$ is a dense Zariski open in G. The B^+ orbits in G/B^- are Bruhat cells, so there is a Zariski dense on G/B^-. The differential of the B^+-action at the point eB^-/B^- is surjective, so the orbit through eB^-/B^- is dense. Let $U^\pm$ be the unipotent radical of the corresponding $B^\pm$. Then $B^+\times B^-\cong U^+\times T\times U^-\cong\mathbb{C}^r\times(\mathbb{C}^*)^s$. By writing fact 3 above for G, we see that line bundles L on G are trivial for high enough tensor powers $L^{\otimes n}$.

Let $L\to X$ be a line bundle over X and $E:=a^*\mathcal{L}|_{e\times X}$. By the lemma, for $M=G$, there is a line bundle F on G such that $a^*L\cong(p_G^*F)\otimes(p_X^*E)$. There exists an n such that $F^{\otimes n}$ is trivial, so that $(a^*L)^{\otimes n}\cong p_X^*(L^{\otimes n})$. Construct Φ in a way such that

$$\begin{array}{ccc} G\times L & \xrightarrow{\Phi} & L \\ {\scriptstyle \mathrm{id}\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ G\times X & \xrightarrow{a} & X \end{array}$$

commutes. Then $\Phi|_{e\times L}$ is an automorphism of L as a line bundle, which we write as multiplication by a function $\phi\colon X\to\mathbb{C}^*$. Replace Φ by $(1/\pi^*\phi)\Phi$ in order to make Φ satisfy condition 3 of the definition of an equivariant sheaf. ☐

Lemma 2.1.6. *If $\mathcal{F}$ is an equivariant sheaf, then $\Gamma(X,\mathcal{F})$ has a natural structure of a G-module.*

Proof. Note that there is a natural chain of isomorphisms

$$\Gamma(X,\mathcal{F})\xrightarrow{a^*}\Gamma(G\times X,a_X^*\mathcal{F})\xrightarrow{I}\Gamma(G\times X,p_X^*\mathcal{F})=\mathbb{C}[G]\otimes\Gamma(X,\mathcal{F}).$$ ☐

Remark. Recall that L is **ample** if, for any coherent sheaf $\mathcal{F}$, there exists an $n=n(\mathcal{F})$ such that $\mathcal{F}\otimes L^{\otimes n}$ is generated by global sections. Any quasi-projective variety has an ample line bundle, namely the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Corollary 2.1.7. *Let G be an algebraic linear group and X a smooth quasi-projective G-variety. Then there exists a G-equivariant ample line bundle on X.*

Theorem 2.1.8 (Equivariant projective embedding)**.** *Let G be a linear algebraic group and X a normal quasi-projective G-variety. Then there exists a finite-dimensional vector space V and an algebraic group homomorphism $\rho\colon G\to\mathrm{GL}(V)$ and an equivariant embedding $i\colon X\to\mathbb{P}(V)$.*

Proof. Embed $X \hookrightarrow \bar{X}$ where $\bar{X}$ is projective. Let L be an ample line bundle on $\bar{X}$. Global sections of L separate points and tangent vectors. Let $H_x \subset \Gamma(\bar{X}, L)$ be the hyperplane of sections vanishing at $x \in X$. Then $x \mapsto H_x$ gives a morphism $i: \bar{X} \to \mathbb{P}(\Gamma(\bar{X}, L)^\vee)$. Replace L by $L^{\otimes n}$ such that $L^{\otimes n}|_X$ is equivariant. We know $\Gamma(X, L)$ is a G-module. However $\Gamma(X, L)$ may be infinite-dimensional. We know $\Gamma(\bar{X}, L)$ is finite-dimensional, but it is not a G-module. But we can pick a G-equivariant $V \subset \Gamma(X, L)$ containing $\Gamma(\bar{X}, L)$. Then V separates points and tangents on X, so $X \to \mathbb{P}(V)$ is an equivariant embedding. ☐

Lemma 2.1.9. *Let X be a smooth (or more generally normal) quasi-projective variety. Then any G-equivariant coherent sheaf $\mathcal{F}$ on X is a quotient of a G-equivariant locally free sheaf.*

Proof. Let $X \subset \bar{X}$ and L be ample on $\bar{X}$. Let $\bar{\mathcal{F}}$ be an extension of $\mathcal{F}$ to $\bar{X}$, which may not be equivariant. There exists an n such that $\bar{\mathcal{F}} \otimes L^{\otimes n}$ is generated by (a finite number of) global sections $\{s_i\}$ on $\bar{X}$. Assume $L^{\otimes n}|_X$ is equivariant (perhaps by making n bigger). Let V be generated by the global sections $\{s_i\}$. Then $V \otimes (L^*|_X)^{\otimes n} \to \mathcal{F}$ is surjective by construction. ☐

Proposition 2.1.10. *If $\mathcal{F}$ is a G-equivariant coherent sheaf on X, then there exists a G-equivariant extension $\bar{\mathcal{F}}$ such that $\bar{\mathcal{F}}|_X = \mathcal{F}$. If $f: \mathcal{F} \to \mathcal{G}$ is a G-equivariant morphism of sheaves on X, then there exists a G-equivariant extension $\bar{f}: \bar{\mathcal{F}} \to \bar{\mathcal{G}}$ between two given extensions $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$.*

Corollary 2.1.11. *If X is a smooth quasi-projective G-variety, then G-equivariant coherent sheaf $\mathcal{F}$ on X has a finite locally free G-equivariant resolution.*

Proof. By the lemma, we can find $0 \to \mathcal{F}' \hookrightarrow \mathcal{F}_1 \to \mathcal{F} \to 0$ such that $\mathcal{F}_1$ is a G-equivariant locally free sheaf. Now apply the same procedure to $\mathcal{F}'$ and repeat. ☐

Remark. We can compute cohomology using these finite locally free G-equivariant resolutions. Hence there is a G-action on $H^i(\mathcal{F})$ for all i.

2.2 Equivariant K-theory

Definition 2.2.1. Let $\mathsf{Coh}^G(X)$ be the category of G-equivariant coherent sheaves on X. Let $K^G(X)$ be the Grothendieck group of $\mathsf{Coh}^G(X)$; this is **equivariant K-theory** $K_0^G(X) \coloneqq K^G(X)$. We almost never talk about higher equivariant K-theory.

Remark. If we take $G = \{1\}$, then we recover ordinary K-theory. If we take $X = \{\mathrm{pt}\}$, then we recover the representation ring $\mathrm{Rep}(G)$ (of finite-dimensional representations). Via the character of a representation, there is an embedding $\mathrm{Rep}(G) \hookrightarrow \mathcal{O}(G)^G$ into class functions. In fact $\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Rep}(G) \xrightarrow{\sim} \mathcal{O}(G)^G$ is an algebra isomorphism if G is reductive.

Proposition 2.2.2. *Let X be a G-variety. If $G = G_1 \times G_2$ and G_1 acts trivially on X, then*

$$\mathsf{Coh}_G(X) = \mathsf{Coh}_{G_2}(X) \otimes \mathsf{Coh}_{G_1}(X) = \mathsf{Coh}_{G_2}(X) \otimes \mathrm{Rep}(G_1).$$

Hence $K^G(X) = \mathrm{Rep}(G_1) \otimes_{\mathbb{Z}} K^{G_2}(X)$.

Definition 2.2.3 (Pullbacks)**.** Let $f: Y \to X$ be a G-equivariant morphism.

1. If f is an open (or more generally, flat) embedding, then there exists a **pullback** $f^*: K^G(X) \to K^G(Y)$ induced by pullback $\mathcal{F} \mapsto f^*\mathcal{F} \coloneqq \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F}$ in Coh.

2. If f is a closed embedding, then we still want a pullback induced by $f^*\mathcal{F} = \mathcal{F}/\mathcal{I}_Y \otimes \mathcal{F} \cong \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F}$. But f^* is not exact. For the smooth case, we use the finite locally free resolution $\mathcal{F}^\bullet \coloneqq \cdots \to \mathcal{F}_1 \to \mathcal{F} \to 0$ and define the **pullback**

$$f^*[\mathcal{F}] \coloneqq \sum_k (-1)^k [f^* \mathcal{H}^k (f_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)]$$

where $\mathcal{H}^k$ denotes cohomology sheaves.

Definition 2.2.4 (Pushforward)**.** If X, Y are quasi-projective G-varieties and $f\colon X \to Y$ is proper, then there is a **pushforward** $f_*\colon K^G(X) \to K^G(Y)$. To define it, note that the complex $R^i f_* \mathcal{F}$ vanishes for $i \gg 0$. So define

$$f_*[\mathcal{F}] \coloneqq \sum_k (-1)^k [R^k f_* \mathcal{F}].$$

The long exact sequence associated to $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ shows that $f_*[\mathcal{F}] = f_*[\mathcal{F}'] + f_*[\mathcal{F}'']$.

Definition 2.2.5 (Tensor product)**.** If X, Y are G-varieties with sheaves $\mathcal{F}$ and $\mathcal{F}'$ respectively, then there is an **external tensor product** $\mathcal{F} \boxtimes \mathcal{F}' \coloneqq p_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} p_Y^* \mathcal{F}'$ on $X \times Y$. If $\mathcal{F}, \mathcal{F}'$ are both on X, then

$$[\mathcal{F}] \otimes [\mathcal{F}'] = \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$$

where $\Delta\colon X \to X \times X$ is the diagonal map. This gives a **tensor product** $K^G(X) \otimes K^G(X) \to K^G(X)$. (Because we are using the diagonal map Δ, we require smoothness.)

Chapter 3

Geometric representation theory

3.1 Borel–Weil

We will see that for a reductive group G over $\mathbb{C}$, the irreps of G are realized as $H^0(G/B, \mathcal{O}(\lambda))$ for some line bundle $\mathcal{O}(\lambda)$ on the flag variety G/B. (Recall that G/B is projective.) The higher cohomologies H^i in fact vanish, so $h^0(G/B, \mathcal{O}(\lambda)) = \chi(G/B, \mathcal{O}(\lambda))$. We can compute $\chi(G/B, \mathcal{O}(\lambda))$ by localization, and thereby obtain the Weyl character formula.

This is not unexpected: V^λ is the space of linear functions on $(V^\lambda)^*$, which arises from $H^0(X, \mathcal{O}(1))$ where X is any subvariety of $\mathbb{P}(V^{\lambda^*})$. In $\mathbb{P}(V^{\lambda^*})$, we can pick v fixed by B; such a line exists uniquely because V^λ is an irreducible representation. Hence there is a map $X = G/B \to G \cdot v$.

Proposition 3.1.1. *This map $V^{\lambda^*} \to H^0(X, \mathcal{O}(1))$ is an isomorphism.*

Proof. First show $H^0(G/B, \mathcal{L})$ is irreducible. This is because it has at most one highest vector. Otherwise if s_1, s_2 are both scaled by B with characters χ_1, χ_2, then both χ_1, χ_2 are invariant under U where U is the unipotent radical $1 \to U \to B \to T \to 1$. (Here T is the maximal torus.) So $f = s_1/s_2 \in \mathbb{C}(G/B)^U$. But U acts on G/B with an open orbit, by looking at the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Hence f is a constant on an dense open subset, and therefore is constant. Now note that the map $(V^\lambda)^* \to H^0(X, \mathcal{O}(1))$ has no kernel, because $\operatorname{span} G \cdot x = V^{\lambda^*}$. ☐

Alternate proof. The Peter–Weyl theorem says $\mathbb{C}[G] = \bigoplus_\lambda V^\lambda \otimes V^{\lambda^*}$. Inside $V^\lambda \otimes V^{\lambda^*}$ we can look at $V^\lambda \otimes v$, where v is the unique B-eigenvector of weight λ^*. Equivalently, $V^\lambda \otimes v$ is the set of solutions to $f(gb) = f(g)\chi^{\lambda^*}(b)$. These are precisely the global sections of the appropriate line bundle on G/B. ☐

Example 3.1.2. If $X = \mathbb{P}^1 = \mathrm{SL}(2, \mathbb{C})/B$, then

$$H^0(\mathcal{O}(d)) = \begin{cases} V^d & d \geq 0 \\ 0 & d \leq -1. \end{cases}$$

If we look at the first cohomology,

$$H^1(\mathcal{O}(d)) = \begin{cases} 0 & d \geq -1 \\ V^{-d-2} & d < -1. \end{cases}$$

where we get V^{-d-2} by noting that $H^i(X, K_X)$ is a trivial representation of G (by Hodge decomposition).

Remark. Every line bundle on G/B arises from a character λ. A character λ of B is the same as a weight of G.

Lemma 3.1.3. *For general $X = G/B$, its canonical bundle is $K_X = -2\rho$ where $\rho = (1/2)\sum_{\alpha>0} \alpha$.*

Definition 3.1.4. A line bundle L is **numerically effective (nef)** if the pairing $(\mathcal{L}, C) \coloneqq \deg(\mathcal{L}|_C)$ is non-negative for all effective curves C. It is **big** if $c_1(\mathcal{L})^{\dim X} > 0$.

Lemma 3.1.5. *$H^i(\mathcal{L}(\lambda) \otimes K_X) = 0$ for dominant integral weights λ.*

Proof. We apply the Kodaira vanishing theorem: if $\mathcal{L}$ is a line bundle which is big and nef (in particular, ample) on X smooth projective (over a field of characteristic 0), then $H^i(X, \mathcal{L} \otimes K_X) = 0$ for $i > 0$. In our case, $K_X = \mathcal{O}(-2\rho)$, and $\mathcal{L} = \mathcal{O}(\rho + \lambda)$ where λ is non-negative. So $\mathcal{L} \otimes K_X$ is ample + canonical, and therefore Kodaira vanishing is applicable. □

3.2 Localization

Let $t \in T \subset G$ be an element in the maximal torus. Then by cohomology vanishing,

$$\operatorname{tr}_{V^\lambda} t = \operatorname{tr}_{H^0(G/B, \mathcal{L}(\lambda))} t = \operatorname{tr}_{\chi(G/B, \mathcal{L}(\lambda))} t.$$

Let $X = G/B$. If we pick t sufficiently generic, $X^t = X^T$.

Proposition 3.2.1. *$X^T = W$.*

Proof. Let $gB \in G/B$ be a T-fixed point. Then $Tg = gT'$ where $T' \subset B$. But $T' = uTu^{-1}$ where $u \in U$ is unipotent. Hence $Tg = guTu^{-1}$. Rearranging, $Tgu = guT$. Hence $gu \in W$ is a normalizer of T. □

Example 3.2.2. Cover $\mathbb{P}^1 = \{[x_0 : x_1]\}$ by two charts $U_0 = \{x_0 \neq 0\} = \mathbb{A}^1$ and $U_\infty = \{x_1 \neq 0\} = \mathbb{A}^1$. Then $U_{0\infty} = \mathbb{C}^*$. If we are interested in $\chi(\mathbb{P}^1, \mathcal{F})$ for some sheaf $\mathcal{F}$, then from the Čech complex, we get $\chi(\mathbb{P}^1, \mathcal{F}) = \chi(U_0, \mathcal{F}) + \chi(U_\infty, \mathcal{F}) - \chi(U_{0\infty}, \mathcal{F})$. If we have a torus action on a manifold, we can choose the cover such that each piece is torus-invariant. The main idea is that $\chi(U_{0\infty}, \mathcal{F}) \in \widehat{K_T(\{\text{pt}\})}_{\text{tors}}$. For example, let $T = \operatorname{diag}(1, t)$ act by scaling. Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}$ be the structure sheaf. Then

$$\chi(U_0, \mathcal{F}) = 1 + 1/t + 1/t^2 + \cdots, \quad \chi(U_\infty, \mathcal{F}) = 1 + t + t^2 + \cdots,$$

and on the intersection we get $\chi(U_{0\infty}, \mathcal{F}) = \sum_{n \in \mathbb{Z}} t^n$. This is annihilated by $(1-t)$. So there is an embedding

$$K_T(\{\text{pt}\}) \hookrightarrow \widehat{K_T(\{\text{pt}\})}[1/(1 - \chi(t))].$$

So if we want to compute in $K_T(\{\text{pt}\})$, then we may as well compute in the completion mod torsion (where in the equation above we were specific about what the torsionis). But in there, we can neglect the term $\chi(U_{0\infty}, \mathcal{F})$. Hence

$$\chi(U_0, \mathcal{F}) = \frac{1}{1 - 1/t}, \quad \chi(U_\infty, \mathcal{F}) = \frac{1}{1-t}, \quad \chi(X, \mathcal{F}) = \frac{-t}{1-t} + \frac{1}{1-t} = 1.$$

The general setting: suppose G acts on a projective variety X, and let $\mathcal{F}$ be a coherent sheaf on X. In particular, the example to have in mind is the flag variety $X = G/B$, with $\mathcal{F} = \mathcal{O}(\lambda)$ where λ is a dominant weight. If $\mathcal{F}$ is G-equivariant, then $\chi(X, \mathcal{F})$ is a virtual representation of G, i.e. a formal linear combination of representations (since $H^i(\mathcal{F})$ is a representation of G by G-equivariance). We would like to compute $\operatorname{tr}_{\chi(X, \mathcal{F})} g$. A general principle called localization is computed in terms of X^g, the g-fixed points on X.

Example 3.2.3 (Finite groups)**.** Let $G = \mathrm{GL}(n, \mathbb{F}_q)$ with its Borel $B \subset G$ of upper triangular matrices. Given a character χ of B, then we get an induced module $V = \operatorname{Ind}_B^G \chi = \mathbb{C}[G] \otimes_{\mathbb{C}[B]} \mathbb{C}_\chi$ (equivalently, functions on G that transform as $f(gb) = \chi(b) f(g)$). Note that $\dim V = |G/B|$. The matrix $g|_V$ will be a permutation matrix, with non-zero entries given by the permutation action of g on G/B. So if we compute $\operatorname{tr} g$, it will only receive contributions from fixed points of g acting on G/B.

Theorem 3.2.4 (Thomasen)**.** *Let $T \subset G$ be a maximal torus and X be a scheme. The inclusion $i\colon X^T \to X$ induces*

$$i_*\colon K_T(pt) \otimes_{\mathbb{Z}} K(X^T) = K_T(X^T) \to K_T(X).$$

The kernel and cokernel of this map are both torsion. Then the diagram

$$\begin{array}{ccc} K_T(X^T) & \xrightarrow{i_*} & K_T(X) \\ \downarrow & & \downarrow \chi \\ K_T(pt) & = & K_T(pt) \end{array}$$

says if $\mathcal{F} = i_*\mathcal{G}$ mod torsion, then $\chi(\mathcal{F}) = \chi(\mathcal{G})$ (where since $K_T(pt)$ is a free module, this really is an equality). But $\mathcal{G} = \sum w_i \otimes \mathcal{G}_i$ where w_i are characters and $\mathcal{G}_i$ are sheaves with trivial actions. Hence

$$\chi(\mathcal{F}) = \sum w_i \chi(\mathcal{G}_i).$$

Proof sketch. Why is the cokernel torsion? We argue by induction: factor $i_*\colon K_T(X^T) \to K_T(Y) \to K_T(X)$ where $Y \subset X$ is a T-invariant subvariety. In general we can do this because if $X \neq X^T$, then there exists an open $U \subset X$ (with $U = X - Y$) and $f \in \mathcal{O}(U)^\times$ of non-trivial torus weight w. If this is the case, then $K_T(X) \twoheadrightarrow K_T(U)$ with kernel $K_T(Y)$ (and in fact we can continue the sequence

$$\cdots \to K_T(Y) \to K_T(X) \to K(TU) \to 0$$

to the higher K-groups). In $K_T(U)$, we have $\mathcal{F}|_U = w\mathcal{F}|_U$ via multiplication by f. Hence $(1-w)\mathcal{F} = 0$ in $K_T(X)$ modulo terms in the image of $K_T(Y)$. Suppose $\mathcal{V}$ i a vector bundle on X of rank r. Then $\mathcal{V}$ is trivial in some open $U \subset Y$, i.e. $\mathcal{V}|_U = \mathcal{O}(U)^{\oplus r}$. If $\mathcal{G}$ is a coherent sheaf on Y, then

$$\mathcal{V} \otimes \mathcal{G} = r\mathcal{G} + \text{terms in higher codimension}.$$

This proves that the operator $(\mathcal{V} \otimes -) - r$ is nilpotent (in non-equivariant K-theory). □

Remark. Note that in this situation there is no natural pullback. For pushforward to exist, we only require the map to be proper. For pullbacks to exist, we require a condition on the resolutions: finite Tor dimension. For example, $(\mathbb{C}^*)^2$ acting on the coordinate cross (xy) has fixed point $(0,0)$, but $\mathrm{Tor}^i(\mathcal{O}_0, \mathcal{O}_0) \neq 0$ for all i.

Remark. Given $\mathcal{F}$, how do we find $\mathcal{G}$? This is in general a hard problem. So we make an important simplifying assumption: assume X is smooth. Then general theory says the fixed loci of a reductive group is still smooth, so X^T is still smooth. Hence we also get a pullback map $i^*\colon K_T(X) \to K_T(X^T)$. So we can ask: what is $i^* i_* \mathcal{G}$? Take $X = \mathbb{C} \supset \mathbb{C}^* = T$ with the weight of the action denoted q. Then $i_*\mathcal{O}_{X^T} = \mathcal{O}_0$. The resolution of $\mathcal{O}_0$ is $0 \to x\mathcal{O} \to \mathcal{O} \to \mathcal{O}_0 \to 0$, where the first term has weight q^{-1} and the second has weight q^0. Hence Tor_0 has character 1, and Tor_1 has character q^{-1}. It follows that $i^* i_* \mathcal{O}_{X^T} = (1-q^{-1})\mathcal{O}_{X^T}$. In general, if $w_1, \ldots, w_n$ are non-trivial weights of the torus, and $X = \mathbb{C}^n$, then the Koszul resolution gives

$$i^* i_* \mathcal{O}_{X^T} = \prod (1 - w_i^{-1}) \mathcal{O}_{X^T}.$$

In general, if $Y \subset X$ is some smooth sub-variety, then

$$\mathrm{Tor}^i_X(\mathcal{O}_Y, \mathcal{O}_Y) = \bigwedge^i \mathcal{N}^\vee_{X/Y}.$$

Hence for a general locally free sheaf $\mathcal{G}$ on Y, then

$$\mathrm{Tor}^i_X(\mathcal{G}, \mathcal{O}_Y) = \mathcal{G} \otimes \bigwedge^i \mathcal{N}^\vee_{X/Y}.$$

Hence we can compute

$$i^* i_* \mathcal{G} = \mathcal{G} \otimes \sum (-1)^i \bigwedge^i \mathcal{N}^\vee_{X/Y}$$

where all operations (in particular the tensor) is in K-theory.

Lemma 3.2.5. *The element* $\bigwedge_{-} \mathcal{N}^\vee_{X/X^T} \coloneqq \sum (-1)^i \bigwedge^i \mathcal{N}^\vee_{X/X^T}$ *is invertible in* $K_T(X^T)[1/(1-w_i^{-1})]$.

Proof sketch. Decompose the normal bundle $\mathcal{N}^\vee_{X/X^T}$ according to the characters: $\mathcal{N}^\vee_{X/X^T} = \bigoplus_{w_i} w_i \mathcal{N}_i$ where w_i are characters and $\mathcal{N}_i$ are non-equivariant bundles. So if we can invert this when all the $\mathcal{N}_i$ are trivial, then we are done. But we adjoined the inverse of $(1 - w_i^{-1})$ already. □

Corollary 3.2.6. *For* X *smooth,* $\mathcal{F} = i_* \left(\frac{i^* \mathcal{F}}{\bigwedge_{-} \mathcal{N}^\vee_{X/X^T}} \right)$.

Example 3.2.7. Let $X \coloneqq \mathrm{GL}(2)/B \cong \mathbb{P}^1$. Let $g \coloneqq \mathrm{diag}(x_1, x_2)$. On the fixed points $[1:0]$ and $[0:1]$, it has weights x_2/x_1 and x_1/x_2 respectively. In general, if we look at $B \in G/B$, then $T_B(G/B) = \mathfrak{n}_-$ and $(G/B)^T = wB$ where $w \in W$, the Weyl group. If we look at $\mathcal{F} \coloneqq \mathcal{O}(d)$ on $\mathbb{P}^1$, then

$$\mathcal{F}|_{[1:0]} = \text{polynomials of degree } d \text{ on the line through } [1:0]$$

so the element g does nothing on the second variable and has weight x_1^{-d} on the first variable. Similarly, $\mathcal{F}|_{[0:1]}$ has weight x_2^{-d}. Hence

$$\chi(\mathcal{F}) = \frac{x_1^{-d}}{1 - x_1/x_2} + \frac{x_2^{-d}}{1 - x_2/x_1} = \frac{x_1^{-d} - x_2^{-d}(x_1/x_2)}{1 - x_1/x_2} = x_1^{-d} + x_1^{-d+1} x_2^{-1} + \cdots + x_2^{-d}.$$

In general, $\chi(G/B, \mathcal{O}(\lambda))$ computed via localization gives precisely the Weyl character formula:

$$\chi(G/B, \mathcal{O}(\lambda)) = \sum_{w \in W} w \frac{x^\lambda}{\prod_{\alpha < 0} (1 - x^\alpha)}.$$

3.3 Borel–Weil–Bott

Theorem 3.3.1 (Borel–Weil–Bott)**.** *Let* $\mathcal{O}(\lambda)$ *be the line bundle on the flag variety* G/B *corresponding to* λ *in the weight lattice.*

1. *If there exists* k *such that* $\langle \lambda, \alpha_k \rangle = -1$*, then* $H^i(G/B, \mathcal{O}(\lambda)) = 0$ *for all* i*.*
2. *Otherwise let* w *be such that* $w \star \lambda \coloneqq w(\lambda + \rho) - \rho$ *is in the dominant cone, then* $H^{\ell(w)}(G/B, \mathcal{O}(\lambda)) = V^{w \star \lambda}$*, the irreducible highest weight module of weight* $w \star \lambda$*, and all other cohomologies vanish.*

Remark. This is a consequence of Serre duality for $\mathbb{P}^1$. Recall what Serre duality says in general: given two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a smooth X, we have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^{\dim X - i}(\mathcal{G}, \mathcal{F} \otimes K_X)^\vee$. In particular, for $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{G}$ locally free, we get cohomology: $H^i(\mathcal{G}) = H^i(\mathcal{G}^\vee \otimes K_X)^\vee$. For $\mathbb{P}^1$, this says $H^i(\mathcal{O}(m)) = H^{1-i}(\mathcal{O}(-m-2))^\vee$.

Proof. Let α be a simple root. Corresponding to that root we have a map $\mathfrak{sl}(2)_\alpha \to \mathfrak{g}$ and $\mathrm{SL}(2)_\alpha \to G$, and a corresponding parabolic subgroup $P \coloneqq \langle \mathrm{SL}(2)_\alpha, B \rangle$. Then there is a natural map $\pi \colon G/B \to G/P$ with fiber $P/B \cong \mathbb{P}^1$. The restrictions of $O(\lambda)$ on the fibers of this map are $\mathcal{O}(\langle \lambda, \alpha \rangle)$, because we are restricting the character λ to $\mathrm{SL}(2)_\alpha$. Note that $\pi_* \mathcal{O}(\lambda) \in D^b \mathsf{Coh}(G/P)$. This is a sheaf (cohomology only in degree 0) on G/P_i if $\langle \lambda, \alpha \rangle > -1$, and is a sheaf shifted by one (cohomology only in degree -1) if $\langle \lambda, \alpha \rangle < -1$. In particular, because Serre duality holds in families, if $\langle \lambda, \alpha \rangle < -1$ then $\pi_* \mathcal{O}(\lambda) = \pi_* \mathcal{O}(s \star \lambda)[1]$ where $s \in W$ is the reflection corresponding to α. Hence $H^*(G/B, \mathcal{O}(\lambda)) = H^*(G/B, \mathcal{O}(w \star \lambda))[\ell(w)]$. □

Example 3.3.2. Take the Grassmannian $\mathrm{Gr}(k, n)$. Recall that it embeds into $\mathbb{P}(\wedge^k \mathbb{C}^n)$ via the Plücker embedding, by mapping $L \subset \mathbb{C}^n$ of dimension k to $v \coloneqq \mathbb{C} \wedge^k L$. Choose a basis so that $L = \langle e_1, \ldots, e_k \rangle$. Then $\mathrm{Stab}(L) = \mathrm{Stab}(v)$, which consists of all matrices of the form

$$P = \begin{pmatrix} k \times k & * \\ 0 & (n-k) \times (n-k) \end{pmatrix} \subset G = \mathrm{GL}(n).$$

Let $\wedge^k\mathbb{C}^n = V^{(1,1,\dots,1,0,0,\dots,0)} \ni v$ be the unique highest vector. Note that $\alpha_i = (0,\dots,1,-1,\dots,0)$. We will compute the degree of Gr in this embedding by computing

$$\chi(X \otimes \mathcal{O}(m)) = \chi(G/P, \mathcal{O}(m,m,\dots,m,0,0,\dots,0)),$$

because on $\mathbb{P}(\wedge^k\mathbb{C}^n)$ we have $\mathcal{O}(1) = \mathcal{O}(1,1,\dots,1,0,0,\dots,0)$. But this is precisely $\dim V^{(m,m,\dots,m,0,0,\dots,0)}$. The Weyl denominator formula says

$$\dim V^{(m,m,\dots,m,0,0,\dots,0)} = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i} = \prod_{1\le i\le k}\prod_{k<j\le n} \frac{m+j-i}{j-i} = \frac{m^{k(n-k)}}{\prod_{i\le k}\prod_{k<j}(j-i)} + \cdots .$$

So we have computed the degree.

3.4 Hecke algebras

Let G be a complex linear algebraic group with Weyl group W. We will deform W into a finite Hecke algebra. Let $T \subset B \subset G$ be the maximal torus and Borel subgroup.

Definition 3.4.1. Define the **finite Hecke algebra** H_W to be $\mathbb{Z}[q,q^{-1}][T_w : w \in W]$ mod the relations $\{(T_s+1)(T_s-q) = 0\}$ if s is a simple reflection, and $T_yT_w = T_{yw}$ if $\ell(y)+\ell(w) = \ell(yw)$.

Remark. The relation $(T_s+1)(T_s-q)$ is a deformation of the relation $T_s^2 = 1$ to make the two eigenvalues different. In principle we can introduce multiple variables q, but one suffices for our purposes.

Remark. Hecke algebras appear in two important contexts:

1. the equivariant K-theory of Steinberg varieties (with the character lattice);
2. Iwahori–Hecke algebras for groups over non-archimedean fields (with the co-character lattice).

These two situations are related by Langlands duality. Let $G = \mathbb{G}(\mathbb{F}_q)$ be a finite group with a Borel $B = \mathbb{B}(\mathbb{F}_q)$. Let V be an irreducible complex representation of G. Then V^B is an irreducible representation of $\mathbb{C}(B\backslash G/B)$. So if we want to study modules with a B-fixed vector, we can also study modules over $\mathbb{C}(B\backslash G/B)$. What is this algebra? Compute

$$\dim \mathbb{C}(B\backslash G/B) = \#\{B\text{-orbits on } G/B\}$$

using the Bruhat decomposition $\mathbb{G}/\mathbb{B} = \bigcup_{w\in W_G} \mathbb{B}w\mathbb{B}/\mathbb{B}$, with each cell isomorphic to $\mathbb{A}^{\ell(w)}$. Let $\{s_i\}$ be the simple reflections, and define

$$T_i \coloneqq (1/|B|)1_{Bs_iB} \subset {}^B\mathbb{C}(G)^B,$$

which is just $1_{Bs_iB} \in {}^B\mathbb{C}(G/B)$. In general, define $T_w \coloneqq 1_{BwB} \in {}^B\mathbb{C}(G/B)$. Clearly if $\ell(w_1w_2) = \ell(w_1)+\ell(w_2)$, then $T_{w_1}T_{w_2} = T_{w_1w_2}$. To compute T_i^2, we do it in SL_2. It is some linear combination of T_i and 1. But

$$B\begin{pmatrix}0&-1\\1&0\end{pmatrix}B\begin{pmatrix}0&-1\\1&0\end{pmatrix}B = B\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&*\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}B = \begin{cases} B & *=0 \\ B\begin{pmatrix}0&-1\\1&0\end{pmatrix}B & *\ne 0.\end{cases}$$

Hence $T_i^2 = (q-1)T_i + 1$. Hence we have recovered the finite Hecke algebra.

Definition 3.4.2. Let P be the weight lattice of G. Then $K^T(\{\mathrm{pt}\}) = R(T) = \mathbb{Z}[P]$, the representation ring of T. The **affine Hecke algebra** $\mathbb{H}$ is $\mathbb{Z}[q,q^{-1}][e^\lambda T_w : w \in W,\ \lambda \in P]$ mod the relations:

1. $\{T_w\}$ generate a sub-algebra isomorphic to H_w;

2. $\{e^\lambda\}$ form a $\mathbb{Z}[q,q^{-1}]$-subalgebra isomorphic to $R(T)[q,q^{-1}]$;
3. for $s = s_\alpha$ with $\langle \lambda, \alpha^\vee\rangle = 0$, then $T_s e^\lambda = e^\lambda T_s$;
4. for $s = s_\alpha$ with $\langle \lambda, \alpha^\vee\rangle = 1$, then $T_s e^{s(\lambda)} T_s = q e^\lambda$.

Take a split reductive group G with maximal torus A, and let $\widehat{G} \coloneqq G(F)$ where $F \coloneqq \mathbb{C}((t))$. It is a bad idea to look at $A(F)$ and $B(F)$. Instead, we look at the torus $T \coloneqq A \times \mathbb{C}^\times_{\text{loop}}$. With respect to this torus,

$$\begin{aligned} \operatorname{Lie} \widehat{G} &= \operatorname{Lie}(T) \oplus \mathfrak{n}_+^{\text{aff}} \oplus \mathfrak{n}_-^{\text{aff}} \\ \mathfrak{n}_+^{\text{aff}} &\coloneqq \mathfrak{n}_+ \oplus \mathfrak{g} \otimes t\mathbb{C}[[t]] \\ \mathfrak{n}_-^{\text{aff}} &\coloneqq \mathfrak{n}_- \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[[t^{-1}]]. \end{aligned}$$

Let $\mathcal{O} \coloneqq \mathbb{C}[[t]]$. Define $I \coloneqq \{g(t) \in G(\mathcal{O}) : g(0) \in B\}$, called the **Iwahori subgroup**. The affine Hecke algebra is then $\mathbb{C}_0^\infty(I\backslash \widehat{G}/I)$, i.e. smooth compactly supported (non-zero in only finitely many double cosets). (We have to take F to be a local field, e.g. $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$, but for us this does not matter.) By the "same" argument as before,

$$I\backslash \widehat{G}/I \leftrightarrow (\widehat{G}/I)^T = \widehat{W} \coloneqq N(T)/T.$$

This group $\widehat{W}$ is the semidirect product $W_G \ltimes \operatorname{cochar}(A)$, where $\operatorname{cochar}(A) \coloneqq \operatorname{Hom}(\mathrm{GL}(1), A)$ is the group of co-characters of G. It is clear that both W_G and $\operatorname{cochar}(A)$ are in $\widehat{W}$. Take $g(t) \in \widehat{G}$, and let $r_z \in R$ be rotation by z. Then $g(t) r_z g(t)^{-1} \in T$. So $g(t)g(zt)^{-1} r_z \in T$. Hence $g(t)g(zt)^{-1} \in A$ and in particular does not depend on t. Define $a(t) \coloneqq g(t)g(1)^{-1}$. Then $a(tz) = a(t)a(z)$, i.e. $a(t) \in \operatorname{cochar}(A)$. The element $g(1)$ normalizes A, i.e. $g(1)Ag(1)^{-1} \subset A$.

The analogue of Bruhat decomposition involves the Iwahori subgroup I instead of the Borel B. (Note that $\operatorname{Lie} I = \mathfrak{b}_+ \oplus \mathfrak{g} \cdot t\Bbbk[[t]]$.) The decomposition is now

$$\widehat{G} = \bigsqcup_{w \in \widehat{W}} IwI.$$

The group $\widehat{G}$ has a Haar measure μ with respect to which I has finite measure. In the field F, we have the units $\mathcal{O} = \Bbbk[[t]]$. The subgroups $t^k\mathcal{O}$ form a basis for the topology, and we can set $\mu(\mathcal{O}) = 1$ and $\mu(t^k\mathcal{O}) = 1/q^k$ since $\mathcal{O}$ consists of q^k cosets of $t^k\mathcal{O}$. Analogously, $G(\mathcal{O}) \subset \widehat{G}$ is a compact subgroup, (warning: not all maximal compact subgroups are conjugate), and we can normalize $\mu(G(\mathcal{O})) = 1$. Given a subgroup $\Gamma \subset G(\mathcal{O})$, the measure $\mu(\Gamma)$ is therefore $1/|G(\mathcal{O})/\Gamma|$. In particular, $\mu(I) = 1/|G(\mathcal{O})/I|$. But we can view $G(\mathcal{O})$ as matrices of polynomials in t, and I as matrices whose upper-triangular block is restricted to polynomials in k. Hence $G(\mathcal{O})/I = G(\Bbbk)/B(\Bbbk)$.

There is a multiplication map $\widehat{G} \times \widehat{G} \to \widehat{G}$. The product of two finite-measure sets is still finite-measure. Hence convolution on $C_0^\infty(I\backslash G/I)$ is well-defined, and it becomes an algebra. Note that there is a map

$$I\backslash G(\mathcal{O})/I \xrightarrow{\text{mod } t} B(\Bbbk)\backslash G(\Bbbk)/B(\Bbbk),$$

which induces a map $H_q \to \mathbb{H}_q$ of functions on these spaces (from the finite Hecke algebra to the affine Hecke algebra).

For convenience, normalize $\mu(I) = 1$. If $s_i \in W$ is a simple reflection, the corresponding parabolic $P_i = Bs_iB \sqcup B$

3.5 Convolution

Example 3.5.1. Consider the equivariant K-theory of $K^{G\times\mathbb{C}^\times}(T^*(G/B))$, where the cotangent bundle has an induced B-action from the B-action on G/B. The torus $\mathbb{C}^\times$ acts only on cotangent fibers, i.e. it has trivial action on the zero section. So there is a Thom isomorphism $K^{G\times\mathbb{C}^\times}(T^*(G/B)) \cong K^{G\times\mathbb{C}^\times}(G/B)$. By equivariant descent,

$$K^{G\times\mathbb{C}^\times}(G/B) \cong K^{B\times\mathbb{C}^\times}(\{\text{pt}\}) = K^B(\{\text{pt}\})[q, q^{-1}].$$

(Note that $K^{\mathbb{C}^\times}(\{\text{pt}\}) = R(\mathbb{C}^\times) = \mathbb{Z}[q, q^{-1}]$ where $q\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the standard representation given by $z \mapsto z$.) We can mod out by the unipotent radical to get

$$K^B(\{\text{pt}\})[q, q^{-1}] = K^T(\{\text{pt}\})[q, q^{-1}] = \mathbb{Z}[P][q, q^{-1}].$$

Theorem 3.5.2 (Lusztig, "Equivariant K-theory and representations of Hecke algebras")**.** *There is an action of the affine Hecke algebra $\mathbb{H}$ on $\mathbb{Z}[P][q, q^{-1}]$ given by*

$$T_{s_\alpha} e^\lambda \mapsto \frac{e^\lambda - e^{s_\alpha \lambda}}{e^\alpha - 1} - q\frac{e^\lambda - e^{s_\alpha\lambda+\alpha}}{e^\alpha - 1}$$

and e^λ acts by multiplication by $e^{-\lambda}$.

The goal is to geometrize this action, i.e. give a geometric interpretation of this affine Hecke algebra $\mathbb{H}$, and to construct the above action. (Using this action, we can classify all the irreducible representations of the affine Hecke algebra; this is the Deligne–Langlands conjecture, proved by Kazhdan–Lusztig and Ginzburg.)

Definition 3.5.3. We want to define **convolution in equivariant K-theory**. Toy model: if we take M_1, M_2, M_3 to be finite sets, we get a map

$$\begin{aligned} &\mathbb{C}(M_1 \times M_2) \times \mathbb{C}(M_2 \times M_3) \to \mathbb{C}(M_1 \times M_3) \\ &\qquad (f_{12}, f_{23}) \mapsto f_{12} * f_{23} \colon (m_1, m_3) \mapsto \sum_{m_2 \in M_2} f_{12}(m_1, m_2) f_{23}(m_2, m_3). \end{aligned}$$

Note that for the sum to be well-defined, we require M_2 finite. Now replace replace M_1, M_2, M_3 by smooth quasi-projective G-varieties. Denote the projections by $p_{ij}\colon M_1 \times M_2 \times M_3 \to M_i \times M_j$. Suppose $Z_{12} \subset M_{12}$ and $Z_{23} \subset M_{23}$ are G-invariant sub-varieties such that

$$p_{13}\colon p_{12}^{-1}(Z_{12} \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is a proper map, so that pushforward p_{13*} in equivariant K-theory is well-defined. (This is the analogue of requiring M_2 to be finite.) Denote the image of p_{13} here to be $Z_{12} \circ Z_{23}$. Then we have a map

$$\begin{aligned} &K^G(Z_{12} \otimes K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23}) \\ &\qquad (\mathcal{F}_{12}, \mathcal{F}_{23}) \mapsto \mathcal{F}_{12} * \mathcal{F}_{23} \coloneqq p_{13*}(p_{12}^*\mathcal{F}_{12} \otimes p_{23}^*\mathcal{F}_{23}). \end{aligned}$$

Remark. Note that whenever we have a cohomology theory with proper pushforwards and pushbacks, we can always define such a convolution operation.

Example 3.5.4. If M_i are finite sets and $G \coloneqq \{1\}$ is the trivial group, then $K^G(M_i) = \mathbb{Z}[M_i]$ is just functions on the varieties, where the isomorphism is given by $\mathcal{F} \mapsto (f_\mathcal{F}\colon m_i \mapsto \dim \mathcal{F}|_{m_i})$. Hence convolution becomes

$$\begin{aligned} K^G(Z_{12}) \otimes K^G(Z_{23}) &\xrightarrow{*} K^G(Z_{12} \circ Z_{23}) \\ \mathbb{Z}[Z_{12}] \otimes \mathbb{Z}[Z_{23}] &\xrightarrow{\text{toy model}} \mathbb{Z}[Z_{12} \circ Z_{23}]. \end{aligned}$$

Example 3.5.5. Let $M_1 = M_2 = M$ and $M_3 = \{\text{pt}\}$, and let $Z_{12} \subset M \times M$ and $Z_{23} = M \times \{\text{pt}\}$. Then $Z_{12} \circ Z_{23} \hookrightarrow M$ embeds, and the construction gives the following map:

$$K^G(Z_{12}) \otimes K^G(M) \to K^G(Z_{12} \circ Z_{23}) \to K^G(M).$$

So in particular we get a map $K^G(Z_{12}) \to \operatorname{End}_{K^G(\{\text{pt}\})} K^G(M)$. From this construction we see that we can get the equivariant K-theory of one variety to act on the equivariant K-theory of some other variety.

Example 3.5.6. Let $M_1 = M_2 = M_3 = M$, and $Z_{12} = Z_{23} = Z_{12} \circ Z_{23} = Z$. Then from the construction we get

$$K^G(Z) \otimes K^G(Z) \to K^G(Z),$$

which puts an algebra structure on $K^G(Z)$.

Theorem 3.5.7. *The product in the preceding example is the same as the tensor product on $K^G(Z)$.*

Proof. Omitted. (Just some diagram-chasing using the projection formula.) □

Definition 3.5.8. Consider the cotangent bundle $T^*(G/B)$. It is a symplectic variety and has a map to the **nilpotent cone** $\mathcal{N} \subset \mathfrak{g}^* \cong \mathfrak{g}$ consisting of all the nilpotent elements in the Lie algebra. This map arises from the moment map $\mu\colon T^*(G/B) \to \mathfrak{g}^*$. (One can show the image lies in $\mathcal{N}$.) This is a resolution of singularities called the **Springer resolution**.

Example 3.5.9. If $G = \mathrm{SL}(2, \mathbb{C})$, the flag variety is $G/B \cong \mathbb{P}^1$, and so

$$T^*\mathbb{P}^1 \to \mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x^2 + yz = 0 \right\}.$$

Definition 3.5.10. Form $Z \coloneqq T^*(G/B) \times_\mu T^*(G/B)$, called the **Steinberg variety**. Since $Z \circ Z = Z$ (which can be checked very easily), we are in the case $M_1 = M_2 = M_3 = M$ and $Z_1 = Z_2 = Z_3 = Z$. Hence $K^{G\times\mathbb{C}^*}(Z)$ has an algebra structure, and it acts on $K^{G\times\mathbb{C}^*}(G/B)$ (by the second example).

Theorem 3.5.11 (Kazhdan–Lusztig, Ginzburg)**.** $K^{G\times\mathbb{C}^*}(Z) \cong \mathbb{H}$*, the affine Hecke algebra.*

Proof. We will see the proof in the most simple case, when $G = \mathrm{SL}(2, \mathbb{C})$. Then

$$\mathbb{H} = \mathbb{C}[q, q^{-1}][T, X, X^{-1}]/\{(T + 1)(T - q) = 0,\ XX^{-1} = X^{-1}X = 1,\ TX^{-1} - XT = (1 - q)X\}$$

Change notation: let $c \coloneqq -(T + 1)$, so that $c^2 = -(q + 1)c$ and $cX^{-1} - Xc = qX - X^{-1}$. In this case, the Steinberg variety is $Z = T^*\mathbb{P}^1 \times_\mu T^*\mathbb{P}^1 = Z_1 \cup Z_2$, where $Z_1 \coloneqq \Delta(T^*\mathbb{P}^1)$ and $Z_2 \coloneqq \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{pr}_1\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be projection onto the first factor. Let $Q \coloneqq \Omega_{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1}$ be the sheaf of relative differentials, and $\pi_\Delta\colon Z_1 \to \mathbb{P}^1_\Delta$, the diagonal. Write $\mathcal{O}_n \coloneqq \pi_\Delta^*\mathcal{O}(n)$. Define the map

$$\begin{gathered} \theta\colon \{c, X, X^{-1}\} \to K^{G\times\mathbb{C}^*}(Z) \\ X \mapsto [\mathcal{O}_{-1}], \quad X^{-1} \mapsto [\mathcal{O}_1], \quad c \mapsto [qQ]. \end{gathered}$$

Claim: θ can be extended to an algebra homomorphism, i.e. we have to check that the following relations hold:

$$[qQ] * [qQ] = -(q+1)[qQ], \quad [qQ] * [\mathcal{O}_1] - [\mathcal{O}_{-1}] * [qQ] = [q\mathcal{O}_{-1}] - [\mathcal{O}_1], \quad [\mathcal{O}_1] * [\mathcal{O}_{-1}] = [\mathcal{O}_{-1}] * [\mathcal{O}_1] = [\mathcal{O}_0].$$

The last relation is obvious, so we have check the first two. This is just a huge calculation. We first establish a few formulas to help us with the calculation.

First note that if M_1, M_2, M_3 are smooth G-varieties with $Y', Y'' \subset M_2$ are G-stable with $Y' \cap Y''$ is proper, then $(M_1 \times Y') \circ (Y'' \times M_3) = M_1 \times M_3$. So we get the map

$$K^G(M_1 \times Y) \otimes K^G(Y'', M_3) \to K^G(M_1 \times M_3)$$

Take elements $\mathcal{F}_1 \in K^G(M_1)$ and $\mathcal{F}_3 \in K^G(M_3)$ and $\mathcal{G}' \in K^G(Y')$ and $\mathcal{G}'' \in K^G(Y'')$. Let $p\colon Y' \cap Y'' \to \{\mathrm{pt}\}$. Fact:

$$(\mathcal{F}_1 \boxtimes \mathcal{G}') * (\mathcal{G}'' \boxtimes \mathcal{F}_3) = \langle \mathcal{G}', \mathcal{G}'' \rangle (\mathcal{F}_1 \boxtimes \mathcal{F}_3), \quad \langle \mathcal{G}', \mathcal{G}'' \rangle \coloneqq p_*(\mathcal{G}' \otimes \mathcal{G}'').$$

Also, note that for all $n \in \mathbb{Z}$, we have $p_*\mathcal{O}(n) = (X^{n+1} - X^{-(n+1)})/(X - X^{-1})$ in $R(T)^W = K^G(\{\mathrm{pt}\})$. This is because $p_*\mathcal{O}(n) = \chi(\mathbb{P}^1, \mathcal{O}(n))$.

Let's begin the calculation. Start with the first relation $qQ * qQ = -(q+1)qQ$. Write $Q = \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}$. Let $\pi\colon (V := T^*\mathbb{P}^1) \to \mathbb{P}^1$ with inclusion of the zero section $i\colon \mathbb{P}^1 \to V$. Then we get a Koszul complex

$$0 \to \pi^* \wedge^1 V^\vee \to \mathcal{O}_V \to i_*\mathcal{O}_{\mathbb{P}^1} \to 0.$$

Tensoring with $\pi^*\Omega_{\mathbb{P}^1}$ and using the projection formula gives

$$0 \to \mathcal{O}_{T^*\mathbb{P}^1} \to \pi^*\Omega_{\mathbb{P}^1} \to i_*\Omega_{\mathbb{P}^1} \to 0.$$

Now apply $\mathcal{O}_{\mathbb{P}^1} \boxtimes -$ to get

$$0 \to \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \boxtimes i_*\Omega_{\mathbb{P}^1} \to 0.$$

These now live on the Steinberg variety Z. However, the differential δ of the Koszul complex drops degree by 1, and therefore it cannot be $\mathbb{C}^*$-equivariant. To restore $\mathbb{C}^*$-equivariance, Hence we multiply by q in the first term, to account for the degree shift:

$$0 \to q^{-1}\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \boxtimes i_*\Omega_{\mathbb{P}^1} \to 0$$

is now $\mathbb{C}^*$-equivariant. Hence in K-theory, we get

$$qQ = q(\mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega_{\mathbb{P}^1}) - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1}.$$

Using the two formulas stated earlier (for the convolution of a tensor and for p_*),

$$\begin{aligned} qQ * qQ &= (q(\mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega_{\mathbb{P}^1}) - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1}) * (q\mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega_{\mathbb{P}^1}) \\ &= q^2\langle \pi^*\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}\rangle(\mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega_{\mathbb{P}^1}) - \langle \mathcal{O}_{T^*\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}\rangle(q\mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega_{\mathbb{P}^1}) \\ &= q(p_*\Omega_{\mathbb{P}^1})qQ - (p_*\mathcal{O}_{\mathbb{P}^1})qQ \\ &= q^2Q - qQ = -(q+1)qQ. \end{aligned}$$

Now we show the second equation $qQ * \mathcal{O}_1 - \mathcal{O}_{-1} * qQ = q\mathcal{O}_{-1} - \mathcal{O}_1$. We transform the computation to $\mathbb{P}^1 \times \mathbb{P}^1$, using the Künneth formula $K^G(\mathbb{P}^1 \times \mathbb{P}^1) \cong K^G(\mathbb{P}^1) \otimes K^G(\mathbb{P}^1)$, using

$$Z \xrightarrow{\bar{\pi}\mathbb{P}^1 \times T^*\mathbb{P}^1} \xleftarrow{\bar{i}} \mathbb{P}^1 \times \mathbb{P}^1$$

where $\bar{\pi} := (\pi, \mathrm{id})$ and $\bar{i} := (\mathrm{id}, i)$. Fact: $\Phi := \bar{i}^*\bar{\pi}_*\colon K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(\mathbb{P}^1 \times \mathbb{P}^1)$ is an injective algebra homomorphism. It is therefore enough to check the equation after applying this map. Compute:

$$\Phi(qQ) = q\mathcal{O} \boxtimes \mathcal{O}(-2) - \mathcal{O} \boxtimes \mathcal{O}.$$

Using this formula, the computation of $\Phi(qQ) * \Phi(\mathcal{O}_1) - \Phi(\mathcal{O}_{-1}) * \Phi(qQ)$ is easy once we get $\Phi(\mathcal{O}_1)$ using the **Beilinson resolution**

$$0 \to \mathcal{O}(-1) \boxtimes \Omega^1_{\mathbb{P}^1}(1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

on $\mathbb{P}^1 \times \mathbb{P}^1$. Tensoring this sequence by $\mathcal{O}(-1)$, we get a formula for $\Phi(\mathcal{O}_1)$. Then the remaining computation is straightforward; we omit it. □

Theorem 3.5.12. *The algebra homomorphism $\theta\colon \mathbb{H} \to K^{G\times\mathbb{C}^\times}(Z)$ is actually an isomorphism.*

3.6 Difference operators

Let Λ' be the lattice given by the kernel of the differential in $\widehat{G} = G(\Bbbk((t)))$. Let Λ be a larger lattice that normalizes $\widehat{W}$, the affine Weyl group. Then $\widehat{W} \supset \Pi \cong \Lambda/\Lambda'$, where Π consists of elements of length 0. Consider $\mathbb{Z}\Lambda$, the group algebra of Λ, with symbols e^λ for $\lambda \in \Lambda$. Define

$$T_i \coloneqq qs_i + \frac{q-1}{1-e^{-\alpha_i^\vee}}(1-s_i),$$

which acts on $\mathbb{Z}\Lambda$. We see that e^λ maps to some stuff with a factor $(e^\lambda - e^{s_i\cdot\lambda})/(1-e^{-\alpha_i^\vee})$. Note that $\alpha_i(\lambda)$ is an integer, so

$$\frac{e^\lambda - e^{s_i\cdot\lambda}}{1-e^{-\alpha_i^\vee}} = e^\lambda \frac{1-e^{-\alpha_i(\lambda)\alpha_i^\vee}}{1-e^{-\alpha_i^\vee}} = 1 + e^{-\alpha_i^\vee} + \cdots + e^{-(\alpha_i(\lambda)-1)\alpha_i^\vee}$$

since $s_i(x) = x - \alpha_i(x)\alpha_i^\vee$. The operators T_i form a faithful representation of $\mathbb{H}$, because it is faithful for $q = 1$, and therefore faithful for any q, and we can check that they satisfy the appropriate relations.

Theorem 3.6.1. *These operators T_i satisfy the relations of a Hecke algebra:*

$$(T_i - q)(T_i + 1) = 0, \quad T_iT_jT_i\cdots = T_jT_iT_j\cdots, \quad \pi T_i \pi^{-1} = T_{\pi(i)} \quad \ell(\pi) = 0$$

where $\pi \in \Lambda/($coroot lattice$)$ is an element of length 0.

View H as $\mathbb{B}/((T_i-q)(T_i+1) = 0)$ where $\mathbb{B}$ is the braid group. The relation $T_iT_jT_i\cdots = T_jT_iT_j\cdots$ means that for any $w \in \widehat{W}$, we can define $T(w) \subset B$ such that $T(w_1)T(w_2) = T(w_1w_2)$ if $\ell(w_1)+\ell(w_2) = \ell(w_1w_2)$. If we translate by λ in the positive cone, we see $\ell(\lambda)$ is linear in λ, and $T(\lambda_1)T(\lambda_2) = T(\lambda_1+\lambda_2)$. For any $\mu \in \Lambda$, we define $Y^\mu \coloneqq T(\lambda_1)T(\lambda_2)^{-1}$ where $\mu = \lambda_1 - \lambda_2$ with $\lambda_1, \lambda_2 > 0$. This is well-defined and commutative. So the braid group $\mathbb{B}$ already contains a lattice. By enlarging Λ if necessary, we may assume $\mu \in \Lambda$ where $\alpha_i(\mu_j) = \delta_{ij}$. We want to compute commutation relations between $Y_i \coloneqq Y^{\mu_i}$ and simple reflections $s_1, \ldots, s_n$ through 0.

Proposition 3.6.2. *$T_iY_j = Y_jT_i$ for $i \neq j$, and $Y_i = T_iY^{s_i(\mu_i)}T_i$.*

Proof. The assumption was that $\alpha_i(\mu_j) = 0$, so μ_j translates parallel to the hyperplane defined by s_i. Hence both $T_iY_j = T_iT(\mu_j)$ and $Y_jT_i = T(\mu_j)T_i$ are both $T(s_i\mu_j)$, and are reduced of length $\ell(\mu_j)+1$.

Set $\eta \coloneqq \mu_i + s_i(\mu_i) = 2\mu_i - \alpha_i^\vee$, so that $\ell(\eta) = 2\ell(\mu_i) - 2$. Compute $\alpha_j(\eta) = 2\delta_{ij} - \alpha_j(\alpha_i^\vee)$. The second term is either 2 or non-positive, so $\alpha_j(\eta) \geq 0$. In $\widehat{W}$, we have $s_i\eta = \mu_is_i\mu_i$. Let $\ell \coloneqq \ell(\mu_i)$, so that the length of the lhs is $1 + 2\ell - 2$, and the length of the rhs is $\ell - 1 + \ell$, and both are reduced decompositions. Hence $T_iT(\eta) = Y_iT_i^{-1}Y_i$, by noting that $Y_i = T(\mu_is_i)T_i$. Hence $T_iT(\eta)Y_i^{-1}T_i = Y_i$, as desired. ☐

Proposition 3.6.3 (Lusztig)**.** $Y^\lambda T_i - T_iY^{s_i\lambda} = (q-1)(Y^\lambda - Y^{s_i\lambda})/(1-Y^{-\alpha_i^\vee})$.

Proof. If this formula holds for Y^μ and Y^λ, then it is true for $Y^{\mu+\lambda}$, because

$$Y^{\lambda+\mu}T - TY^{\lambda+\mu} = Y^\lambda(Y^\mu T - TY^{s_i\mu}) + (Y^\lambda T - TY^{s_i\lambda})Y^{s_i\mu} = (q-1)\left(Y^\lambda \frac{Y^\mu - Y^{s_i\mu}}{1-Y^{\alpha_i^\vee}} + \frac{Y^\lambda - Y^{s_i\lambda}}{1-Y^{\alpha_i^\vee}}Y^{s_i\mu}\right).$$

Therefore, it is enough to check for Y_i and T_j. For $i \neq j$, we get $Y_iT_j - T_jY_i = 0$ as desired. For $i = j$, compute

$$Y_iT_j - T_jY_iY^{-\alpha_i^\vee} = (q-1)Y_i\frac{1-Y^{-\alpha_i^\vee}}{1-Y^{-\alpha_i^\vee}}.$$

Rearranging, $Y_i(T_j - (q-1)) = T_jY_i^{-\alpha_i^\vee}$. (We are missing a factor of q, but $T_j - (q-1)$ is supposed to be T_j^{-1}.) ☐

Corollary 3.6.4. *The finite Hecke algebra $H_W = \langle T_1, \ldots, T_n\rangle$ has a finite representation $T_i \mapsto q$. The affine Hecke algebra $\mathbb{H}$ has an induced representation $\mathbb{H}/(T_i - q)_{i=1,\ldots,n} \cong \mathbb{Z}[q^{\pm 1}]\Lambda$ given by $Y^\mu \mapsto (-)\cdot e^\mu$. In particular, $T_i1 = q\cdot 1$.*

3.7 Equivariant K-theory of Steinberg variety

Let G be a reductive group, and $F \coloneqq G/B$ be the flag variety. Let $X \coloneqq T^*F$ and $\mathrm{St} \coloneqq X \times_{X_0} X \subset X \times X$ be the Steinberg variety, where X_0 is the nilpotent cone of $\mathfrak{g}$. Note that $K_G(F) = K_B(\{\mathrm{pt}\}) = K_T(\{\mathrm{pt}\})$. Let $R \coloneqq K_{T\times\mathbb{C}_q^\times}(\{\mathrm{pt}\})$. If $\Delta \subset \mathrm{St}$ is the diagonal, then $R \subset K_{G\times\mathbb{C}_q^\times}(\mathrm{St})$ as $K_{\mathrm{equivariant}}(\Delta)$.

Theorem 3.7.1. $K_{G\times\mathbb{C}_q^\times}(\mathrm{St}) = H_q(\widehat{W})$, *the affine Hecke algebra of* $W_G \ltimes \mathrm{cochar}(T)$.

Note that $K(X \times_{X_0} X) \otimes_{\mathbb{Z}} R = \bigoplus_{x_0\in X_0} \mathrm{Mat}(|f^{-1}(x_0)|, R)$. Explcitly, we have

$$X = \{(\mathfrak{b}, \xi) : \mathfrak{b} \subset \mathfrak{g} \text{ Borel}, \xi \in \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]\},$$

where $\mathfrak{b} \in F$ specifies a point, and $T_{\mathfrak{b}}F = \mathfrak{n}_-$ so that $T^*_{\mathfrak{b}}F = \mathfrak{n}$. There is a map $X \to X_0$ given by projection onto ξ. Note that X is a symplectic variety, and in fact $X \to X_0 \subset \mathfrak{g}^*$ is a moment map. In more generality, if X is (smooth) symplectic and X_0 is affine, and $f\colon X \to X_0$ is projective and birational, then we say X is a **equivariant symplectic resolution** if there is an action of $\mathbb{C}_q^*$ on f that contracts X_0 to a singe point.

Theorem 3.7.2. *For* $X = T^*G/P$ *in general, the Steinberg variety is always isotropic, so* $\dim \mathrm{St} \le \dim X$.

We have an explicit description of the Steinberg variety in the case $X = T^*G/B$:

$$\mathrm{St} = \{(\mathfrak{b}_1, \mathfrak{b}_2, \xi) : \xi \in [\mathfrak{b}_1, \mathfrak{b}_1] \cap [\mathfrak{b}_2, \mathfrak{b}_2]\}.$$

There is a map $\varphi\colon \mathrm{St} \to F \times F$ that forgets ξ but remembers $(\mathfrak{b}_1, \mathfrak{b}_2)$. Observation: the fiber of φ is the conormal to the (diagonal) G-orbit through $(\mathfrak{b}_1, \mathfrak{b}_2)$. But G-orbits on $F \times F$ are equivalent to B-orbits on F, which are in bijection with the Weyl group W_G. For example, the diagonal in $F \times F$ corresponds to $1 \in W_G$.

Example 3.7.3. For $\mathfrak{sl}_2$, the nilpotent cone X_0 consists of matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with determinant 0. So it is a quadric cone. The flag variety is $\mathbb{P}^1$. The blow-up at the singularity is a degree -2 bundle, so $X = \mathcal{O}_{\mathbb{P}^1}(-2) = T^*\mathbb{P}^1$. The Steinberg variety is therefore $\Delta \cup (\mathbb{P}^1 \times \mathbb{P}^1)$.

The closure $\overline{\mathrm{St}}_w$ of an orbit is $\overline{\mathrm{St}}_w = \bigcup_{w' \le w} \mathrm{St}_{w'}$. Hence St has a stratification and we can decompose $K_{\mathrm{eq}}(\mathrm{St}) = \bigoplus_w K_{\mathrm{eq}}(\mathrm{St}_w)$. Note that although each orbit BwB is not *canonically* isomorphic to affine space, the attracting manifolds are canonically isomorphic to a tower of bundles, each of which has affine-linear transition functions. But we know the K-theory of projective bundles. Hence $K(\mathrm{St})$ is a free module over $K(\Delta)$ with basis $\mathcal{O}_{\overline{\mathrm{St}}_w}$ for $w \in W$. (In fact, for dimension reasons, $\mathcal{O}_{\overline{\mathrm{St}}_{s_i}}$ generate.) There is a short exact sequence

$$0 \to K(\mathrm{St}_{<w}) \to K(\mathrm{St}_{\le w}) \to K(\mathrm{St}_w) \to 0.$$

That this sequence terminates on the left like this follows by noting that whatever the kernel is, it has to be smaller than $K(\mathrm{St}_{<w})$. Note that $K(\mathrm{St}_1) = K(\Delta)$, the diagonal, and acts by multiplication by $R \coloneqq K_{\tilde{G}}(F)$ (= Laurent polynomials in $\mathrm{rank}\, G + 1$ variables) where $\tilde{G} \coloneqq G \times \mathrm{GL}(1)$. In $K(\mathrm{St}_{\le s_i})$, we would like to find T_i that acts by $qs_i + (q-1)/(1-e^{-\alpha_i})(1-s_i)$. Write $u \coloneqq e^{-\alpha_i}$. We will realize

$$T_i + 1 = (q - \frac{q-1}{1-u})s_i + (\frac{q-1}{1-u} + 1) = \frac{1-qu}{1-u}s_i + \frac{1-u^{-1}q}{1-u^{-1}} \cdot 1$$

geometrically, because T_i has eigenvalues q and -1 and therefore $T_i + 1$ is a projector. Note that the coefficients are related by $u \mapsto 1/u$, so this is starting to look like a localization formula. Recall that $K_{\tilde{G}}(F) = R$ by the fiber at $B \in G/B$. The $\mathbb{P}^1$ contained in $\overline{\mathrm{St}}_{s_i}$ arises as $\mathrm{SL}(2)_i/B \to B$, which has two fixed points B and s_iB. An element $f \in R$ has tangent weight u at 1 and u^{-1} at s_i. But $\mathbb{P}^1$ sits inside $T^*\mathbb{P}^1$, so the weights in the cotangent direction are qu^{-1} and qu respectively. So we get $\frac{1-1/(qu^{-1})}{1-u^{-1}}$ and $\frac{1-1/(qu)}{1-u}$ respectively, if we put the structure sheaf on $\mathbb{P}^1$. This is not quite what we want, but we are free to put whatever sheaf we want on $\mathbb{P}^1$. In particular we can put the structure sheaf of $T^*\mathbb{P}^1$ itself. This gives extra factors of q/u and qu respectively:

$$\frac{1-1/(qu^{-1})}{1-u^{-1}}\frac{q}{u} = -\frac{1-q/u}{1-u^{-1}}, \quad \frac{1-1/(qu)}{1-u}qu = -\frac{1-qu}{1-u},$$

which are the desired coefficients. This is the geometric realization of $T_i + 1$.

The isomorphism $\mathbb{H}_q(\widehat{W}) \cong K_{\tilde{G}}(\mathrm{St})$ essentially exhibits $\mathbb{H}_q(\widehat{W})$ as a collection of matrix algebras. View this algebra as a deformation of $\mathbb{Z}W \times R$. Both algebras are over $K_{\tilde{G}}(\mathrm{pt}) = R^W$. In any irreducible representation, R^W therefore acts by scalars. For example, take T to be a torus acting on X. Then there is a map $\operatorname{Spec} K_T(X) \to T = \operatorname{Spec} K_T(\mathrm{pt})$. The fiber over a point $g \in T$ is $K(X^g)$, the K-theory of the fixed locus. Recall that if $Y \subset X$ is torus invariant and smooth, and $\{a\}$ are the weights of the normal bundle to X, then the composition

$$K_T(Y) \to K_T(X) \to K_T(Y)$$

is an isomorphism as long as $a_i - 1$ are invertible. (This is just localization.) Hence in the diagram

$$\begin{array}{ccc} \operatorname{Spec} K_T(X) & \longleftarrow & \operatorname{Spec} K(X^g) \\ \downarrow & & \downarrow \\ T & \longleftarrow & g, \end{array}$$

the upper arrow is an isomorphism whenever the corresponding characters on T are invertible.

Example 3.7.4. Let $T = \mathbb{C}^\times \ni t$ act on $X := \mathbb{P}^2 = \mathbb{P}(1, t, t^2)$. Then $K_T(X)$ is generated by $s := \mathcal{O}(1)$. There are three fixed points, and the weights of s at these points are 1, t^{-1}, t^{-2}. So the minimal polynomial of s is

$$(1-s)(1-ts)(1-t^2s) = 0.$$

If $t \neq \pm 1$, then the fiber $\operatorname{Spec} K(X^T)$ is 3 distinct points. If $t = 1$, then we get $(1-s)^3 = 0$, so the fiber is a triple point $\operatorname{Spec} \mathbb{C}[\ln s]/(\ln s)^3$. Finally, if $t = -1$, then we get $(1-s)^2(1+s) = 0$, so the fiber is a double point plus a single point.

This is great, because now we can look at $\mathbb{H}_q(\widehat{W}) \otimes_{\text{center}} \text{field}$. The center is just $\operatorname{Ad}(G)$-invariant functions on $\tilde{G}$. Closed points are closed orbits, which are orbits of semisimple elements. Hence this is just $K(\mathrm{St}^g)$, because we killed all the equivariance. So now we are in the situation of X smooth with a proper morphism to Y, and $K(X \times_Y X)$ acts on $K(X)$ by correspondence, and we would like to know the irreducible modules. Note that the Chern character ch gives a homomorphism $\mathrm{ch}\colon K(X \times_Y X) \otimes \mathbb{Q} \to H_\bullet(X \times_Y X)$.

Example 3.7.5. Suppose $X \to Y$ is finite, and we might as well assume it is surjective. There is a stratification in Y given by the number of points that ramify. If we pick a generic $y \in Y$ with fiber $F := f^{-1}(y)$, then there is a monodromy representation (by going around ramification points). Then $H_{\text{top}}(Y) \otimes \mathbb{C} = \operatorname{End}(\mathbb{C}^F)^{\pi_1}$. We can write $\mathbb{C}^F = \bigoplus_{L_i} L_i \otimes M_i$, where L_i range over irreps of π_1 (i.e. local systems), and M_i is some sort of multiplicity space. Hence $\operatorname{End}(\mathbb{C}^F)^{\pi_1} = \bigoplus_{L_i} \operatorname{End}(M_i)$, since $\dim \operatorname{Hom}(L_i, L_j)^{\pi_1} = \delta_{ij}$.

3.8 Quantum groups and knots

Note that the Hecke algebra $\mathbb{H}_q(\widehat{W})$ is a q-deformation of $\mathbb{Z}[\widehat{W}]$.

Definition 3.8.1. Define $[n]_q := (1-q^n)/(1-q)$. We can view this as either:

1. $|\mathbb{P}^{n-1}(\mathbb{F}_q)|$, where q is the eigenvalue of the Frobenius; equivalently, this is $|\mathbb{A}^n - \{0\}|/|\operatorname{GL}(1)|$;
2. $\operatorname{tr} H^\bullet(\mathbb{P}^1, \mathcal{O}(n-1))$, where q is the weight of $T_0\mathbb{P}^1$, which is the span of $1, 1/x, 1/x^2, \ldots, 1/x^{n-1}$, which has weights $1, q, q^2, \ldots, q^{n-1}$.

Let $C = (a_{ij})$ be a Cartan matrix with Kac–Moody Lie algebra $\mathfrak{g}$. Recall the construction is via generators h_i, e_i, f_i, where

$$[h, e_j] = \alpha_j(h)e_j, \quad [h, f_j] = -\alpha_j(h)f_j, \quad [e_i, f_j = \delta_{ij}h_i.$$

There is a miraculous q-deformation of this construction. We will make a torus out of h. Introduce symbols q^h where $q^{h_1}q^{h_2} = q^{h_1+h_2}$. If we choose a basis h_i, we can define $K_i \coloneqq q^{h_i}$. The appropriate relation on the q^h is obtained by exponentiating the Lie algebra relations above:

$$q^h e_i q^{-h} = q^{\alpha_i(h)}e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)}f_i, \quad [e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

where $q^{\alpha_i(h)}$ is now in our base ring.

Remark. Why choose the term $(q^{h_i} - q^{-h_i})/(q - q^{-1})$ instead of, say, $(q^{d_i h_i} - q^{-d_i h_i})/(q^{d_i} - q^{-d_i})$ for some d_i? The answer is: choose d_i so that $(d_i\alpha_j(h_i))$ is symmetric.

Do we really need the denominator $q - q^{-1}$? We can absorb it into either e_i or f_i. Then note that h_i does not appear anywhere else, so we can rescale h_i to absorb d_i. Hence we are left with

$$K_i e_j K_i^{-1} = q^{\alpha_j(h_i)}e_j, \quad K_i f_j K_i^{-1} = q^{-\alpha_j(h_i)}f_j, \quad [e_i, f_j] = \delta_{ij}(K_i - K_i^{-1})$$

where now $(\alpha_j(h_i))$ is symmetric.

Definition 3.8.2. This construction gives the **quantum groups** $U_q(\mathfrak{g})$. They act on $\mathbb{C}^n$ by

$$e_i = e_i, \quad f_i = f_i, \quad h_i = h_i.$$

For example, in $U_q(\mathfrak{sl}_2)$, we have

$$q^h = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \frac{q^h - q^{-h}}{q - q^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that $U\mathfrak{g}$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, is a Hopf algebra, i.e. has coproduct and antipode homomorphisms

$$\Delta \colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}, \quad \xi \mapsto \xi \otimes 1 + 1 \otimes \xi$$
$$S \colon U\mathfrak{g} \to U\mathfrak{g}, \quad \xi \mapsto -\xi$$

satisfying appropriate relations.

Proposition 3.8.3. *$U_q(\mathfrak{g})$ is a Hopf algebra, but is not co-commutative.*

Proof. The coproduct is given by

$$K_i \mapsto K_i \otimes K_i, \quad e_i \mapsto e_i \otimes 1 + K_i \otimes e_i, \quad f_j \mapsto f_j \otimes K_j^{-1} + 1 \otimes f_j.$$

To check that it is a homomorphism, compute that

$$[\Delta e_i, \Delta f_j] = [e_i, f_j] \otimes K_j^{-1} + K_i \otimes [e_i, f_j] + K_i f_j \otimes e_i K_j^{-1} - f_j K_i \otimes K_j^{-1} e_i.$$

But using the symmetry of $(\alpha_i(h_j))$, we have

$$K_i f_j \otimes e_i K_j^{-1} = q^{-\alpha_j(h_i) + \alpha_i(h_j)} f_j K_i \otimes K_j^{-1} e_i.$$

Hence the last two terms vanish and we end up with

$$[\Delta e_i, \Delta f_j] = \delta_{ij}((K_i - K_i^{-1}) \otimes K_i^{-1} + K_i \otimes (K_i - K_i^{-1})) = \delta_{ij}(K_i \otimes K_i - K_i^{-1} \otimes K_i^{-1}) = \delta_{ij}\Delta(K_i - K_i^{-1}). \quad \square$$

What about the Serre relations? Define $U_{\geq 0}(\mathfrak{g}) \subset U_q(\mathfrak{g})$ to be the sub-algebra generated by K_i and e_j. By the proposition, it is a Hopf algebra.

Definition 3.8.4. A **Hopf pairing** (a, b) satisfies

$$(a, bc) = (\Delta a, b \otimes c)$$

i.e. if $\Delta a = \sum a_i' \otimes a_i''$, then $(a, bc) = \sum (a_i', b)(a_i'', c)$.

To define a Hopf pairing, it suffices to define it on generators, and check that the relations among the generators pair to 0.

Proposition 3.8.5. *There is a non-degenerate Hopf pairing on* $U_{\geq 0}(\mathfrak{g})$ *given by*

$$(K_i, K_j) \coloneqq q^{-(h_i, h_j)}, \quad (K_i, e_j) \coloneqq 0, \quad (e_i, e_j) \coloneqq \delta_{ij}.$$

The Serre relations are the radical of the Hopf pairing on the entire $U_q(\mathfrak{g})$. Using the Hopf pairing, the dual to $U_{\geq 0}(\mathfrak{g})$ is $U_{\leq 0}(\mathfrak{g})$.

Definition 3.8.6. Since $U_q(\mathfrak{g})$ is a Hopf algebra, its category of modules is a:

1. **rigid**, i.e. has left/right duals $M \mapsto M^\vee$ on which the module acts by the antipode,
2. **tensor**, i.e. has a bifunctor $(M_1, M_2) \mapsto M_1 \otimes M_2$, on which the module acts via the coproduct, and $M^\vee \otimes M \xrightarrow{\sim}$ trivial,

category. (It is not necessarily the case that $S^2 = 1$, but it is true that S^2 is an inner automorphism; this is why we have left and right duals.)

Remark. The issue with a general Hopf algebra is that its category of modules is completely non-commutative. The miracle is that for $U_q(\mathfrak{g})$, its category of modules is basically commutative, i.e. there is an isomorphism $R^\vee \colon M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1$. This operator $R^\vee$ comes from a universal expression in $U_q(\mathfrak{g}) \widehat{\otimes} U_q(\mathfrak{g})$.

Take an oriented framed link and assign a module to each link. Insert the $R^\vee$ matrix at every crossing, and at every local minimum or local maximum we insert $M^\vee \otimes M \to$ trivial.

Definition 3.8.7. Consider $U_q(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}}$ an affine Lie algebra. This is a Hopf algebra deformation of $U(\mathfrak{g}[t^{\pm 1}])$. This algebra $\mathfrak{g}[t^{\pm 1}]$ has the **loop rotation** automorphism $t(d/dt)$, which still as a GL(1)-automorphism in $U_q(\widehat{\mathfrak{g}})$. In other words, for any module M and $a \in \mathrm{GL}(1)$, we can define $M(a) \coloneqq M \circ a$.

So now modules have labels, and we should rewrite in more generality:

$$R^\vee(a_1/a_2) \colon M_1(a_1) \otimes M_2(a_2) \xrightarrow{\sim} M_2(a_2) \otimes M_1(a_1)$$

where the isomorphism is for generic a_1, a_2. Hence $R^\vee(a_1/a_2)$ should now be viewed as a rational function of a_1/a_2.

Definition 3.8.8. A tensor category is **braided** if it has such an operator $R^\vee \colon M_1 \otimes M_2 \to M_2 \otimes M_1$ that does not necessarily square to 1. The **fiber functor** gives the underlying vector space of modules.

Theorem 3.8.9. *Any braided tensor category with a fiber functor is the category of modules for some quantum group.*

There is a completely pedestrian way to reconstruct the quantum group from such a category by looking at the universal expression for the R-matrix. It is convenient to look at the operator $R \colon M_1 \otimes M_2 \to M_2 \otimes_{\mathrm{opp}} M_1$ which uses the opposite coproduct. Using it, we can take matrix elements in M_1 and get operators in M_2 via

$$M_1^* \otimes M_1 \to (U_q \mathfrak{n}_-)^* \to U_q \mathfrak{n}_+, \quad \phi \mapsto \mathrm{tr}_{M_1}(\varphi R) =: \psi.$$

So the pairing between φ and ψ is $\langle\psi,\phi\rangle := \mathrm{tr}_{M_1\otimes M_2}((\psi\otimes\phi)R)$. The braid relation gives the **Yang–Baxter equation** $R_{12}R_{13}R_{23} = R_{23}R_{13}R12$ (which is strictly more powerful than the relation appearing in the Hecke algebra relations). This gives the commutation relations for operators in $M_1\otimes M_2$: to commute them, just change basis by R_{12}. To get the coproduct, we want to act in $M_2\otimes M_3$, so take matrix elements in M_1 for $R_{M_1,M_2\otimes M_3} = R_{M_1,M_3}R_{M_1,M_2}$. So really all we need is a "trigonometric R-matrix" $R(a_1/a_2)$ that solves the Yang–Baxter equation.

Let $\tilde{G} := \mathrm{GL}(2)\times\mathbb{C}^*$ where $\mathrm{GL}(2)\ni\mathrm{diag}(a_1,a_2)$ and $\mathbb{C}^*\ni q$. Consider $K_{\tilde{G}}(T^*\,\mathrm{Gr}(?,2))$. Note that $\mathrm{Gr}(?,2)$ decomposes as $\bigsqcup_k \mathrm{Gr}(k,2)$, which is just $\mathrm{pt}\sqcup T^*\mathbb{P}^1\sqcup\mathrm{pt}$. Let $X := G(n)$ and $X(k) := \mathrm{Gr}(k,n)$. Then, just as for G/B, for G/P we have the "Steinberg variety"

$$X(k)\times_{X_0(k)} X(k) = \bigsqcup (T^\perp\,\mathrm{GL}(n)\text{-orbits on }\mathrm{Gr}\times\mathrm{Gr}).$$

We can realize an action of $U_q(\widehat{\mathfrak{gl}(2)})$ on here. The K-theory of $TG(n)$, via this correspondence, can be turned into the category of modules for $U_q(\widehat{\mathfrak{g}})$.