Notes for Enumerative geometry seminar (Spring 2019)

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Webpage: `http://www.math.columbia.edu/~ccliu/Seminars/EG_S19.html`

Abstract

These are my live-texed notes for the Spring 2019 enumerative geometry seminar. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 Jan 29 (Renata): Gromov–Witten invariants of stable maps with fields

No notes, sorry!

2 Feb 05 (Renata): Gromov–Witten invariants of stable maps with fields

No notes, sorry!

3 Feb 19 (Zhengyu): (Equivariant) mirror symmetry for non-Hamiltonian torus actions

Let's first review all-genus mirror symmetry for toric CY 3-folds (or orbifolds), of finite type. This means that the fan contains only finitely many cones. In this setting, there is the *remodeling conjecture*, which is an all-genus statement for mirror symmetry. It was first introduced by Bouchard–Klemm–Mariño–Pasquetti (BKMP).

Before we talk about all-genus mirror symmetry, let's first review genus-0 mirror symmetry for general toric manifolds/orbifolds.

1. In the smooth case, this was first proved by Givental and also by Lian–Liu–Yau (in 1997).
2. In the orbifold case, there are two independent proofs:
   (a) by Coates–Corti–Iritani–Tseng (in 2013) generalizing the proof in the smooth case;
   (b) by Cheung–Ciocan–Fontanine–Kim (in 2014) using quasimap theory.

The statement of mirror symmetry is the correspondence of the following data:

A-model	Landau–Ginzburg B-model
Semi-projective toric orbifold X of dim r	T-equivariant superpotential $W^T\colon (\mathbb{C}^*)^r \to \mathbb{C}$
torus action $T = (\mathbb{C}^*)^r$ on X	the critical locus $\mathrm{Jac}(W^T)$
equivariant quantum cohomology $QH^*_{CR,T}(X)$	a residue pairing (f,g)
quantum product $\star_t$	
the T-equivariant Poincaré pairing $(-,-)_{X,T}$	

(The construction of W^T comes purely from the combinatorial data of the fan of X.)

Theorem 3.1 (Genus-0 mirror symmetry)**.** *There is an isomorphism of Frobenius structures*

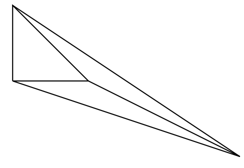
$$(QH^*_{CR,T}(X), \star_t, (-,-)_{X,T}) \cong (\mathrm{Jac}(W^T), \cdot, (-,-)).$$

Now we want to generalize this to the higher-genus case. In higher genus, we also have A-model and B-model.

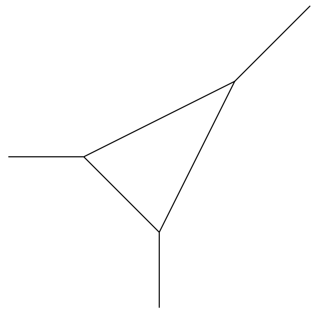
A-model	B-model
X a toric CY 3-orbifold	affine mirror curve $C \subset (\mathbb{C}^*)^2$
a Lagrangian $L \subset X$ (Aganagic–Vafa brane)	Eynard–Orantin topological recursion producing
open GW potential $F^{X,L}_{g,n}$ of (X,L)	a symmetric n-form $\Omega_{g,n} \in H^0(\mathbb{C}^n, K_C \otimes \cdots)^{S_n}$

Theorem 3.2 (Remodeling conjecture)**.** *If we expand $\Omega_{g,n}$ under suitable local coordinates on the mirror curve C, we obtain the open GW potential $F^{X,L}_{g,n}$ under the mirror map.*

Example 3.3. Consider $X = \mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$. The fan is the cone over the polytope



From this polytope, we can draw the *toric diagram* by drawing edges in a dual graph perpendicular to corresponding edges in the polytope:



The Lagrangian L will intersect with one of the three non-compact edges. On the mirror side, we get

$$H(X,Y) \coloneqq X + Y + 1 + qX^3Y^{-1} = 0.$$

Here this q is the parameter for complex structures. The equation comes from writing down an monomial for each point in the polytope. The topological type of the mirror curve C is determined by the toric diagram, by fattening. We put a puncture at $(X,Y) = (0,-1)$. If we expand $\Omega_{g,n}$ around $(0,-1)$, we will get the A-model open GW potential.

Proof strategy. A priori, the A-model and B-model pictures are of very different objects. The strategy is to realize both A-model and B-model higher-genus potentials as quantizations on two isomorphic semisimple Frobenius structures. The diagram is

$$\begin{array}{ccc}
\text{genus 0 A-model} & \xrightarrow{\text{genus 0 mirror symmetry}} & \text{genus 0 B-model} \\
\Big\downarrow {\scriptstyle\text{orbifold Givental quantization}} & & \Big\downarrow {\scriptstyle\text{dimensional reduction}} \\
 & & \text{genus 0 data of mirror curve} \\
 & & \Big\downarrow {\scriptstyle\text{Eynard–Orantin recursion}} \\
\text{higher genus A-model} & \xrightarrow{\text{graph sum formula}} & \text{higher genus B-model}
\end{array}.$$

□

Today we want to generalize this to toric CY3s Y of **infinite type**, i.e. Y is no longer an algebraic variety in the usual sense. In the toric case, this is equivalent to saying that there are infinitely many cones in the toric fan. But the fan is still always a cone over a triangulation of a *non-compact* polyhedron in $\mathbb{R}^2$. On the B-model side we still have a curve

$$\hat{C} \coloneqq \{H(X,Y) = 0\} \subset (\mathbb{C}^*)^2.$$

In the original case, the monomials in the equation $H(X,Y)$ corresponded to points in the toric polytope, but now there are infinitely many points. So $H(X,Y)$ will not be an algebraic equation, and there will be a convergence issue. It turns out convergence is not an issue. We can still run topological recursion to get $\Omega_{g,n}$.

Another more interesting case we want to study is when there exist a certain symmetry on the toric variety Y. Specifically, assume there exists an action of Γ on Y, for a certain discrete group Γ. This induces a Γ-action on the mirror curve $\hat{C}$. We want to consider the pair

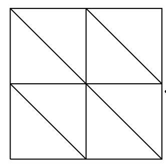
$$X \coloneqq Y/\Gamma, \quad C \coloneqq \hat{C}/\Gamma,$$

so C is the mirror curve for X. The picture is summarized in the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{MS} & \hat{C} \subset (\mathbb{C}^*)^2 \\ \downarrow & & \downarrow \\ X = Y/\Gamma & \xrightarrow{MS} & C = \hat{C}/\Gamma. \end{array}$$

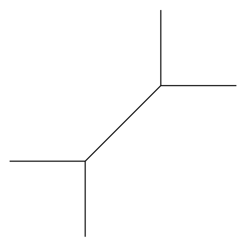
Of course, we want to assume the Γ action commutes with the torus action on Y. So X will have a non-Hamiltonian torus action in general. This is because the moment map $Y \to \mathbb{R}^2$ descends to $X \to \mathbb{R}^2/\Gamma$, which is in general not a vector space.

Example 3.4. The polyhedral Δ is given by the (infinite) periodic tiling which looks like



.

The toric diagram arising from this polyhedral cone is a hexagonal lattice. There is a $\mathbb{Z}^2$-action on Δ given by translation $(a,b) \mapsto (a+m, b+n)$, which induces an action on the resulting toric variety Y. The fundamental diagram in the quotient $X = Y/\mathbb{Z}^2$ is



Here the two vertical edges are identified, and the two horizontal edges are identified, for a total of three T-invariant $\mathbb{P}^1$'s. This X is sometimes called the local banana manifold. The generalized moment map is

$$\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = U(1)^2,$$

i.e. it is Lie group valued. It turns out there is a generalization of Hamiltonian actions called *quasi-Hamiltonian* T-actions. This is when the target is Lie group valued and the moment map satisfies certain conditions.

The mirror curve lies in $A \coloneqq \mathbb{C}^2/\mathbb{Z}^2$, which is an abelian surface. If we write its period matrix as

$$P = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix},$$

then the mirror curve is given by

$$H(X,Y) \coloneqq \Delta(q)\Theta_2\left[(-\tau/2, -\rho/2)\right](-x/2\pi i, -y/2\pi i; \rho),$$

where Θ_2 is the genus-2 Riemann zeta function. So the mirror curve is a genus-2 curve in the abelian surface A.

4 Feb 26 (Song): Open-closed Gromov–Witten invariants of toric CY3-orbifolds

We'll start with some basics about toric orbifolds. Recall that toric varieties can be defined via the combinatorial data of *fans*. Toric orbifolds arise from *stacky* fans, encoding data of the stabilizers. In the algebraic setting, these are called smooth toric DM stacks.

Definition 4.1. A **stacky fan** is the following data:

1. a finitely-generated free abelian group $N \cong \mathbb{Z}^n$ (i.e. a lattice of finite rank);
2. a simplicial fan $\Sigma \subset N_{\mathbb{R}} \coloneqq N \otimes_{\mathbb{Z}} \mathbb{R}$;
3. vectors $b_1, \dots, b_{r'}$ such that the set of 1-cones in Σ are generated by $\{\mathbb{R}_{\geq 0} b_i\}$;
4. additional vectors $b_{r'+1}, \dots, b_r$ such that $N = \langle b_1, \dots, b_r \rangle$.

So we have a short exact sequence

$$0 \to (\mathbb{Z}^{r-n} \cong L) \to (\mathbb{Z}^r \cong \tilde{N}) \xrightarrow{\tilde{b}_i \mapsto b_i} N \to 0$$

Applying $\otimes \mathbb{C}$, we get

$$0 \to G \to \tilde{T} \to T \cong (\mathbb{C}^*)^r \to 0.$$

This inclusion $G \to \tilde{T}$ induces an action of G on $\mathbb{C}^r$. The **toric orbifold** $\mathfrak{X}$ is the quotient by G of a subvariety of $\mathbb{C}^r$ as follows.

1. Define the **anti-cones**
$$\mathcal{A} \coloneqq \{I' \cup \{r'+1, \dots, r\} : I' \subset \{1, \dots, r'\} \text{ s.t. } \sum_{i \notin I'} \mathbb{R}_{\geq 0} b_i \text{ is a cone in } \Sigma\}.$$
2. Remove anti-cones:
$$U_{\mathcal{A}} \coloneqq \mathbb{C}^r \smallsetminus \bigcup_{I \in \mathcal{A}} Z(\prod_{i \in I} x_i).$$
3. Define the stacky quotient $\mathfrak{X} \coloneqq [U_{\mathcal{A}}/G]$.

Remark. If $b_1, \dots, b_{r'}$ are *minimal* generators, in the sense that on each ray we select the smallest integral point, $\mathfrak{X}$ will be a simplicial toric variety, not a DM stack. We disallow torsion in N because that gives rise to generic stabilizer.

Example 4.2. Let $N = \mathbb{Z}$ and Σ be the complete fan on $\mathbb{R}$. Choose $b_1 = 3$ and $b_2 = 2$. To generate N, pick $b_3 = 1$. The result is

$$\mathfrak{X} = \mathbb{P}[3, 2].$$

Definition 4.3. In the toric case, we have a nice description of stabilizers in terms of the stacky fan data.

1. Let $\Sigma(d)$ be the set of all d-dimensional cones in Σ.
2. Given $\sigma \in \Sigma(d)$, let
$$I_{\sigma'} \coloneqq \{i \in \{1, \dots, r'\} : \mathbb{R}_{\geq 0} b_i \in \sigma\}$$
$$I_{\sigma} \coloneqq \{1, \dots, r\} \smallsetminus I_{\sigma'}.$$
3. Define the $n - d$-dimensional closed substack
$$V(\sigma) \coloneqq [(U_{\mathcal{A}} \cap Z(x_i : i \in I_{\sigma'}))/G].$$

   Note that this is$T$-invariant. In general, it is not an orbifold. Smaller orbits will have larger generic stabilizers.

Let $G_\sigma \leq G$ be the generic stabilizer of $V(\sigma)$. All of these arise combinatorially as follows.

Lemma 4.4. *There is an identification of G_σ with*

$$\mathrm{Box}(\sigma) := \{\sum_{i\in I_{\sigma'}} c_i b_i \in N : 0 \le c_i \le 1\}.$$

Example 4.5. Let $N = \mathbb{Z}^2$ with fan Σ just the first quadrant. Pick

$$b_1 = (2,0), \quad b_2 = (0,3)$$

with additional vectors to generate the lattice. Then we have one 2-dimensional cone σ and two 1-dimensional cones τ_1, τ_2. The box elements are therefore

$$\mathrm{Box}(\tau_1) = \{(0,0),(1,0)\}, \quad \mathrm{Box}(\tau_2) = \{((0,0),(0,1),(0,2)\}.$$

These are identified with stabilizers $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. What about $\mathrm{Box}(\sigma)$? It is

$$\mathrm{Box}(\sigma_1) = \mathrm{Box}(\tau_1) \cup \mathrm{Box}(\tau_2) \cup \{(1,1)\}.$$

This is identified with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Remark. We see that in general $\mathrm{Box}(\sigma)$ is always actually a box. If $\tau \subset \sigma$ then $\mathrm{Box}(\tau)$ is the projection of $\mathrm{Box}(\sigma)$ in the appropriate direction.

Remark. The most interesting case is $\tau \prec \sigma$ where $\tau \in \Sigma(n-1)$ and $\sigma \in \Sigma(n)$. This is a T-fixed point on a T-fixed curve. The quotient is always a cyclic group $\mu_{r(\tau,\sigma)}$, where

$$r(\tau,\sigma) := \frac{|G_\sigma|}{|G_\tau|}.$$

Definition 4.6. For $v := \sum_{i\in I_{\sigma'}} c_i b_i \in \mathrm{Box}(\sigma)$, define the **age**

$$\mathrm{age}(v) := \sum c_i.$$

Definition 4.7. We can package all this data into an object called the **inertia stack** $I\mathfrak{X}$. Formally, it is defined via a Cartesian square

$$\begin{array}{ccc} I\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X}\times\mathfrak{X} \end{array}.$$

Example 4.8. In our case, because we understand the stabilizers well, we can identify all the components in the inertia stack. There is a bijection

$$\{g \in G : g \text{ fixes a point in } U_{\mathcal{A}}\} \leftrightarrow \mathrm{Box}(\Sigma) := \bigcup_{\sigma\in\Sigma(n)} \mathrm{Box}(\sigma).$$

The inertia stack will be

$$I\mathfrak{X} := \coprod_{v\in\mathrm{Box}(\Sigma)} \mathfrak{X}_v$$

where $\mathfrak{X}_v$ is the maximal closed substack fixed by the element $g \in G$ corresponding to v.

Definition 4.9. The inertia stack allows us to upgrade our usual cohomology theory to **Chen–Ruan orbifold cohomology**

$$H^*_{CR}(\mathfrak{X}) := \bigoplus_{v\in\mathrm{Box}(\Sigma)} H^*(\mathfrak{X}_v)[2\,\mathrm{age}(v)].$$

Example 4.10. Let $\mathfrak{X} = \mathbb{P}[3,2]$. In this case, the boxes are $\mathrm{Box}(\sigma_1) = \{0,1,2\}$ and $\mathrm{Box}(\sigma_2) = \{0,-1\}$.

1. The generic stabilizer of the point corresponding to σ_1 is therefore $\mathbb{Z}/3\mathbb{Z}$. Hence $V(\sigma_1) = B\mu_3$.
2. The generic stabilizer of the point corresponding to σ_2 is therefore $\mathbb{Z}/2\mathbb{Z}$. Hence $V(\sigma_2) = B\mu_2$.

The inertia stack $I\mathfrak{X}$ therefore has four components

$$I\mathfrak{X} = \mathfrak{X}_0 \sqcup \mathfrak{X}_1 \sqcup \mathfrak{X}_2 \sqcup \mathfrak{X}_{-1}.$$

1. $\mathfrak{X}_0 = \mathfrak{X}$, since the identity element fixes everything. The age is 0.
2. $\mathfrak{X}_1 = \mathfrak{X}_2 = B\mu_3$, corresponding to σ_1. The age is 1/3 in both cases.
3. $\mathfrak{X}_{-1} = B\mu_2$, corresponding to σ_2. The age is 1/2.

Hence the Chen–Ruan orbifold cohomology, as a $\mathbb{Q}$-vector space, is

$$H^*_{CR}(\mathfrak{X}) = \mathbb{Q}1_0 \oplus \mathbb{Q}1_{1/3} \oplus \mathbb{Q}1_{1/2} \oplus \mathbb{Q}1_{2/3} \oplus \mathbb{Q}H.$$

Remark. Note that $I\mathfrak{X}$ has an involution given by

$$\text{inv} \coloneqq (x, \varphi) \mapsto (x, \varphi^{-1}).$$

This essentially maps $\mathfrak{X}_v$ to $\mathfrak{X}_{v^{-1}}$. Here "v^{-1}" means the inverse of the group element corresponding to v.

Definition 4.11. The product structure on $H^*_{CR}(\mathfrak{X})$ is defined via the **orbifold Poincaré pairing**

$$(\alpha, \beta) \coloneqq \int_{I\mathfrak{X}} \alpha \cup \text{inv}^*(\beta).$$

If α, β are pure in that they both come from single strata $\mathfrak{X}_{v_1}$ and $\mathfrak{X}_{v_2}$, then

$$(\alpha, \beta) = \begin{cases} \int_{\mathfrak{X}_{V(\alpha)}} \alpha \cup \text{inv}^*(\beta) & V(\beta) = \text{inv}(V(\alpha)) \\ 0 & \text{otherwise.} \end{cases}$$

Remark. For toric orbifolds, Borisov–Reisner–Smith writes down a presentation for the orbifold cohomology ring which looks like a Stanley–Reisner presentation.

Definition 4.12. Now let's talk about open-closed GW invariants. Specialize to $\mathfrak{X}$ a toric CY3-orbifold. So $N = \mathbb{Z}^3 = \langle u_1, u_2, u_3 \rangle$. The CY condition says that

$$b_i \in \text{hyperplane } N'_1 \coloneqq \{u_3 = 1\}.$$

In this situation, the data of the stacky fan is realized on the N'_1 plane as a planar graph. This is equivalent to giving the data called the **toric graph**, consisting of a point for each top-dimensional cone and a ray for each edge.

Definition 4.13. Since we want to study open invariants, we need **Aganagic–Vafa branes**. These are Lagrangian sub-orbifolds $\mathcal{L}$, preserved under the compact CY torus $T' \subset T$. Such $\mathcal{L}$ intersect unique T-fixed curves $V(\tau)$.

1. If $\mathcal{L}$ intersects a non-compact orbit, call it **outer**.
2. If $\mathcal{L}$ intersects a compact orbit, call it **inner**.

The computations for the two cases will be very similar.

Remark. In general, $\mathcal{L} = [S^1 \times \mathbb{C}/G_\sigma]$. This is saying that in addition to the winding numbers around $\mathcal{L}$, we also need to know stabilizers. We get a sequence

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_\tau & & \\
 & & & & & & \downarrow & & \\
 & & & & H_1(\mathcal{L},\mathbb{Z}) & \longrightarrow & G_\sigma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot r} & \mathbb{Z} & \longrightarrow & \mu_{r(\tau,\sigma)} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
.
$$

Definition 4.14. Now we can define **stable maps to** $(\mathfrak{X}, \mathcal{L})$

$$\overline{\mathcal{M}}_{(g,h),n}(\mathfrak{X}, \mathcal{L}|\beta', \vec{\mu}).$$

1. The domain is a bordered Riemann orbi-surface

   $$(\Sigma, x_1, \dots, x_n)$$

   with n marked points. The compact components are genus g, with h disks attached to these components. We require that orbifold points only come from marked points or the nodes.

2. The map is

   $$(\Sigma, \partial\Sigma) \xrightarrow{u} (\mathfrak{X}, \mathcal{L}).$$

   The data of this map is

   $$\beta' \coloneqq u_*[\Sigma] \in H_2(\mathfrak{X}, \mathcal{L})$$

   and $\vec{\mu} = (\mu_1, \dots, \mu_n)$ where

   $$\mu_j \coloneqq u_*([\partial D_j]) \in H_1(\mathcal{L}, \mathbb{Z}).$$

Remark. We can write the usual perfect obstruction theory, but evaluation maps are more subtle than in the non-orbifold case. Fixed points $[\mathrm{pt}/\mu_r]$ can have non-trivial stabilizer, so evaluation maps must keep track of where the generator of the stabilizer group is sent. So

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{(g,h),n}(\mathfrak{X}, \mathcal{L}|\beta', \vec{\mu}) \to I\mathfrak{X}.$$

Hence we must pull back classes in $H^*(I\mathfrak{X}) = H^*_{CR}(\mathfrak{X})$. In general, $\mathfrak{X}$ is not compact, so we must use the torus action of T' to define invariants via localization:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g,\beta',\vec{\mu}} \coloneqq \int_{[F]^{\mathrm{vir}}} \frac{\prod \mathrm{ev}_i^*(\gamma_i)|_F}{e_{T'}(N_F^{\mathrm{vir}})}.$$

How do we compute invariants? We can decompose the data of a stable map into two parts:

1. invariants on the closed components, with $n + h$ fixed points total;

2. (disk factors) invariants of the h open components.

In the case of disk factors, our moduli space is simple since $(h,n) = (1,1)$. Consider the case that $\mathcal{L}$ is inner, on an edge $V(\tau)$ between fixed points $V(\sigma^+)$ and $V(\sigma^-)$. It splits the edge into two orbi-disks $D^\pm$. Let $\alpha \in H_2(\mathfrak{X},\mathcal{L})$ be the class of $V(\tau)$, and let β and $\alpha - \beta$ be the classes $[D^+]$ and $[D^-]$ respectively. Given the data

$$(d_0,\lambda) \in \mathbb{Z} \times G_\tau,$$

define $h^\pm(d_0,\lambda) \in G_{\sigma^\pm}$ so that $(d_0, h^+, h^-) \in H_1(\mathcal{L},\mathbb{Z})$. Define the **disk factor**

$$D_{d_0,\lambda} \coloneqq \begin{cases} \langle 1_{h^+(d_0,\lambda)} \rangle^{(\mathfrak{X},\mathcal{L})}_{0,d_0 b,(d_0,\lambda)} & d_0 > 0 \\ \langle 1_{h^-(d_0,\lambda)} \rangle^{(\mathfrak{X},\mathcal{L})}_{0,-d_0 b,(d_0,\lambda)} & d_0 < 0. \end{cases}$$

These are still a little difficult to compute. We reinterpret them as relative invariants and compute them that way. Idea: degenerate the orbit such that the generic stabilizer G_τ of the resulting node p_0 is the generic stabilizer of the entire orbit. So for $d_0 > 0$, we can compute

$$D_{d_0,\lambda} = \int_{[\overline{\mathcal{M}}_{0,1}(I_+/p_0,(d_0,\lambda))]^{\mathrm{vir}}} \mathrm{ev}^*(1_{h^+(d_0,\lambda)}) e(V^+)$$

where $e(V^+)$ is the contribution of the obstruction bundle. Explicitly,

$$V^+ \coloneqq R\pi_* f^*(\text{degeneration of } N_{V(\tau)/\mathfrak{X}} \text{ to } I^+).$$

So in the localization calculation, all we need are the additional torus weights at the degeneration point on $V(\tau)$, and along the divisor corresponding to the degeneration.

5 Mar 05 (Song): Mirror symmetry and crepant resolution conjecture for disks

Last time we talked about open-closed GW invariants. These are roughly just virtual counts of holomorphic orbifold maps

$$f\colon \begin{pmatrix} \text{bordered prestable} \\ \text{orbifold Riemann surface} \end{pmatrix} \to (\mathfrak{X},\mathcal{L})$$

where $(\mathfrak{X},\mathcal{L})$ is a toric CY3-orbifold. This means we have the following data.

1. The action of a 3d complex torus T on $\mathfrak{X}$, with compact real sub-torus $T'_{\mathbb{R}} \coloneqq U(1)^2$, called the **Calabi–Yau torus**.
2. Combinatorially, the data of $\mathfrak{X}$ is given by the **stacky fan**. By the CY condition, this amounts to some two-dimensional fan, or equivalently a **toric graph**.
3. $\mathcal{L}$ is a Lagrangian called the **Aganagic–Vafa brane**. It must be invariant under the $T'_{\mathbb{R}}$ action. It intersects a unique T-invariant curve of $\mathfrak{X}$.
   (a) If the curve is non-compact, we call $\mathcal{L}$ **outer**.
   (b) If the curve is compact, we call $\mathcal{L}$ **inner**.

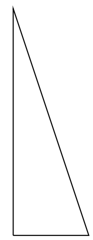
Because we are in the orbifold, each fixed point σ and curve τ will have stabilizers. There is a short exact sequence

$$1 \to G_\tau \to G_\sigma \to \mu_{r(\tau,\sigma)} \to 1$$

where $r(\tau,\sigma)$ is the appropriate index. We saw last time that we can get $r(\tau,\sigma)$ from the box description. We also get a bijection

$$H_1(\mathcal{L},\mathbb{Z}) \leftrightarrow \mathbb{Z} \times \mu_m.$$

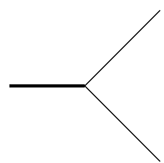
Example 5.1. Let $\mathfrak{X} \coloneqq [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}$. The first factor is an $\mathcal{A}_2$-singularity. In terms of the stacky fan, we have



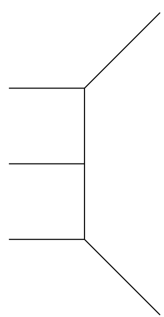
The stacky fan data is from the five interior lattice points

$$1 \to L \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ -2 & 1 \\ 1 & -1 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \to 1$$

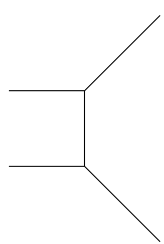
The non-trivial stabilizer is $G = \mathbb{Z}/3$ and all other stabilizers are trivial. The corresponding toric 1-skeleton is



The thick leg is a **gerby leg**, because it has stabilizer G. We want to resolve this singularity, to get the full resolution



We can also look at partial resolutions, given in the obvious way, e.g.



Definition 5.2. The **moduli space** is denoted

$$\mathcal{M} \coloneqq \overline{\mathcal{M}}_{(g,h),n}(\mathfrak{X}, \mathcal{L} | \beta', \vec{\mu}).$$

It parametrizes maps $(\Sigma, \partial\Sigma) \xrightarrow{u} (\mathfrak{X}, \mathcal{L})$ such that

$$\beta' = u_*[\Sigma] \in H_2(\mathfrak{X}, \mathcal{L})$$
$$\mu_j = u_*[R_j] \in H_1(\mathcal{L})$$

where R_j is the j-th boundary component of Σ. There are also **evaluation maps**

$$\mathrm{ev}_i\colon \mathcal{M} \to I\mathfrak{X}$$

landing in the *inertia stack*. Because we have a compact CY torus $T'_{\mathbb{R}}$ acting on $(\mathfrak{X}, \mathcal{L})$, we get an action by $T'_{\mathbb{R}}$ on $\mathcal{M}$. **Open-closed invariants** are defined by localization as

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta',\vec{\mu}}^{\mathfrak{X},\mathcal{L},T'_{\mathbb{R}}} \coloneqq \int_{[F]^{\mathrm{vir}}} \frac{\prod_i \mathrm{ev}_i^*(\gamma_i)}{e_{T'_{\mathbb{R}}}(N_F^{\mathrm{vir}})} \in \mathbb{Q}(v'_1, v'_2)$$

where v'_1, v'_2 are the weights of $T'_{\mathbb{R}}$. Here $\gamma_i \in H^*_{\mathrm{CR}}(\mathfrak{X}, \mathbb{Q})$ are elements in *orbifold* cohomology.

There are two types of contributions to open-closed invariants: from the open disk components (called disk factors), and from the closed components. We computed disk factors by viewing them as relative invariants, for $\mathbb{P}^1$ relative to a point. If we pick a *framing* $f \in \mathbb{Z}$, we get a 1-dimensional torus

$$T'_f \coloneqq \ker(v'_2 - f v'_1) \hookrightarrow T'.$$

Pulling back to this torus, disk factors basically become number $\cdot\, v$ where v is the weight of T'_f. These numbers are very explicit (Ross '14, Fang–Liu–Tseng). Write

$$\beta' = \beta + \sum_{d_j > 0} d_j b + \sum_{d_j < 0} (-d_j)(\alpha - b)$$

where $(b, \alpha - b)$ are the classes the two sides of the curve cut by the Lagrangian $\mathcal{L}$. So $\beta \in H_2(\mathfrak{X})$. If we restrict the map u to each of the disk components, then we get maps

$$u|_{D_j}\colon (D_j, \partial D_j) \to (\mathfrak{X}, \mathcal{L})$$

which are accounted for by the disk factors. So the remaining problem is the contribution from the closed component C, i.e. the map

$$[u|_C\colon (C, x_i, y_j) \to (\mathfrak{X}, \mathcal{L})] \in \overline{\mathcal{M}}_{g,n+h}(\mathfrak{X}, \beta) \eqqcolon \hat{\mathcal{M}}.$$

The data of fixed loci in $\hat{\mathcal{M}}$ consists of decorated graphs $\hat{\Gamma}$, just like the data of fixed loci in $\mathcal{M}$ consists of decorated graphs Γ. Given Γ, we get $\hat{\Gamma}$ by requiring that

$$\mathrm{ev}_{n+j}(\hat{\Gamma}) = (p_{\sigma_\pm}, h_1(d_0, \lambda)),$$

i.e. we put constraints on where the extra marked points (corresponding to the disks) map to. Actually, F_Γ and $F_{\hat{\Gamma}}$ differ by a finite map, i.e.

$$[F_\Gamma]^{\mathrm{vir}} = ? \cdot [F_{\hat{\Gamma}}]^{\mathrm{vir}},$$

and we need to know the extra factor. It contains contributions from:

1. automorphisms of $u|_{D_j}$;
2. node smoothing at y_j.

The former already shows up in disk factors. The latter is essentially some ψ class insertions. We get

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta',\vec{\mu}}^{\mathfrak{X},\mathcal{L}} = \sum_{\hat{\Gamma}} ? \cdot D_{(d_0\lambda, f)} \int_{[F_{\hat{\Gamma}}]^{\mathrm{vir}}} \frac{\prod_{i=1}^n \mathrm{ev}_i^* \gamma_i \prod_{j=1}^h \mathrm{ev}_{n+j}^* \phi_j}{\prod_j \left(\frac{v}{r_j d_j} - \frac{\overline{\psi}_{n+j}}{r_i} \right) e(N_{F_{\hat{\Gamma}}}^{\mathrm{vir}})}$$

where ? is some combinatorial automorphism factor.

To compute disk invariants, we will compute the whole generating function for $(g,h) = (0,1)$. Define

$$F^{\mathfrak{X},(\mathcal{L},f)}_{(0,1)}(\tau_0,\tau_1,\ldots,\tau_k) \coloneqq \sum_{(d_0,\lambda)\in H_1(\mathcal{L},\mathbb{Z})} \sum_{\beta,n\geq 0} \cdots \in H^*_{\mathrm{CR}}(B\mu_m;\mathbb{C}).$$

One part of this comes from genus-0 closed GW invariants $J(\tau_1,\ldots,\tau_k,z)$, where $\tau_1,\ldots,\tau_k$ are formal variables keeping track of coefficients in front of a basis of $H^2_{\mathrm{CR}}(\mathfrak{X})$. The remaining τ_0 keeps track of the open part. In genus-0, this J function is related to the I function

$$I(q_0,q_1,\ldots,q_k,z) \in H^*_{\mathrm{CR}}(\mathfrak{X})[[z]].$$

The $q_1,\ldots,q_k$ keep track of curve classes, and q_0 keeps track of the open part. The relation is as follows.

Theorem 5.3 (Coates–Corti–Iritani–Tseng, equivariant genus-0 mirror symmetry)**.**

$$e^{\frac{\tau_0}{2}} J(\tau,z) = I(q_0,q,z)$$

under the closed mirror map.

We can look at $F^{\mathfrak{X},(\mathcal{L},f)}_{(0,1)}$ under the mirror map, to get

$$W^{\mathfrak{X},(\mathcal{L},f)}(q_0,q,z).$$

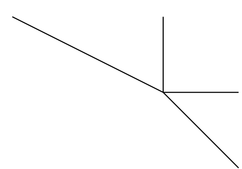
This W can be written very explicitly via the work of Fang–Liu–Tseng.

Example 5.4. Return to the example $\mathfrak{X} = [\mathbb{C}^2/\mathbb{Z}_3]\times\mathbb{C}$. Then the rows of the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ -2 & 1 \\ 1 & -1 \end{pmatrix}$$

are the D_i.

1. The Kähler cone, in terms of the D_i, is described by the picture



   and is given by the interior of the cone spanned by D_4 and D_5. The other chambers correspond to the partial resolutions written earlier. Together, they form a fan covering the whole plane. The whole system is called the **secondary fan**.

2. The **Mori cone** parametrizes effective curve classes. This means we take the dual. Let β_1,β_2 be dual to D_4,D_5.

Then the extended parameter space is

$$K_{\mathrm{eff}}(\mathfrak{X},\mathcal{L}) \coloneqq \{(d, c_1\beta_1, c_2\beta_2) : d > 0,\ c_1,c_2 \geq 0\}.$$

This is what we sum over in the Fang–Liu–Tseng formula.

This is useful for the crepant resolution conjecture. If $\mathfrak{X}$ is a Gorenstein orbifold and $Y \xrightarrow{\pi} \mathfrak{X}$ is a crepant resolution, then the enumerative invariants of Y and $\mathfrak{X}$ should roughly match up.

Conjecture 5.5 (Ruan)**.** *If $Y \xrightarrow{\pi} \mathfrak{X}$ is crepant, then there is an isomorphism of rings*

$$\mathrm{QH}^*(Y, \mathbb{C}) \cong \mathrm{QH}^*_{CR}(\mathfrak{X}, \mathbb{C}).$$

Here QH^*_{CR} means the orbifold cohomology with the full quantum product.

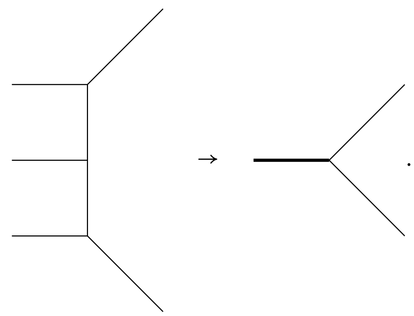
This roughly boils down to identifying the Gromov–Witten invariants of Y and $\mathfrak{X}$. In our case, if $\mathfrak{X}$ is a toric CY3-orbifold, then the partial resolutions are given by subdivision of cones and are all crepant.

Theorem 5.6 (Ke–Zhou)**.** *In the case where $\mathcal{L}$ is effective, i.e. $\mathcal{L}$ intersects a non-gerby leg, for $(\hat{\mathfrak{X}}, \hat{\mathcal{L}}, \hat{f}) \to (\mathfrak{X}, \mathcal{L}, f)$ we can match up the two disk potentials, i.e.*

$$W^{\hat{\mathfrak{X}},(\hat{\mathcal{L}},\hat{f})}(\hat{x}, \hat{q}) = W^{\mathfrak{X},(\mathcal{L},f)}(x, q)$$

with a change of variables.

Example 5.7. Take our previous example. We have the full resolution



Put the Lagrangian on a non-gerby leg, so that the resolution doesn't affect the Lagrangian. (This is what makes this case easier.) Then the strategy is to identify

$$K_{\mathrm{eff}}(\hat{\mathfrak{X}}, \hat{\mathcal{L}}) \leftrightarrow K_{\mathrm{eff}}(\mathfrak{X}, \mathcal{L})$$

by doing a change of basis to identify D_2, D_5 with D_4, D_5.

6 Mar 12 (Clara): Klein TQFTs

We will first recap the complex picture, of local curves. Then we will move to the real Gromov–Witten setting, where we get a Klein TQFT.

Definition 6.1. Consider GW theory for Y a non-singular quasi-projective algebraic 3-fold:

$$\overline{\mathcal{M}}^\bullet_h(Y, \beta).$$

We would like to define invariants via

$$Z'(Y)_\beta \text{``} := \text{''} \int_{[\overline{\mathcal{M}}^\bullet_h(Y,\beta)]^{\mathrm{vir}}} 1.$$

Since Y is not compact, we instead need to define invariants via localization:

$$Z'(Y)_\beta := \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{\mathcal{M}}_h(Z,\beta)]^{\mathrm{vir}}} \frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})}$$

where $Z \subset Y$ is a compact submanifold.

Definition 6.2 (Local curves)**.** Now specialize to the following setting. Let X be a curve of genus g, with two line bundles L_1, L_2. Let

$$Y \coloneqq \operatorname{tot}(L_1 \oplus L_2 \to X)$$

be the total space. There are two tori $\mathbb{C}^\times$ acting on L_1 and L_2 by scaling fibers. All torus-invariant curves to Y will therefore land in X, so

$$\overline{\mathcal{M}}_h^\bullet(Y, d[X])^T = \overline{\mathcal{M}}_h^\bullet(X, d).$$

The difference between the obstruction theories is the term $\mathrm{Norm}^{\mathrm{vir}} \coloneqq R\pi_* \operatorname{ev}^*(L_1 \oplus L_2)$.

Definition 6.3 (Relative stable maps)**.** In order to use degeneration, we generalize to **relative stable maps**, i.e. maps $C \to Y$ with prescribed ramification profiles over points in $X \subset Y$. Degeneration then tells us that if $Y = Y_1 \cup_D Y_2$, then we should reconstruct the GW theory of Y from gluing the GW theories of Y_1 and Y_2 along D. Let $\lambda_1, \dots, \lambda_r$ be partitions of the degree d. Define the **moduli** of relative stable maps

$$\overline{\mathcal{M}}_h^\bullet(X, \lambda_1, \dots, \lambda_r).$$

Denote the invariants by

$$\mathrm{GW}(g|k_1, k_2)^{\lambda^1,\dots,\lambda^t}_{\mu^1,\dots,\mu^s}$$

where k_1, k_2 are the **levels**, i.e. the degrees of the line bundles L_1, L_2 over X.

Theorem 6.4 (Degeneration and gluing)**.** *Pick splittings $g = g' + g''$ and $k_i = k_i' + k_i''$. Then*

$$\mathrm{GW}(g|k_1, k_2)^{\lambda^1,\dots,\lambda^t}_{\mu^1,\dots,\mu^s} = \sum_{\nu \vdash d} \mathrm{GW}(g'|k_1', k_2')^{\lambda^1,\dots,\lambda^t}_{\nu} \, \mathrm{GW}(g''|k_1'', k_2'')_{\mu^1,\dots,\mu^s}.$$

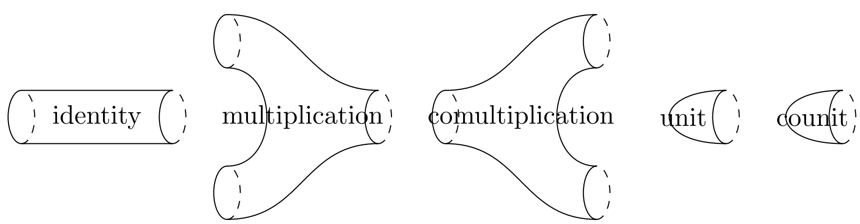
Definition 6.5. The GW partition functions for relative maps into local curves forms a TQFT. A **TQFT** is a symmetric monoidal functor

$$F\colon \mathsf{2Cob} \to \mathsf{Mod}(R).$$

The category $\mathsf{2Cob}$ has:

1. objects which are compact oriented 1-manifolds, i.e. $\bigsqcup_{i=1}^n S^1$ for every n;
2. morphisms which are oriented cobordisms.

Hence there are generators:



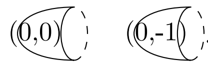
Example 6.6. We describe the **GW local curves TQFT** F. Take the ring $R \coloneqq \mathbb{Q}(t_1, t_2)((q))$, where q will keep track of Euler characteristic and t_1, t_2 are equivariant weights. Define the functor F as follows.

$$F(S^1) \coloneqq H \coloneqq \bigoplus_{\rho \vdash d} R e_\rho, \quad F(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{r \text{ copies}}) \coloneqq H^{\otimes r}.$$

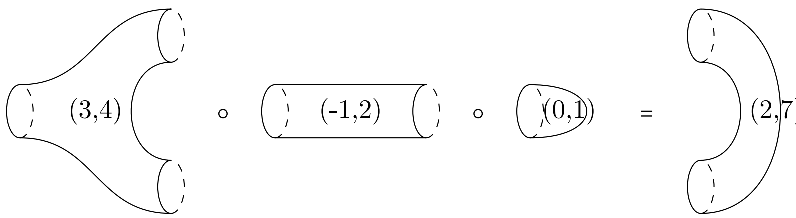
On a cobordism between s circles and t circles, the morphism will be

$$e_{\eta^1} \otimes \cdots \otimes e_{\eta^s} \mapsto \sum_{\mu^1,\dots,\mu^t \vdash d} \mathrm{GW}(g|0,0)^{\mu^1,\dots,\mu^t}_{\eta^1,\dots,\eta^s} e_{\mu^1} \otimes \cdots \otimes e_{\mu^t}.$$

Gluing cobordisms corresponds to gluing GW partition functions via the degeneration formula. To cover all levels (k_1, k_2), i.e. degrees of the line bundles L_1, L_2, we decorate the cobordisms with the levels of the line bundles over them. So for example, we will have a $(0,0)$ cap and a $(0,-1)$ cap:



Levels are additive with respect to gluing. For example



So now our cobordism category is enriched with the data of labels. Denote it $2\mathsf{Cob}^{k_1,k_2}$. Now to solve the local curves TQFT, it suffices to compute a few $(0,0)$ and $(0,-1)$ cobordisms.

Now we will move to the real case. Let Σ be a Riemann surface. The moduli

$$\overline{\mathcal{M}}^{\phi,\bullet}_{d,\chi}(\Sigma),$$

will parametrize real curves $C \to \Sigma$ with degree d and Euler characteristic χ. There is an involution ϕ representing complex conjugation.

Definition 6.7 (Real GW)**.** Let (X, ω) be a symplectic manifold. Let ϕ be an anti-symplectic involution on X, i.e. $\phi^*\omega = -\omega$. To have a *real* map, we also need a real domain. A **symmetric Riemann surface** (C, σ) is a Riemann surface C with an anti-holomorphic involution σ. Then σ splits the domain into:

1. *real* components, invariant under it, and
2. complex-conjugate **doublets**, which we write as $C_0 \sqcup \overline{C_0}$.

A **real stable map** is

$$f\colon (C, \sigma) \to (X, \phi)$$

compatible with σ and ϕ.

Definition 6.8 (Real local curve)**.** An **R-bundle** on (X, ϕ) is

$$(V, \varphi) \to (X, \phi)$$

where $V \to X$ is a vector bundle and φ is an involution lifting ϕ fiber-wise. To get the setting of local curves, take a holomorphic line bundle $L \to (\Sigma, \sigma)$ and pull back σ to form

$$(L \oplus \sigma^* L, \sigma) \to (\Sigma, \sigma).$$

This is the setting of **real local curves**. Invariants are denoted

$$RZ^{\phi}_{d,\chi} \coloneqq \int_{[\overline{\mathcal{M}}^{\phi,\bullet}_{d,\chi}(\Sigma)]^{\mathrm{vir}}} \frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})}.$$

Ramification data λ now comes in conjugate pairs $(\lambda, \overline{\lambda})$, via applying the involution σ.

Example 6.9 (Doublet target)**.** We want to compute

$$RZ^{\sigma}_{d,\chi}(D\Sigma|D(L_1,L_2))_{\lambda^1,\dots,\lambda^s}.$$

Here

$$D\Sigma \coloneqq (\Sigma \sqcup \overline{\Sigma}, \sigma) = (\Sigma_1 \sqcup \Sigma_2, \sigma)$$

and $D(L_1, L_2)$ is a single line bundle given by

$$D(L_1,L_2)|_{\Sigma_1} = L_1, \quad D(L_1,L_2)|_{\Sigma_2} = \sigma^* \overline{L}_2.$$

Then we look at the bundle $D(L_1,L_2) \oplus \sigma^* D(L_1,L_2)$. There is an isomorphism of moduli spaces

$$D\colon \overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\lambda^1,\dots,\lambda^s} \leftrightarrow \overline{\mathcal{M}}^{\phi,\bullet}_{d,2\chi}(D\Sigma)_{\lambda^1,\dots,\lambda^s}$$

by doubling. The virtual classes compare as

$$[\overline{\mathcal{M}}^{\phi,\bullet}_{d,2\chi}(D\Sigma)_{\lambda^1,\dots,\lambda^s}]^{\mathrm{vir}} = (-1)^{dm_2+\ell_2} D_* [\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\lambda^1,\dots,\lambda^s}]^{\mathrm{vir}}.$$

The normal bundles also differ up to some sign.

7 Mar 26 (Mark): Integral transforms and quantum correspondences

Start with $\overline{\mathcal{M}}_{g,n}(X,d)$ for a smooth projective variety/orbifold X. The general question; if two varieties X, Y are related in some way, how does their GW theory relate?

Example 7.1 (K-equivalence)**.** If X_1, X_2 are birational

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X_1 & & X_2 \end{array}$$

such that $\pi_1^* K_{X_1} = \pi_2^* K_{X_2}$, we say they are **K-equivalent**. Then it is known in a lot of generality that the GW theories of X_1 and X_2 are related, called the **crepant transformation conjecture** (CTC). The relationship is complicated, involving analytic continuation of partition functions. This analytic continuation is compatible, in a precise sense, with a Fourier–Mukai transform $D^b(X_1) \to D^b(X_2)$.

Example 7.2 (Quantum Serre duality)**.** Let $Z = Z(W)$ be a degree-d (smooth) hypersurface in $\mathbb{P}^r$. Let

$$Y \coloneqq \mathrm{tot}(\mathcal{O}_{\mathbb{P}^r}(-d)).$$

Then Y is non-compact of dimension $r+1$, and Z is compact of dimension $r-1$. There is a relation called **quantum Serre duality** that says the GW theory of Y determines the GW theory of Z in genus 0. (The procedure is to take a derivative of the partition function of Y and then do a change of variables.) This is not too surprising, because:

1. GW theory of Y involves $H^1(C, f^*(\mathcal{O}(-d))(-p_2))$;
2. GW theory of Z involves $H^0(C, f^*(\mathcal{O}(d))(-p_1))$.

On the smooth locus $C = \mathbb{P}^1$, these are related by Serre duality.

Example 7.3. Now let's look at an example where both are in play. Let W be a degree-d polynomial in d variables, i.e. corresponding to a section $s \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(d))$, so that

$$Z := \{W = 0\}$$

is CY. Then we have a diagram of relations

$$\begin{array}{ccccc} & & \mathrm{GW}_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{crepant transformation}} & \mathrm{GW}_{g=0}(Y) \\ & \swarrow & & & \swarrow \text{ quantum Serre duality} \\ \mathrm{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \xrightarrow{W} \mathbb{C}) & & \xrightarrow{\text{LG/CY}} & & \mathrm{GW}_{g=0}(Z) \end{array}$$

Here the potential W defines a function on $[\mathbb{C}^d/\mathbb{Z}_d]$. The enumerative theory for such singularities with potential is called FJRW theory.

Theorem 7.4 (YP, Nathan, Mark)**.** *There exists a correspondence on the left of the above diagram.*

Remark. This theorem means we can understand LG/CY in terms of CTC. But how geometric are these correspondences? More precisely, CTC comes from some FM transform at the level of derived categories, but do the rest?

Definition 7.5. Let $K(Y) \to \mathrm{GW}_{g=0}(Y)$ be Iritani's integral structure, then we can draw another square

$$\begin{array}{ccccc} & & K([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{Fourier–Mukai}} & K(Y) \\ & & \downarrow & & \downarrow \\ & & \mathrm{GW}_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{crepant transformation}} & \mathrm{GW}_{g=0}(Y) \\ & \swarrow & & & \swarrow \text{ quantum Serre duality} \\ \mathrm{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \xrightarrow{W} \mathbb{C}) & & \xrightarrow{\text{LG/CY}} & & \mathrm{GW}_{g=0}(Z) \end{array}.$$

Iritani's **integral structure map** comes from

$$\begin{aligned} L(t,z)\colon H^*(X) &\to H^*(X)[[q]][z^{-1}] \\ \alpha &\mapsto \alpha + \sum_d q^d \sum_{n \geq 0} \langle \frac{\alpha}{-z-\psi}, t, \ldots, t, \phi_i \rangle_{0,n+2} \phi^i. \end{aligned}$$

Then $L(t,z)(\alpha)$ is a solution to the Dubrovin connection

$$\nabla_i = \partial_i + \frac{1}{z}\phi_i \star^X (-).$$

Iritani's map is given by

$$\begin{aligned} K(X) &\to \text{solutions to the qDE} \\ E &\mapsto L(t,z)(\mathrm{ch}(E)\Gamma(TX)) \end{aligned}$$

where $\Gamma(TX)$ is a square root of the Todd class.

Definition 7.6. We can add the same square for K-theory of $[\mathbb{C}^d/\mathbb{Z}_d]$ twisted by W, and Orlov tells us we have an equivalence $K([\mathbb{C}^d/\mathbb{Z}_d], W) \to K(Z)$. So we now have a diagram

$$\begin{array}{ccccc}
 & & K([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{Fourier–Mukai}} & K(Y) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{GW}_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & \xrightarrow{\text{crepant transformation}} & \mathrm{GW}_{g=0}(Y) \\
 & \swarrow & & & \swarrow{\scriptstyle \text{quantum Serre duality}} \\
\mathrm{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \xrightarrow{W} \mathbb{C}) & & \xrightarrow{\text{LG/CY}} & \mathrm{GW}_{g=0}(Z) &
\end{array}.$$

If we restrict $K([\mathbb{C}^d/\mathbb{Z}_d])$ to the ones supported only on the origin $[0/\mathbb{Z}_d]$ and the same for $K(Y)$ to the ones supported only on $\mathbb{P}^{d-1}$, then we get arrows

$$K([\mathbb{C}^d/\mathbb{Z}_d])|_{[0/\mathbb{Z}-d]} \to K([\mathbb{C}^d/\mathbb{Z}-d], W)$$
$$K(Y)|_{\mathbb{P}^{d-1}} \to K(Z)$$

The final diagram is

$$\begin{array}{ccccccc}
 & & K([\mathbb{C}^d/\mathbb{Z}_d]) & & \xrightarrow{\text{Fourier–Mukai}} & & K(Y) \\
 & \swarrow & & & & \swarrow & \\
K([\mathbb{C}^d/\mathbb{Z}_d], W) & & \xrightarrow{\text{Orlov}} & & K(Z) & & \downarrow \\
 & & \downarrow & & & & \\
\downarrow & & \mathrm{GW}_{g=0}([\mathbb{C}^d/\mathbb{Z}_d]) & & \xrightarrow{\text{crepant transformation}} & & \mathrm{GW}_{g=0}(Y) \\
 & \swarrow & & & \downarrow & \swarrow{\scriptstyle \text{quantum Serre duality}} & \\
\mathrm{FJRW}([\mathbb{C}^d/\mathbb{Z}_d] \xrightarrow{W} \mathbb{C}) & & \xrightarrow{\text{LG/CY}} & & \mathrm{GW}_{g=0}(Z) & &
\end{array}.$$

Theorem 7.7 (YP, Nathan, Mark)**.** *The back square implies the front square.*

Definition 7.8. What are the arrows in the top square? Look in the simple case of $d = 2$. Let $j: Z \to \mathbb{P}^1$ be the inclusion, and $\pi: Y \to \mathbb{P}^1$ be the projection. Then the diagram is

$$\begin{array}{ccccc}
K([\mathbb{C}^2/\mathbb{Z}_2])_{[0/\mathbb{Z}_2]} & & \xrightarrow{\cong} & & K(\mathrm{tot}(\mathcal{O}_{\mathbb{P}^1}(-2)))_{\mathbb{P}^1} \\
 & \nwarrow{\scriptstyle \in} & & \nearrow{\scriptstyle \in} & \\
 & \mathcal{O}_{[0/\mathbb{Z}_2]} = [\mathbb{C} \xrightarrow{\binom{x_2}{-x_1}} \mathbb{C}^2 \xrightarrow{(x_1 \;\; x_2)} \mathbb{C}] & & [\mathcal{O}_Y(2) \to \mathcal{O}_Y] = \mathcal{O}_{\mathbb{P}^1} & \\
\downarrow & \downarrow & & \downarrow & \downarrow{\scriptstyle j^*\pi_*} \\
 & \mathcal{O}_{[0/\mathbb{Z}_2]} = [\mathbb{C} \underset{\binom{x_2}{-x_1}}{\overset{(x_2 \;\; -x_1)}{\longleftrightarrow}} \mathbb{C}^2 \underset{(x_1 \;\; x_2)}{\overset{\binom{x_1}{x_2}}{\longleftrightarrow}} \mathbb{C}] & & [\mathcal{O}_Y(2) \to \mathcal{O}_Y] = \mathcal{O}_Z & \\
 & \swarrow{\scriptstyle \in} & & \searrow{\scriptstyle \in} & \\
K([\mathbb{C}^2/\mathbb{Z}_2], x_1^2 + x_2^2) & & \xrightarrow{\cong} & & K(Z)
\end{array}.$$

Here we are viewing $\mathbb{P}^1 = \{\ell = 0\} \subset Y$ and taking the Koszul resolution.

1. In general, if $f: X \to Y$ is a proper map, then there is a pushforward

$$f_*: K(X, f^*W) \to K(Y, W)$$

for potentials $W: Y \to \mathbb{C}$. Using this, the vertical left arrow comes from

$$[\mathbb{C}^2/\mathbb{Z}_2] \leftrightarrow_i^p [0/\mathbb{Z}_2].$$

by the composition

$$K([\mathbb{C}^2/\mathbb{Z}_2]) \xrightarrow{p_*} K([0/\mathbb{Z}_2], 0) \xrightarrow{i_*} K([\mathbb{C}^2/\mathbb{Z}_2], x_1^2 + x_2^2).$$

2. Under the isomorphism $K(Z) = K(Y, \tilde{W})$, the structure sheaf $\mathcal{O}_Z$ becomes a factorization version of the Koszul complex:

$$[\mathcal{O}_Y(2) \leftrightarrow \mathcal{O}_Y].$$

Then the map from $K(\mathrm{tot}(\mathcal{O}(-2))) \to K(Y, W)$ arises the same way as the vertical left map.

Remark (Connection to p-fields). Let $Y = \mathcal{O}_{\mathbb{P}^r}(-d)$. In genus 0, recall that the GW theory of Y has virtual class

$$[\overline{\mathcal{M}}_{0,n}(Y, D)]^{\mathrm{vir}} = e(R^1\pi_*\mathcal{L}^{\otimes -d}) \in A_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)).$$

For a hypersurface Z, we can use p-fields:

$$[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, D)^p] = e(R^1\pi_*(\mathcal{L}^{\otimes -d} \otimes \omega_{\log})) \in A_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, D)).$$

If $n = 2$ and C is smooth, then $\omega_{\log} = \mathcal{O}_C$. Hence on the smooth locus, the two virtual classes are identical. This perspective generalizes beyond complete intersections.

8 Apr 02 (Clara): Klein TQFTs

Recall that the **moduli of stable real maps** is

$$\overline{\mathcal{M}}^{n,c}_{d,\chi}(\Sigma)_{\lambda^1,\dots,\lambda^r}.$$

We continue with the notation and setup of last time. Start with a symmetric connected Riemann surface Σ. Real maps have involutions on domains, so we get a splitting of the domain into components of two types:

1. (doublets) pairs of components swapped by the involution;
2. (connected components) components fixed by the involution.

So we need to compute these individually, and then exponentiate. We can relate both to complex invariants.

Example 8.1 (Doublet domains)**.** For doublet domains, the first step is to pass to the double cover of the domain. The **doublet moduli space**

$$\widetilde{D\mathcal{M}}_{d,h}(\Sigma)_{\lambda^1,\dots}$$

has domains with fixed first and second components. There is a forgetful map

$$\varphi: \widetilde{D\mathcal{M}}_{d,h}(\Sigma) \to \overline{D\mathcal{M}}_{d,h}(\Sigma)$$

which gives

$$\overline{D\mathcal{M}}_{d,h}(\Sigma) = \frac{1}{2}\varphi_*[\widetilde{D\mathcal{M}}_{d,h}(\Sigma)]^{\mathrm{vir}}.$$

Then we need to distribute ramification patterns. Let f split as (f_1, f_2) over the two components in the domain. If f_1 has ramification λ^+ over x^+, then f_2 has ramification λ^- over x^+, such that the original is $\lambda = \lambda^+ \sqcup \lambda^-$. So we have a decomposition of the doublet moduli space into pieces

$$\widetilde{D\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}^+,\vec{\lambda}^-}$$

where $\vec{\lambda}^{\pm}$ are ramifications over $x^{\pm}$ in the *first component*. (This fully determines the ramification over the second component.) These pieces are related to the complex moduli $\overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}^+,\vec{\lambda}^-}$ by doubling, and their virtual classes agree up to sign.

Example 8.2 (Sphere relative to a point)**.** Define a *shifted* version of the partition function such that the level-0 real GW series has no non-zero terms of positive degree. (This is a non-trivial vanishing result.)

1. (Doublet) Upon doubling, this becomes connected GW invariants for a complex tube. Such invariants vanish unless both ramification points are fully ramified.
2. (Connected components) By the vanishing result, we need $h - 1 + \ell(\lambda) = 0$. So $\ell(\lambda) = 1$ and $h = 0$.

(Notes stop here, sorry.)

9 Apr 09 (Yunfeng): Vafa–Witten invariants via surface Deligne–Mumford stacks

Definition 9.1. Let's first discuss the background of Vafa–Witten for surfaces. Let S be a smooth projective surface. If we fix some polarization $\mathcal{O}_S(1)$, then we can consider the moduli space $\mathcal{M}^{(r,c_1,c_2)}(S)$ of stable torsion-free sheaves of rank r and Chern classes c_1, c_2 on S. In Donaldson theory, if we fix r, c_1 and let c_2 move, we can write down a generating function

$$F(q) \coloneqq \sum_{c_2 \in \mathbb{Z}} \chi(\mathcal{M}^{(r,c_1,c_2)}) q^{c_2}.$$

S-duality from physics implies that $F(q)$ is a modular form (Vafa–Witten, 94).

Example 9.2. If $S = \mathbb{P}^2$, for $r = 1$ and $c_1 = 0$, then

$$F(q) = \sum_{c_2 \in \mathbb{Z}} \chi(\mathrm{Hilb}^{c_2}(S)) q^{c_2} = \frac{1}{\prod_{k>0}(1-q^k)^3}.$$

If $r = 2$ and $c_1 = 0$, then

$$F(q) = \frac{1}{\prod_{k\geq 1}(1-q^k)^6} \sum_{m,n\geq 1} \frac{q^{mn+m+n}}{-q^{m+n}}.$$

This method can be generalized to any smooth toric surface.

Example 9.3 (Blow-up formula)**.** Let $S = \mathbb{P}^2$ and let $\tilde{S} \to S$ be a blow-up at one point. Then

$$\sum_n \chi(\mathcal{M}^{H_\infty,c_1,n}(\tilde{S})) q^{c_2 - \tilde{c}_1^2/4} = \frac{\sum_{n\in\mathbb{Z}} q^{(n+c_1/2)^2}}{\prod_k (1-q^k)^2} \sum_n \chi(\mathcal{M}^H(c_1,n)) q^{n-c_1^2/4}.$$

Definition 9.4. Let $\mathcal{S}$ be a 2d DM stack. Many examples are interesting here.

1. Take weighted projective planes $\mathbb{P}(a,b,c)$.
2. If S is a smooth surface and $D \subset S$ is a smooth divisor, take the **r-th root stack** $\sqrt[r]{(S,D)}$ of S along D, where we make $D \subset S$ have multiplicity $\mathbb{Z}/r$.
3. Quintic surface has deformations into ADE surfaces, which can be viewed as smooth DM surfaces.

F. Nirumi constructed the moduli stack of stable torsion-free sheaves on $\mathcal{S}$ as follows. Let $p\colon \mathcal{S} \to \mathsf{S}$ be the map to the coarse moduli space. Choose a **generating sheaf** Ξ: a locally free sheaf on $\mathcal{S}$ which is relatively

p-very ample. We use this to define the Hilbert polynomial, so we don't lose any stacky information. For any $\mathcal{E} \in \mathsf{Coh}(\mathcal{S})$, the **Hilbert polynomial** is

$$H_\Xi(\mathcal{E}, m) \coloneqq \chi(\mathcal{S}, \mathcal{E} \otimes \Xi^\vee \otimes p^* \mathcal{O}_{\mathsf{S}}(m)) = \sum_{i=0}^{\dim} \alpha_{\Xi,i} \frac{m^i}{i!}.$$

Define the **reduced** Hilbert polynomial as $h_\Xi(\mathcal{E}) \coloneqq H_\Xi(\mathcal{E})/\alpha_{\Xi,\dim}$. We say $\mathcal{E}$ is **Gieseker stable** if for any subsheaf $\mathcal{F} \subset \mathcal{E}$,

$$h_\Xi(\mathcal{F}) < h_\Xi(\mathcal{E}).$$

Remark. In the root stack case $\mathsf{S} = \sqrt[d]{(S, D)}$, we can choose

$$\Xi \coloneqq \mathcal{O}_S \oplus \mathcal{O}_S(D^{1/d}) \oplus \cdots \oplus \mathcal{O}_S(D^{(d-1)/d}).$$

With this choice, Gieseker stability is equivalent to **parabolic stability** on the pair (S, D). This means we take a parabolic structure, i.e. a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_d = \mathcal{E}(-D)$$

along the divisor D, and require stability with respect to this filtration.

Proposition 9.5 (Nirumi)**.** *Fix the Hilbert polynomial H. Then $\mathcal{M}_H(\mathcal{S})$ exists as a projective scheme.*

Definition 9.6. Let

$$\tilde{\mathrm{ch}}\colon K_*(\mathcal{S}) \to H^*(I\mathcal{S})$$

be the **orbifold Chern character**. In the decomposition $I\mathcal{S} = \bigsqcup_{i \in I_S} \mathcal{S}_i$, write

$$\tilde{\mathrm{ch}}(\mathcal{E}) \coloneqq (\tilde{\mathrm{ch}}_i(\mathcal{E}) \in H^*(\mathcal{S}_i)), \quad (\tilde{\mathrm{ch}}_i)^k \in H^{\dim \mathcal{S}_i - k}(\mathcal{S}_i).$$

Then the generating function we look at is

$$H_{\alpha,\beta}(q) \coloneqq \sum_{\substack{\tilde{\mathrm{ch}}^2(C)=\alpha \\ \tilde{\mathrm{ch}}^1(C)=\beta}} \chi(\mathcal{M}_{\Xi,C}(\mathcal{S})) q^C.$$

Example 9.7. Let $\mathcal{S} = \mathbb{P}(1,1,2)$, so that $I\mathcal{S} = \mathbb{P}(1,1,2) \sqcup B\mu_2$. Let $\Xi \coloneqq \mathcal{O}_{\mathcal{S}} \oplus \mathcal{O}_{\mathcal{S}}(-1)$ be the generating sheaf. If we fix $C \in K(\mathcal{S})$, then

$$(\tilde{\mathrm{ch}}_1)^2 = 2, \quad (\tilde{\mathrm{ch}}_1)^1 = c_1(\mathcal{S}).$$

But we have other terms:

1. q_1 will keep track of $c_2(\mathcal{E}) = (\tilde{\mathrm{ch}}_1)^0$;
2. q_2 keeps track of $(\tilde{\mathrm{ch}}_\zeta)^0$ where $\zeta^2 = 1$ is the generating of μ_2.

The generating function is therefore $F(q_1, q_2) = \mathcal{H}_{2,c_1(\mathcal{S})}(q_1, q_2)$. In the case where $q_1 = q$ and $q_2 = 1$, we can compute it to be

$$\left(\frac{q^{1/6}}{\eta(q)^4} \theta_3(q) \right)^2 \sum_{(w_1,w_2,w_3) \in C_{c_1}} q^{c_1^2/4 + (w_1^2 + w_2^2 + w_3^2)/4}$$

where C_{c_1} is the restrictions $2 \mid c_1 + \sum w_i$ and $2 \mid w_2$ and $w_i < w_j + w_k$ for $\{i,j,k\} = \{1,2,3\}$. The second term counts contributions from the moduli of bundles, while the first term counts the rest. The moduli of bundles arises because any torsion-free sheaf $\mathcal{E}$ embeds into its reflexive hull $\mathcal{E}^{\vee\vee}$, which is locally free for surfaces. (This is a general phenomenon for surfaces.)

Definition 9.8 (Vafa–Witten)**.** A **Higgs bundle** $(\mathcal{E}, \phi)$ on $\mathcal{S}$ is a torsion-free coherent sheaf $\mathcal{E}$ and a section $\phi: \mathcal{E} \to \mathcal{E} \otimes K_{\mathcal{S}}$. The usual spectral construction relates this to DT invariants on 3-folds. Let

$$\mathcal{X} \coloneqq \mathrm{tot}(K_{\mathcal{S}})$$

so that $\mathcal{X}$ is a CY3 DM stack.

Proposition 9.9 (Spectral construction)**.** *There exists an equivalence of abelian categories*

$$\mathsf{Higgs}_{K_{\mathcal{S}}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{Coh}_c(\mathcal{X}).$$

Proof. If we let $\pi: \mathcal{X} \to \mathcal{S}$ be the projection, then there is a decomposition

$$\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_S + K_{\mathcal{S}}^{-1} + K_{\mathcal{S}}^{-2} + \cdots.$$

Then use that $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$ is equivalent to $\pi_* \mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{S}$. □

Remark. We can still define Gieseker invariance for these Higgs bundles, by only imposing the stability condition for ϕ-invariant subsheaves. Using appropriate pullbacks of the polarization, Gieseker stability for $\mathsf{Higgs}_{K_{\mathcal{S}}}(\mathcal{S})$ matches Gieseker stability on $\mathsf{Coh}_c(\mathcal{X})$.

Definition 9.10. The symmetric obstruction theory on $\mathsf{Coh}_c(\mathcal{X})$ now induces a symmetric obstruction theory on $\mathsf{Higgs}_{K_{\mathcal{S}}}(\mathcal{S})$ which is exactly the usual obstruction theory. Vafa–Witten invariants are defined using this obstruction theory:

$$\widetilde{\mathrm{VW}} \coloneqq \int_{[\mathcal{N}^{\mathbb{C}^\times}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}$$

where $\mathcal{N}$ is the moduli. This is not entirely correct: there is a trivial factor in the obstruction theory corresponding to the trace. Actual **Vafa–Witten invariants** are defined on the moduli $\mathcal{N}_L^\perp$ of *fixed* determinant L and trace-free (E, ϕ).

$$\mathrm{VW} \coloneqq \int_{[(\mathcal{N}_L^\perp)^{\mathbb{C}^\times}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}.$$

We can also define

$$\mathrm{vw} \coloneqq \chi(N_L^\perp, \nu)$$

using a Behrend function ν.

Remark. In general, VW and vw are not the same. However, the invariant vw agrees with the one coming from cosection localization, where the cosection comes from the $\mathbb{C}^\times$-action on the symmetric obstruction theory.

Proposition 9.11 (Tanaka–Thomas)**.** VW = vw *if S is Fano.*

Proposition 9.12 (Maulik–Thomas)**.** VW = vw *if S is K3.*

Definition 9.13. To calculate the invariant, we have to study the $\mathbb{C}^\times$-fixed locus on $\mathcal{N}_L^\perp$. There are two kinds of fixed locus.

1. $(\phi = 0)$ This is the moduli $\mathcal{M}_L$ of stable sheaves on S. This is called the **instanton branch** $\mathcal{M}^1$.
2. $(\phi \neq 0)$ This is the case of ϕ nilpotent and is called the **monopole branch** $\mathcal{M}^2$. Gholampour–Thomas proved that $\mathcal{M}^2$ in general is a nested Hilbert scheme on S.

Remark. S-duality from physics swaps the monopole and instanton branches via the transformation $\tau \mapsto -1/\tau$. In general, it is easier to compute on the monopole branch, so this gives a lot of conjectural data about Donaldson invariants coming from the instanton branch.

Remark. Yunfeng computed two cases:

1. root stacks of quintic surfaces;
2. quintic surfaces with ADE singularities. These are deformation invariant to smooth quintics, so VW should remain the same. But vw is *not* deformation invariant, so we expect different answers.

10 Apr 16 (Shuai): Relative Gromov–Witten theory and vertex operators

In many mathematical problems, we have a space $\mathcal{M}$ and an associated linear space $V_{\mathcal{M}}$ on which a Hamiltonian H acts, and we care about solutions to the equation

$$Hf = \frac{\partial}{\partial t} f.$$

For example: modular forms, wave equation, heat equation. If we diagonalize H, we turn the DE problem into a linear algebra problem:

$$\lambda^2 \hat{f} = \frac{\partial}{\partial t} \hat{f}.$$

Shuai's slogans (v1.0):

1. Better universes exist;
2. Representation theory as the exploitation of symmetry;
3. Geometry encodes complicated algebraic information.

Concretely, consider the bundle $T^*\mathbb{P}^1 \to \mathbb{P}^1$, where $\mathbb{P}^1$ has weight a and the bundle has linearization $(\hbar + a, \hbar - a)$. Note that

$$e(T^*\mathbb{P}^1)|_0 = -a(\hbar + a), \quad e(T^*\mathbb{P}^1)|_\infty = a(\hbar - a).$$

On the DT side, $\operatorname{Hilb}(T^*\mathbb{P}^1)$ has divisor classes $M_{(2)}$ and $M_{(1^d)}$.

From elementary number theory, we know that a prime p is the sum of two squares iff $p \equiv 1 \bmod 4$. One way to prove this is to look at the lattice $\mathbb{Z}[i]$. The geometric question to ask is: over which primes p do we have ramification in the map $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$. Alternatively, we want to ask which points in the lattice $\mathbb{Z}[i]$ have norm p.

Now let's return to $T^*\mathbb{P}^1$ and $\mathbb{P}^1$. We understand the equivariant quantum cohomology $\mathrm{QH}^*_T(\mathbb{P}^1)$ very well. Why would we want to look at $\mathrm{QH}^*_T(T^*\mathbb{P}^1)$? It fits nicely into $\bigsqcup_k \operatorname{Gr}(k, 2)$, and the cohomology

$$H^*(\mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt}) = \mathbb{C} \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus \mathbb{C}$$

is a 4-dimensional rep of $\mathfrak{sl}_2$. The operator of quantum multiplication $u\star$ can now be understood in terms of the representation theory of $\mathfrak{sl}_2$. We will find

$$u\star = u \cup + \frac{q}{1-q}[\mathbb{P}^1 \times \mathbb{P}^1].$$

We will see that

$$\mathrm{QH}^*_T(T^*\mathbb{P}^1) = \mathbb{C}[\hbar, a, u]/\langle u^2 - au - \hbar\frac{q}{1-q}(\hbar + a - 2u)\rangle.$$

The projection to $\mathrm{QH}^*_T(\mathbb{P}^1)$ is given by the $\hbar^2 q$ coefficient.

To compute quantum multiplication by u, it suffices to compute

$$(u \star u, 1) = \langle u, u, 1\rangle_{0,3,0} + \sum q^k \langle u, u, 1\rangle_{0,3,n}$$
$$(u \star u, u) = \langle u, u, u\rangle_{0,3,0} + \sum q^k \langle u, u, u\rangle_{0,3,n}.$$

Let's compute the first term.

1. The classical part is

$$\langle u, u, 1\rangle_{0,3,0} = \int_{T^*\mathbb{P}^1} u^2 = a\int_{T^*\mathbb{P}^1} u = a\left(\frac{u}{-a(a+\hbar)} + \frac{a}{a(\hbar - a)}\right) = \frac{a}{\hbar - a}.$$

2. The quantum part is
$$\langle u, u, 1\rangle_{0,3,n} = \langle 1\rangle_{0,1,n} = 0$$
by divisor relation (twice).

Now for the second term.

1. The classical part is
$$\langle u, u, u\rangle_{0,3,0} = a^2 \int_{T^*\mathbb{P}^1} u = \frac{a^2}{\hbar - a}.$$

2. The quantum part is
$$\langle u, u, u\rangle_{0,3,n} = \langle\rangle_{0,0,n}$$
by divisor relation (three times). Then $\overline{\mathcal{M}}_{0,0}(T^*\mathbb{P}^1, 1) = \mathrm{pt}$, but the virtual class is $\hbar$. In general this is true in degree n as well.

So we find
$$(u \star u, 1) = \frac{a}{\hbar - a}, \quad (u \star u, u) = \frac{a^2}{\hbar - a} + \sum q^K \hbar.$$

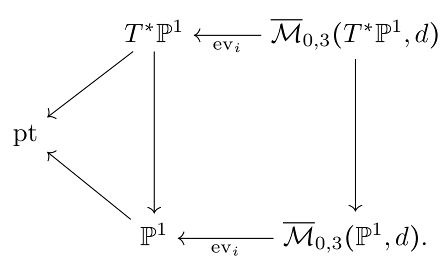
What if we want to compute $c_1\star$? Look at the Steinberg Z. We want to argue that
$$c_1 \star = c_1 \cup + \hbar \frac{q}{1-q}[\mathbb{P}^1 \times \mathbb{P}^1]$$
where $\mathbb{P}^1 \times \mathbb{P}^1 \subset Z$ is the diagonal. Let $\pi_1, \pi_2 : Z \to T^*\mathbb{P}^1$ be the projections. Then we only need to compute the terms
$$\int_{T^*\mathbb{P}^1} \pi_{2*}([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1^*(u)) \cdot 1, \quad \int_{T^*\mathbb{P}^1} \pi_{2*}([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1^*(u)) \cdot u.$$

1. By push-pull, we get
$$\begin{aligned}\int_{T^*\mathbb{P}^1} \pi_{2*}([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1^*(u)) \cdot 1 &= \int_Z [\mathbb{P}^1 \times \mathbb{P}^1]\pi_1^*(u) \\ &= \frac{a}{-a^2} + \frac{a}{a^2} = 0.\end{aligned}$$

2. Again by push-pull, we get
$$\int_{T^*\mathbb{P}^1} \pi_{2*}([\mathbb{P}^1 \times \mathbb{P}^1] \cdot \pi_1^*(u)) \cdot u = 1.$$

So we just have to check that $\int (\hbar + a - 2u) \cdot 1 = 0$ and $\int (\hbar + a - 2u) \cdot u = 1$, which it does. Hence
$$c_1 \star = c_1 \cup + \hbar \frac{q}{1-q}(\hbar + a - 2u).$$

Now let's look at $\overline{\mathcal{M}}_{0,3}(T^*\mathbb{P}^1, d) \to \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, d)$, i.e. how to take the limit in $\hbar$. It fits into a diagram



Let's compare virtual localization on both. In general, localization takes the form

$$\int \prod \mathrm{ev}_i^*(\gamma_i) = \sum_{\text{fixed}} \int_{\mathcal{M}_\Gamma} \frac{\prod i^*(\gamma_i)}{N_{\Gamma/X}^{\mathrm{vir}}}.$$

Because the diagram above commutes, the numerators in both cases are the same. The only difference is a factor $e(f^*\mathcal{O}(-2))$. By Riemann–Roch, this is $1-2d$. So downstairs the coefficients we compute are for q^d, and upstairs the coefficients are for $\hbar^{2d-1}q^d$. It turns out there aren't many terms like $\hbar^{2d-1}q^d$ upstairs. In general, there is a complicated change of variables and limit we have to take.

Now look at 2-pointed relative GW invariants for $X \coloneqq T^*\mathbb{P}^1 \times \mathbb{P}^1$ with relative conditions μ, ν at 0 and ∞. We can compare it with $Y \coloneqq \mathbb{P}^1 \times \mathbb{P}^1$:

$$\int_{\overline{\mathcal{M}}_{g,1}(X,\mu,\nu)} \mathrm{ev}^*(i^*\omega) \eqqcolon \langle \mu | M_{(1,\omega)} | \nu \rangle$$
$$\int_{\overline{\mathcal{M}}_{g,1}(Y,\mu,\nu)} \mathrm{ev}^*(\mathrm{pt}) \eqqcolon \langle \mu | M_{(u,Q)} | \nu \rangle.$$

The difference in virtual normal bundles is still $e(f^*\mathcal{O}(-2))$, which again by Riemann–Roch becomes

$$(1-g)1 + ((dV + mH), -2H) = 1 - g - 2d.$$

Hence coefficients of $u^{2g-2}v^d H^m$ downstairs become coefficients of $u^{2g-2}v^d H^m \hbar^{2d+g-1}$. Let $Z_{(\mu,(1,\omega),\nu)}$ denote the generating function. It is given by

$$u^{-\ell(\mu)-\ell(\nu)} s \frac{d}{ds}\Theta, \quad \Theta = \frac{\hbar}{(\mathrm{Aut}\,\mu)(\mathrm{Aut}\,\nu)} \sum (du)^{\ell(\mu)+\ell(\nu)-2} \frac{\prod_{i=1}^{\ell(\mu)} s(d\mu_i u) \prod_{i=1}^{\ell(\nu)} s(d\nu_i u)}{ds^2(du)}$$

where $s(u) \coloneqq \sin(u/2)/(u/2)$. For example, if $\mu = \nu = \square$, then $Z = \hbar u^{-2} \sum_{d=1}^\infty s^d$, and

$$\langle \square | \mathrm{pt} | \square \rangle = [\hbar u^{-2} s] Z = 1.$$

If $|\mu| = |\nu|$, then we will need the $\hbar u^{-2} s$ coefficient of

$$u^{-2}\hbar \frac{d^{\ell(\mu)+\ell(\nu)-2}}{|\mathrm{Aut}\,\mu||\mathrm{Aut}\,\nu|} \frac{\prod \cdots \prod \cdots}{\prod \cdots}.$$

The $\cdots$ are of the form $\sin(u)/u$, and in the limit contribute just 1.

In Maulik's $\mathcal{A}_n$-resolutions paper, the quantum multiplication operator is written as

$$M_{(1,\omega)} = M_{(1,\omega)} \cup + s\frac{d}{ds} \sum_{k \in \mathbb{Z}} {:}f(k)e(-k){:} \log(1 - q^k s).$$

We can match

$$\sum_{k\in\mathbb{Z}} {:}f(k)e(-k){:} = \sum_{|\mu|=|\nu|} \alpha_{-\mu}\alpha_\nu$$

by showing the LHS is the zeroth order term in the product $\Gamma_+\Gamma_-$ of two vertex operators.

11 Apr 30 (Shuai)

Sorry, no notes!

12 May 07 (Zijun): 3d mirror symmetry and elliptic stable envelopes

Let's begin with some motivation from physics. Mirror symmetry is a phenomenon coming from physics. For 3d theories, it is called symplectic duality mathematically. We start with a 3d N=4 supersymmetric (susy) gauge theory. Physicists found that for some special such theories, there is a *mirror* theory in the following sense. The moduli space of vacua of 3d N=4 theories have two branches: **Higgs** and **Coulomb**. They involve **FI** and **mass** parameters. The mirror theory has Higgs and Coulomb branches exchanged, and FI and mass parameters exchanged.

Mathematically, $N = 4$ susy means the moduli of vacua is hyperkähler. The Higgs branch is a holomorphic symplectic quotient and is well understood. The Coulomb branch was not understood mathematically until recently, by work of Braverman–Finkelberg–Nakajima. FI parameters are Kähler parameters, and mass parameters are equivariant variables.

Example 12.1. Some examples of Higgs/Coulomb branches are as follows.

1. When the gauge group is abelian, the Higgs branch is a hypertoric variety, and the Coulomb branch is the dual hypertoric variety.
2. We will see later what $T^* \mathrm{Gr}$ is dual to.
3. T^*G/B is self-dual (but maybe using different Weyl chambers).
4. $\mathrm{Hilb}\,\mathbb{C}^2$ is self-dual.
5. The moduli $\mathcal{M}(\mathcal{A}_n, r+1)$ of instantons on $\mathcal{A}_n$ surface is dual to moduli $\mathcal{M}(\mathcal{A}_r, n+1)$ of instantons on $\mathcal{A}_r$ surface. (For $r = n = 0$ we get the Hilb case.)

These are very nice examples. In general, the BFN construction of Coulomb branches gives a very singular affine scheme which has non-commutative deformations. Because we want to do enumerative geometry with these spaces as targets, singularities are bad. The examples above are nice cases.

Let X denote the Higgs branch and X' denote the Coulomb branch. Via the symplectic duality conjecture, we implicitly identify X, X' of the original theory with X', X of the mirror theory. If we have enumerative invariants defined on X, we expect them to be related to analogous enumerative invariants on X', but with Kähler and equivariant parameters swapped.

Now let's say something about stable envelopes. The first motivation for stable envelopes is that they form a very nice basis for cohomology (and K-theory and elliptic cohomology). They depend on a choice of cocharacter and satisfy some wall-crossing formulas. The transition matrix arising from the wall-crossing defines an R-matrix, using which we can form quantum group actions on cohomology. The second motivation is special to elliptic stable envelopes. Consider the **vertex** $V(X) \in K_T(X)[[z]]$ where X is a Nakajima quiver variety, defined by counting quasimaps from $\mathbb{P}^1 \dashrightarrow X$. The vertex $V(X)$ is a solution to two different quantum difference equations (qDEs): qKZ, coming from q-shifts of equivariant parameters a, and another coming from q-shifts of Kähler parameters z. The vertex $V(X)$ is holomorphic in z and meromorphic in a. Call a solution with such properties a z**-solution**.

Theorem 12.2 (Aganagic–Okounkov)**.** *Let X be a Nakajima quiver variety.*

1. *Elliptic stable envelopes $(\mathrm{Stab}(p)|_q)_{p,q\in X^T}$ exist.*
2. *$(\mathrm{Stab}(p)|_q)_{p,q\in X^T}$ maps $(V(X)|_p)_{p\in X^T}$ to another set of solutions to the qDEs which are holomorphic in a and meromorphic in z, called a-**solutions**.*

Conjecture 12.3. *These a-solutions are vertex functions $V(X')$ of the dual X'.*

As a corollary of this conjecture, we get the following.

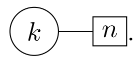
Conjecture 12.4. *There should be relations like*

$$\frac{\mathrm{Stab}(q)|_p}{\mathrm{Stab}(p)|_q} = \frac{\mathrm{Stab}'(\lambda)|_\mu}{\mathrm{Stab}'(\mu)|_\lambda}, \quad p, q \in X^T,\ \lambda, \mu \in (X')^{T'}$$

under a bijection of fixed point sets and exchanging Kähler and equivariant parameters.

Theorem 12.5 (RSVZ)**.** *This second conjecture holds for* $T^*\,\mathrm{Gr}$ *and its dual.*

Now let's go into more detail. Let $n \geq k$. Construct $T^*\,\mathrm{Gr}(k,n)$ as the Nakajima quiver variety associated to



Explicitly, in this case,

$$T^*R = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$$

acted on by $G \coloneqq \mathrm{GL}(k)$. If $(i,j) \in T^*R$, the action is

$$g \cdot (i,j) = (gi, jg^{-1}),$$

giving a moment map

$$\mu\colon (i,j) \mapsto ij \in \mathrm{End}(\mathbb{C}^k).$$

To do holomorphic symplectic reduction, we need a stability condition $\theta \in \mathbb{Z}$, specifying a character $g \mapsto (\det g)^\theta$. For $\theta < 0$, we get

$$\mu^{-1}(0) /\!\!/_\theta\, G = T^*\{k\text{-subspaces of } \mathbb{C}^n\}.$$

For $\theta > 0$, we get

$$\mu^{-1}(0) /\!\!/_\theta\, G = T^*\{k\text{-quotients of } \mathbb{C}^n\}.$$

We will stick with $\theta < 0$. Let $X \coloneqq \mu^{-1}(0) /\!\!/_\theta\, G$.

In X, there is a $(\mathbb{C}^\times)^n$ acting on the $\mathbb{C}^n$, and a $\mathbb{C}^\times_\hbar$ acting on cotangent fibers. Let $T \coloneqq (\mathbb{C}^\times)^n \times \mathbb{C}^\times_\hbar$. Then

$$X^T \coloneqq \{\text{coordinate } k\text{-planes}\}.$$

Index them by k-subsets $p = \{p_1, \dots, p_k\} \subset \{1, \dots, n\}$. If $\mathcal{V}$ is the tautological bundle, then the equivariant K-theory of X is

$$K_T(X) = \mathbb{C}[\underbrace{u_1^\pm, \dots, u_n^\pm}_{\text{equivariant params}}, \underbrace{y_1^\pm, \dots, y_k^\pm}_{\text{Chern roots of } \mathcal{V}}, h^\pm]^{S_k}/(\text{relations}).$$

The relations are given by things like

$$\mathcal{V}|_p = u_{p_1}^{-1} + \cdots + u_{p_k}^{-1}, \quad \forall p \in X^T.$$

In other words, at $p \in X^T$, we set $y_i = u_{p_i}^{-1}$.

Now we want to look at $\mathrm{Spec}\, K_T(X)$. By viewing $\hbar$ as a constant, there are maps

$$\begin{array}{ccc} \mathrm{Spec}\, K_T(X) & \hookrightarrow & \mathrm{Spec}\, \mathbb{C}[u_1^\pm, \dots, u_n^\pm, y_1^\pm, \dots, y_k^\pm]^{S_k} = (\mathbb{C}^\times)^n \times \mathrm{Sym}^k \mathbb{C}^* \\ \downarrow & & \\ \mathrm{Spec}\, \mathbb{C}[u_1^\pm, \dots, u_n^\pm] = \mathrm{Spec}\, K_T(\mathrm{pt}) & & \end{array}.$$

To move to elliptic cohomology, fix $q \in \mathbb{C}$ with $|q| < 1$, and take the elliptic curve $E \coloneqq \mathbb{C}^\times / q^{\mathbb{Z}}$. Then there are maps

$$\begin{array}{ccc} \mathrm{Ell}_T(X) & \hookrightarrow & E^n \times \mathrm{Sym}^k E \ni (u_1, \dots, u_n, y_1, \dots, y_n) \\ \downarrow & & \\ \mathrm{Ell}_T(\mathrm{pt}) = E^n \ni (u_1, \dots, u_n) & & \end{array}.$$

Since X is GKM, X^T is finite and has finitely many 1-dimensional orbits. In K-theory, this means

$$K_T(X) \hookrightarrow \bigoplus_{p \in X^T} K_T(p),$$

and the image is given by

$$\{(f_p) : f_p|_{\ker \chi_C} = f_q|_{\ker \chi_C}\}$$

where C is a 1-dimensional orbit connecting p and q and χ_C is the hyperplane character. This GKM description of $K_T(X)$ actually implies

$$\mathrm{Ell}_T(X) = \bigcup_{p \in X^T} \mathcal{O}_p, \quad \mathcal{O}_p \cong E^n$$

if it is simple normal crossing.

Example 12.6. If $X = T^*\mathbb{P}^n$, then

$$K_T(X) = \operatorname{Spec} \mathbb{C}[u_1^\pm, \ldots, u_n^\pm, y^\pm, \hbar^\pm]/(1 - u_1 y) \cdots (1 - u_n y)$$

and $\operatorname{Spec} K_T(p_i) = \{1 - u_i y = 0\}$.

Let $E_T(X) \coloneqq \mathrm{Ell}_T(X) \times E^{\operatorname{rank} \operatorname{Pic}(X)}$. Choose a cocharacter $\sigma = (1, 2, \ldots, n) \in \mathbb{R}^n$. This induces an ordering of fixed points: $p < q$ iff $p_i < q_i$.

Definition 12.7. For $p \in X^T$, define $\mathrm{Stab}_\sigma(p)$ as the unique section of the line bundle $\mathcal{T}(p)$ on $E_T(X)$ such that:

1. $$\mathrm{Stab}(p)|_p = \prod_{\substack{i \in p, j \notin p \\ i < j}} \theta\left(\frac{u_i}{u_j}\right) \prod_{\substack{i \in p, j \notin p \\ i > j}} \theta\left(\hbar^{-1} \frac{u_i}{u_j}\right);$$

2. $$\mathrm{Stab}(p)|_q = f_{p,q} \cdot \prod_{\substack{i \in q, j \notin q \\ i > j}} \theta\left(\hbar^{-1} \frac{u_j}{u_i}\right)$$

   where $f_{p,q}$ is holomorphic in the u_i.

$\mathcal{T}(p)$ is a line bundle coming from $E^n \times E^k \times E$ equivariant with respect to S_k acting on E^k, thereby descending to $E^n \times \operatorname{Sym}^k E \times E$. Because it is a line bundle on elliptic curves, sections can be given in terms of theta functions. Here our θ is the **Jacobi theta**, satisfying

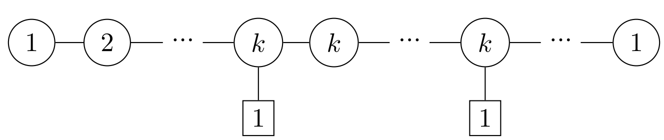
$$\theta(qx) = -q^{-1/2} x^{-1} \theta(x).$$

In other words, it is a section of the line bundle $\mathcal{O}(e)$ on E, where e is the identity.

Remark. There is an explicit formula for the stable envelope coming from the **abelianization** process. It is

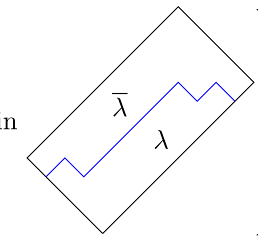
$$\mathrm{Stab}(p) = \mathrm{Sym}_{y_1, \ldots, y_k} \frac{\prod_{\ell=1}^k \left(\prod_{i < p_\ell} \theta(y_\ell u_i \hbar^{-1})\right) \cdot \frac{\theta(u_\ell u_{p_i} z^{-1} \hbar^{k-n+p_i-2\ell})}{\theta(z^{-1} \hbar^{k-n+p_i-2\ell})} \prod_{i > p_i} \theta(u_i^{-1} u_i^{-1})}{\prod_{1 \le i < j \le k} \theta(y_i/y_j) \theta(\hbar y_i / y_j)}.$$

The dual X' for $n \geq 2k$ is given by the A_{n-1} quiver:



We can represent the dimensions of the vertices as a slanted $k \times (n-k)$ rectangle. Let ν_i be the number of boxes in the i-th diagonal. The Chern roots x_I are indexed by boxes I. There are two equivariant parameters a_1, a_2, and $n+1$ Kähler parameters $z_1, \dots, z_{n+1}$. The fixed points are

$$(X')^{T'} = \left\{ \text{partitions in} \right.$$

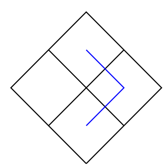


$$\left. \right\}.$$

Such a partition is given by a path between opposite corners. The bijection $(X')^{T'} \leftrightarrow X^T$ is given by taking the path and writing the sequence of direction changes. This is a standard bijection in e.g. the boson-fermion correspondence. For X', we also have an explicit formula for the stable envelope:

$$\mathrm{Stab}'_{\sigma'}(\lambda) = \mathrm{Sym}_{\prod_{i=1}^{n+1} S_n} \left(S_\lambda^{n,k} \sum_{t,\bar{t}:\text{trees}} w(t,\bar{t}) \right)$$

where $S_\lambda^{n,k}$ is a monomial of theta functions, and depends on the partition λ, Chern roots, and equivariant parameters (but is independent of Kähler parameters). The $w(t,\bar{t})$ term is also monomial in theta functions, and depends on λ, Chern roots, and Kähler parameters. Trees must satisfy the constraint that no edges of the form



can appear in any 2×2 box. If λ contains $m(\lambda)$ number of 2×2 boxes, then the number of trees is $2^{m(\lambda)+m(\bar{\lambda})}$.

From the theorem, we can therefore obtain identities of theta functions. For $\mathrm{Gr}(2,4)$, we can recover the standard three-term identity of theta functions.