Notes for Learning Seminar on Symplectic Duality (Spring 2019)

Henry Liu

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Abstract

These are my live-texed notes for the Spring 2019 student learning seminar on symplectic duality. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 Sam (Feb 07): Instantons, ALE spaces, quiver varieties

The starting point is Yang–Mills theory.

Definition 1.1 (Yang–Mills theory)**.** Let (M, g) be a Riemannian/Lorentzian manifold. (The physicists are ultimately interested in the Lorentzian case.) Let G be a compact Lie group. The space of *physical fields* are connections A on a vector bundle E such that $\mathrm{End}(E) = \mathfrak{g}$. Explicitly, a connection will therefore be

$$\nabla_A : \Omega^0(E) \to \Omega^1(E),$$

locally given by a matrix of 1-forms. It extends to a map

$$d_A \colon \Omega^k(E) \to \Omega^{k+1}(E).$$

We care about its square $d_A^2 = F_A \wedge -$, where F_A is some 2-form. In coordinates, it is $F_{ij} = [\nabla_i, \nabla_j]$. The **field strength** associated to a connection A is $|F_A|^2$ (which is a function of the metric g). The **Yang–Mills functional** is

$$S_{\mathrm{YM}}[A] \coloneqq \int_M |F_A|^2 \, d\mu = \int_M \operatorname{tr}(F_A \wedge \star F_A) \, d\mu.$$

Remark. The Euler–Lagrange equation for this functional is $d_A^* F_A = 0$, where $d_A^* = \star d_A \star$ is the adjoint of d_A. By the Bianchi identity, $d_A F_A = 0$. This makes it easy to construct solutions to the Euler–Lagrange equations, because if

$$\star F_A = \pm F_A$$

then we automatically satisfy this.

Definition 1.2 ((A)SD connections)**.** Since $\star|_{\Omega^2}$ satisfies $\star^2 = 1$, we can split Ω^2 into eigenspaces with eigenvalues ± 1, i.e.

$$\Omega^2 = \Omega^+ \oplus \Omega^-,$$

with projections $F_A \mapsto F_A^\pm$. From this perspective, if $F_A = F_A^+$ or $F_A = F_A^-$, then $d_A^* F_A = 0$ and we call such connections **self-dual (SD)** or **anti-self-dual (ASD)**.

Example 1.3 (Electromagnetism)**.** For $G = U(1)$, it turns out we get Maxwell's equations from this. This is by splitting F_A as follows:

$$F_A = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

where B is the magnetic field and E is the electric field.

Definition 1.4 (Instantons)**.** Move now to a Riemannian manifold (M, g), so that now we have a Euclidean version of the action $S_E[A]$ and all of the above theory carries through. **Instantons** are solutions to the Euler–Lagrange equation $d_A^* F_A = 0$ such that the total Euclidean action $S_E[A]$ is finite.

Definition 1.5 (Hyperkähler manifolds)**.** Take a complex structure I on (M, g). It gives a hermitian metric given by

$$h(X, Y) \coloneqq g(X, Y) + ig(X, IY), \quad \omega(X, Y) \coloneqq g(X, IY).$$

Here ω is a real $(1,1)$-form with respect to I. A **hyperkähler structure** is when we have three compatible complex structures I, J, K on (M, g), i.e.

$$I^2 = J^2 = K^2 = IJK = -1$$

and all three are integrable. Doing the same procedure, we get three real $(1,1)$-forms $\omega_I, \omega_J, \omega_K$. In fact, we get an entire 2-sphere of complex structures by quaternionic rotation of I, J, K.

Definition 1.6. We know ω_J is a real $(1,1)$-form with respect to J, so that with respect to I,

$$\omega_J + i\omega_K \in \Omega^{2,0}$$

is a **holomorphic symplectic form**. (This works with respect to J and K as well.)

Proposition 1.7. *The SD 2-forms on a hyperkähler manifold are spanned by ω_I, ω_J, and ω_K.*

Proof. We can count dimensions and see that Ω^+ is rank-3. But locally, $\omega_I = dx_1 dx_2 + dx_3 dx_4$. The Hodge dual $\star$ exchanges dx_1, dx_2 and dx_3, dx_4, so that $\omega_I = \omega_I^+$. This is also true for ω_J and ω_K. Since these are all non-degenerate sections of Ω^+, they form a frame and span Ω^+. ☐

Corollary 1.8. *F_A is ASD iff F_A is a $(1,1)$-form and $F_A \in \omega_I^\perp \subset \Omega^{1,1}$.*

Proposition 1.9. *The connection A is ASD iff A is compatible with the unitary connection on E and $(F_A, \omega_I) = 0$.*

Proof. We can write down the ASD equation in the following form:

$$A \text{ ASD} \quad \Longleftrightarrow \quad \begin{pmatrix} [\nabla_1 + i\nabla_2, \nabla_3 + i\nabla_4] = 0 \\ [\nabla_1, \nabla_2] + [\nabla_3, \nabla_4] = 0 \end{pmatrix}.$$

The top equation is $\bar{\partial}_A^2 = 0$, and the bottom equation is $(F_A, \omega_I) = 0$. We also know that all this works for any of the other complex structures coming from I, J, K. □

The way to actually get solutions to these equations is to reduce the problem to a quadratic linear algebra problem. The way this happens is kind of magical. We will start with some vector space of matrices, and then take a quotient by a Lie group which will preserve some symplectic form on the vector space. The action will give exactly these ASD equations as the moment map.

Definition 1.10 (Symplectic reduction)**.** Let G be a compact Lie group acting on a symplectic manifold (M, ω). The infinitesimal action is

$$\mathfrak{g} \to \{\Gamma(TM) : L_X\omega = 0\}.$$

Make the assumption that all the vector fields created from the infinitesimal action are **Hamiltonian**. From this assumption, the map

$$(\mathfrak{g}, [,]) \to (C^\infty(M), \{-,-\})$$

induces a **moment map**

$$(\mathfrak{g}^*, \{-,-\}) \xleftarrow{\mu} (M, \{-,-\}).$$

We call

$$M /\!\!/ G \coloneqq \mu^{-1}(0)/G$$

the **symplectic reduction**.

Remark. More generally, if G acts on a hyperkähler manifold preserving $\omega_I, \omega_J, \omega_K$, we get three moment maps μ_I, μ_J, μ_K. We will package this as

$$(\mu_I, \mu_J, \mu_K) = (\mu_{\mathbb{R}}, \mu_J + i\mu_K) \colon M \to \mathfrak{g}^* \oplus \mathfrak{g}^* \otimes \mathbb{C}.$$

(For example, we will have $\mathfrak{g} = \mathfrak{u}(n)$ so that the target is $\mathfrak{u}(n) \oplus \mathfrak{gl}(n)$.) The **hyperkähler quotient** is

$$M /\!\!/\!\!/\!\!/ G \coloneqq (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})^{-1}(0)/G.$$

Theorem 1.11. *A hyperkähler quotient is hyperkähler.*

Example 1.12 (ADHM construction)**.** This will be an example of a hyperkähler quotient. The goal of the ADHM construction is to produce instantons on $\mathbb{C}^2$. First view $\mathbb{C}^2 \cong \mathbb{H}$, the quaternions, to give it a hyperkähler structure. We always start with a $4n$-dimensional vector space in this way, viewed as a quaternionic vector space. The data it contains will be the following.

1. Put $\mathbb{C}^2 \subset S^4 = \mathbb{HP}^1$ and consider the fiber E_∞ of the vector bundle at ∞. This is an n-dimensional complex vector space.

2. To allow non-trivial bundles, introduce a complex vector space V of complex dimension k.

3. We include the data of maps

$$i \colon E_\infty \to V, \quad j \colon V \to E_\infty, \quad B_1, B_2 \in \operatorname{End}(V).$$

In other words, the quaternionic vector space we will consider is

$$\mathbb{M} \coloneqq \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,E_\infty) \oplus \mathrm{Hom}(E_\infty,V).$$

By construction, $\mathbb{M}$ has a natural hermitian metric induced from V and E_∞, giving $\omega_\mathbb{R}$. We get $\omega_\mathbb{C}$ by

$$\omega_\mathbb{C}((B,i,j),(B',i',j')) \coloneqq \mathrm{tr}(B_1(B_2')^\dagger) - \mathrm{tr}(B_2(B_1')^\dagger) + \mathrm{tr}(ij' - i'j).$$

The group action is the $U(n)$-action on V, i.e. $G \coloneqq U(V)$. (There is an $\mathrm{SU}(n)$-action on E_∞, but we will not quotient by it, so that we get *framed* instantons.) The G-action is the natural induced one:

$$g \cdot (B,i,j) \coloneqq (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

We can now take the hyperkähler quotient

$$\mathbb{M} /\!/\!/\!/ G.$$

The moment map equations in this case are called the **ADHM equations**:

$$\begin{cases} 0 = \mu_\mathbb{C}(B,i,j) = [B_1,B_2] + ij \in \mathfrak{gl}(V) \\ 0 = \mu_\mathbb{R}(B,i,j) = [B_1,B_1^\dagger] + [B_2,B_2^\dagger] + ii^\dagger - j^\dagger j \in \mathfrak{u}(V). \end{cases}$$

Definition 1.13. These equations look very similar to the ASD equations! So we want to reinterpret B_1, B_2 as connections. The way to do this is, for a point $x \in \mathbb{C}^2$, to take

$$V \xrightarrow{\alpha_x} \mathbb{C}^2 \otimes V \oplus E_\infty \xrightarrow{\beta_x} V$$

and let our bundle (the instanton) be the cohomology of this complex. (For $x = \infty$, the cohomology should just be the E_∞ we started with.) We engineer α_x, β_x such that this is a complex. The right thing to do turns out to be

$$\begin{aligned} \alpha_x &\coloneqq (B_1 - x_1, B_2 - x_2, i) \\ \beta_x &\coloneqq (-B_2 + x_2, B_1 - x_1, j), \end{aligned}$$

so that $\beta_x\alpha_x = 0$ iff the ADHM equations are satisfied. There also is a nondegeneracy condition for the cohomology to be a vector bundle (i.e. constant rank).

Theorem 1.14. *The non-degeneracy condition is equivalent to surjectivity of*

$$R_x \coloneqq \alpha_x^* \oplus \beta_x \colon \mathbb{C}^2 \otimes V \oplus E_\infty \to V \oplus V.$$

Theorem 1.15 (ADHM)**.** *If (B,i,j) satisfies the ADHM equations and this non-degeneracy condition, then the vector spaces*

$$\ker(R_x)$$

glue together to give a vector bundle, which has a natural ASD connection A with finite energy.

Proof. We will prove the forward direction. The converse, that each ASD connection arises from this construction, is much harder.

To show that A is ASD, it suffices to show $\ker(R_x)$ gives a holomorphic vector bundle with respect to I, J, K. For one complex structure, this is equivalent to

$$\beta_x\alpha_x = 0.$$

But this is equivalent to the ADHM equations. Those were symmetric with respect to I, J, K, so we only need to check it for one.

The hard part of the forward direction is that the bundle $\ker(R)$ has finite energy. The way to do it is by extending from $\mathbb{C}^2$ to S^4, which is compact. By fixing a complex structure, we get isomorphisms

$$\mathbb{C}^2 \cong \mathbb{H} = S^- \quad V \oplus V = (\mathbb{C} \oplus \mathbb{C}) \otimes V = \mathbb{H} \otimes V = S^+$$

where $S^{\pm}$ are the spin bundles. Under this repackaging, R_x is actually a map

$$R_x : \mathbb{H} \otimes V \oplus E_\infty \to \mathbb{H} \otimes V, \quad \left(B_1 + jB_2 - q\right)\begin{pmatrix} j \\ i^\dagger \end{pmatrix}$$

where $q \coloneqq x \in \mathbb{H}$. This extends to S^4 because this linear quaternionic polynomial can be turned into a homogeneous quaternionic polynomial on $\mathbb{HP}^1$. □

Remark. In fact, $c_2(\ker(R)) = k = \dim_{\mathbb{C}} V$.

Definition 1.16 (Generalized ADHM construction)**.** Now we generalize to ADE surfaces, which are resolutions of $\mathbb{C}^2/\Gamma$ for a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ of ADE type. To generalize the ADHM description to this setting, we generalize the data of morphisms to diagrams Γ of affine Dynkin type $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_n$.

1. Each node in the Dynkin diagram is a vector space V_i. At each node, add a new node corresponding to E_∞ with corresponding vector space W_i.
2. Every edge in the Dynkin diagram gives two maps $B_{k,1}, B_{k,2}$ in each direction.
3. Every edge from the Dynkin diagram to the extra nodes at infinity gives two maps i_k, j_k.

So in general, we will have

$$\mathbb{M} \coloneqq \bigoplus_{h \in E(\Gamma)} \mathrm{Hom}(\mathrm{in}(h), \mathrm{out}(h)) \oplus \bigoplus_{i=1}^{k} \mathrm{Hom}(V_i, W_i) \oplus \mathrm{Hom}(W_i, V_i).$$

This has an action by $G \coloneqq \prod_i U(V_i)$. We have $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ by the straightforward generalization. The hyperkähler quotient

$$M_\xi(\vec{v}, \vec{w}) \coloneqq (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})^{-1}(\xi)/G$$

is called the **quiver variety** associated to the quiver Γ. (Instead of taking 0, we will take a parameter $\xi \in Z(\mathfrak{g} \oplus \mathfrak{g} \otimes \mathbb{C})$. This is more general.) We label the quiver variety by:

1. a vector $\vec{v}$ giving dimensions of the V_i;
2. a vector $\vec{w}$ giving dimensions of the W_i.

Theorem 1.17 (Nakajima)**.** *Let $\Gamma = \tilde{A}_n, \tilde{D}_n, \tilde{E}_n$, with generalized Cartan matrix C. Let $n \in \ker C$ be a generator such that its first entry (corresponding to the affine root) is normalized to be* 1*.*

1. *$M_\xi(\vec{n}, 0)$ is smooth when ξ is generic, and is isomorphic to the corresponding ADE surface.*
2. *Varying $\xi_{\mathbb{R}}$ gives the resolution of singularities.*
3. *These are gravitational instantons in the sense that $S_{E,gravity}[M_\xi(\vec{v}, 0)] < \infty$.*
4. *All other quiver varieties on the same graph are moduli of instantons on the ADE surface, and all are hyperkähler.*

Theorem 1.18 (Nakajima)**.** *For fixed quiver Γ, the smooth locus*

$$M_\xi^{reg}(\vec{v}, \vec{w})$$

are moduli of ASD finite-energy connections on $M_{-\xi}(\vec{n}, 0)$. They are smooth for $v_0 = 0$ and generic ξ.

2 Semon (Feb 14): Hypertoric varieties and Gale duality

Definition 2.1. A **polarized hyperplane arrangement** is the data $\mathcal{V}$ of

$$V \subset \mathbb{R}^I, \quad \eta \in \mathbb{R}^I/V, \quad \xi \in V^* = (\mathbb{R}^I)^*/V^\perp.$$

We can think of this as giving us some hyperplanes, plus an *affine covector*. Denote V_η to be V translated by η. Define hyperplanes

$$H_i \coloneqq V_\eta \cap \{x_i = 0\} \subset V_\eta.$$

If $\mathcal{V}$ is **rational**, i.e. everything is defined over $\mathbb{Q}$, one can associate a **hypertoric variety** $\mathcal{M}_{V,\eta}$ to this data, and ξ will determine a $\mathbb{C}^*$-action on $\mathcal{M}_{V,\eta}$.

Definition 2.2. Given $\mathcal{V}$, define its **Gale dual** $\mathcal{V}^\vee \coloneqq (V^\perp, -\eta, -\xi)$.

Given $\mathcal{V}$, we can associate to it two combinatorially-defined algebras $A(\mathcal{V})$ and $B(\mathcal{V})$. These are quadratic algebras.

Definition 2.3. A **quadratic algebra** is a ring

$$E \coloneqq T_R(M)/W$$

which is the tensor algebra $T_R(M)$ of some R-module M, modulo an ideal with only quadratic relations. Its **dual** is

$$E^! \coloneqq T_R(M^*)/W^\perp.$$

Example 2.4. Take $E = \operatorname{Sym} V$. Then $E^! = \wedge^*(V^\perp)$ in some canonical way.

Theorem 2.5. *$A(\mathcal{V})$ and $B(\mathcal{V})$ are quadratic algebras, and*

$$A(\mathcal{V})^! = A(\mathcal{V}^\vee) = B(\mathcal{V}).$$

Some motivation for caring about this is as follows. If $E = \bigoplus_{k\geq 0} E_k$ is a graded algebra, then E is **Koszul** if every simple subobject has a linear resolution, i.e. a resolution where the i-th object is generated in degree i. For Koszul algebras, there is the following very deep theorem.

Theorem 2.6 (BGS)**.** *If E is Koszul, then E is quadratic, $E^!$ is also Koszul, and*

$$E^! = \operatorname{Ext}^*_E(E_0, E_0)^{op}$$

where E_0 is the augmentation of E, and

$$D^b(E) \cong D^b(E^!).$$

Definition 2.7 (Hypertoric varieties)**.** Before we continue, we need to define hypertoric varieties. From a hyperplane arrangement, get $V_\mathbb{Z} \coloneqq V \cap \mathbb{Z}^I$ and $V^*_\mathbb{Z} \subset V^*$. Take

$$G \to (S^1)^I \twoheadrightarrow T$$

acting on $\ker \to \mathbb{R}^I/\mathbb{Z}^I \to V^*/V^*_\mathbb{Z}$. Then G acts on $W^* \coloneqq \ker$. The action of G on W gives an action on T^*W in a Hamiltonian way, giving a **moment map**

$$T^*W \to \mathfrak{g}^*, \quad \mu(z,w)(X) = \Omega(g \cdot z, w).$$

Given $\lambda \in Z(\mathfrak{g}^*)$, we know G preserves $\mu^{-1}(\lambda)$. We also need the data of a character $\alpha \in Z(\mathfrak{g}^*)_\mathbb{Z}$. Then we can form the **hyperkähler quotient**

$$T^*W \,/\!\!/\!\!/\!\!/_{\lambda,\alpha} G \coloneqq \mu^{-1}(\lambda) /\!\!/_\alpha G.$$

Usually we take $\lambda = 0$ and $\alpha \in \mathfrak{g}^*$ comes from η. This is because from how we defined G, we get

$$\mathfrak{g} \to \operatorname{Lie}((S^1)^I) \to \operatorname{Lie}(T),$$

acting on $\mathbb{R}^I/V$, $\mathbb{R}^I$, and V respectively.

Example 2.8. Consider $T^*\mathbb{P}^n$. Take

$$\mathbb{C}^* \xrightarrow{t \mapsto (t,\dots,t)} (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^n.$$

Explicitly, the matrix of the second arrow (on the tangent space) is something like (for $n = 2$)

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Say we pick the character $\alpha(t) = t^{-1}$ in $\mathrm{Lie}(\mathbb{C}^*)$. Then (for $n = 2$) we get the hyperplanes

$$\{x_0 = 1\}, \quad \{x_1 = 1\}, \quad \{-x_0 - x_1 = 1\},$$

from unwinding the construction above. Concretely, we transposed the matrix and set it equal to the matrix of the first map. These hyperplanes form the moment polytope of $\mathbb{P}^2$. The resulting hyperkähler quotient gives $T^*\mathbb{P}^2$.

Remark. Note that the choice of ξ gives a 1-parameter subgroup on the quotient torus $(\mathbb{C}^*)^n$. Using ξ, some regions in the moment polytope will have values bounded under the covector ξ. Picking ξ generically, it acts in the standard way on $T^*\mathbb{P}^n$. There is a correspondence between the bounded regions and components of the attracting strata of the action coming from ξ.

Example 2.9. Consider $\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$. Its defining equation is

$$(z, u, v) \in \mathbb{C}^3 : z^{n+1} - uv = 0\}.$$

Define a subtorus

$$K \coloneqq \{(t_1, \dots, t_{n+1}) : \prod t_i = 1\} \subset (\mathbb{C}^*)^{n+1}.$$

The quotient is a 1-dimensional $\mathbb{C}^*$. The maps will be

$$(\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n+1} \to \mathbb{C}^*$$

given by the transposes of the previous matrices for $T^*\mathbb{P}^n$. The resulting hyperplane arrangement is

$$\{x = 1\}, \quad \{x = 2\}, \quad \dots, \quad \{x = n\}.$$

Importantly, these points don't coincide. Fact: this hyperplane arrangement and the previous one are Gale dual.

Definition 2.10. We will now define the algebra $A(\mathcal{V})$. An element $\alpha \in (-1)^I$ determines a component Δ_α of the hyperplane diagram by intersection of half-planes. (Note that these signs do not change under Gale duality.) Let

$$F \coloneqq \{\alpha : \Delta_\alpha \neq 0\}.$$

Define the algebra

$$B \coloneqq \{\alpha : \xi \text{ is bounded on } \Delta_\alpha\},$$

and let $P \coloneqq F \cap B$. We can run this construction on the Gale dual side to get $F^\vee$, and the standard duality from linear programming tells us that

$$F^\vee = B, \quad B^\vee = F, \quad P^\vee = P.$$

The **algebra** $A(\mathcal{V})$ is constructed as follows.

1. Take the quiver Q with vertices F and arrows (both ways) if α and β differ by a single sign.

2. Let A be the path algebra $P(Q) \otimes_{\mathbb{R}} \mathrm{Sym}(V^*)$ modulo relations:

(a) if $\alpha \notin P$, then $e_\alpha = 0$;

(b) going around in a square commutes;

(c) going in a loop $\alpha \to \beta \to \alpha$ gives $t_i e_\alpha$ where t_i is the equation of the hyperplane in V.

Theorem 2.11. *$A(\mathcal{V})$ is quadratic.*

Proof. Consider path algebra $P(Q_P)$ of the subquiver with only vertices from P. There is a map

$$P(Q_P) \to A.$$

Relation (3) lets us expand a function into a path. In particular, there is a surjection $P(Q_F) \to A$ from the path algebra of the *whole* quiver. Then relation (1) says the vertices in $F \smallsetminus P$ die. The relations in this presentation come from relation (2), since the other relations are all gone. But these relations are all quadratic. □

To show that $A(\mathcal{V})^! = A(\mathcal{V}^\vee)$, the idea is to take a free $\mathbb{R}$-module M on paths $p(\alpha\beta)$. We identify $M^\vee$ with M^* by writing down an inner product given by

$$\langle p(\alpha\beta), p(\beta\alpha)\rangle = \pm 1$$

with sign chosen so that going around a commuting square gives -1. (Of course, these choices are not canonical.)

Example 2.12. Let's compute $A(\mathcal{V})$ for $\widetilde{\mathbb{C}^2/\mathbb{Z}_3}$. Let p, q, r, s be the edges. Then the relations are

$$\{pq + sr = 0, \quad rs = 0\}.$$

Definition 2.13. The algebra $B(\mathcal{V})$ is the convolution algebra associated to the cohomology of components of the attracting stratification, which is a union of toric varieties. The components are indexed by $\alpha \in P$.

3 Yasha (Feb 21): Physics of symplectic duality

Let's first look at $N = 2$ supersymmetric 4d theories, with a $\mathrm{SU}(2)_R$ symmetry acting on susy charges Q_d. What is the on-shell spectrum?

1. It contains a hypermultiplet
$$(\phi, \bar{\phi}, \psi, \bar{\psi})$$
where $(\phi, \bar{\phi})$ are complex scalars, and $(\psi, \bar{\psi})$ are Weyl fermions.

2. It contains a vector multiplet
$$(\varphi, \lambda, \bar{\lambda}, A_\mu)$$
where φ is a complex scalar, $(\lambda, \bar{\lambda})$ is a Weyl fermion, and A_μ is a gauge field.

If we want to see the theory in the infrared, we should look at the zeros of the potential. It turns out here are no kinetic terms in the Lagrangian mixing hypermultiplets and vector multiplets. This is why the moduli space decomposes as

$$\mathcal{M} = \mathcal{M}_{\text{hyper}} \times \mathcal{M}_{\text{vector}}.$$

The first term is called the **Higgs branch**, and the second is called the **Coulomb branch**.

Now let's look at 3d $N = 4$ theory. It can be obtained from $N = 1$ 6d theory by dimensional reduction. For example, take super Yang–Mills with gauge group G

$$\int F \wedge \star F + \bar{\psi} \not{\nabla} \psi.$$

In 6d, we have the gauge field A_μ and two Weyl fermions $(\bar{\psi}, \psi)$. The R-symmetry $\mathrm{SU}(2)_R$ acts on the two fermions. We assume the fields are independent on the coordinates x^4, x^5, x^6 which we want to compactify.

1. A_μ splits into $A_{\le 3}, A_4, A_5, A_6$ where A_4, A_5, A_6 are scalars. These three form some connection on 3d space. So out of A_μ, we get a 3d gauge field A_μ and three scalars ϕ_1, ϕ_2, ϕ_3.
2. What is the symmetry of this theory? There is an $\mathrm{SU}(2)_R$ symmetry (which we started with), and a $\mathrm{SU}(2)_N$ and $\mathrm{SU}(2)_E$. Here
$$\mathrm{SU}(2)_N \coloneqq \mathrm{SO}(x^4, x^5, x^6)$$
and $\mathrm{SU}(2)_E = \mathrm{SO}(x^1, x^2, x^3)$ is the Euclidean rotation group of 3d space. As a representation, the fermions decompose as $2 \otimes 2 \otimes 2$.

What is the resulting action? Since $F = dA + A \wedge A$, we get $d\phi + \phi \wedge \phi$ as well. Hence we get an extra potential term in the Lagrangian
$$V = \frac{1}{4e^2} \sum_{i<j} \mathrm{tr}[\phi_i, \phi_j]^2.$$
To get 0 out of this, the ϕ_i should commute. By gauge transformations, they can be put into the Cartan of the gauge group G. So if $\mathrm{rank}\,\mathfrak{g} = r$, there are $3r$ fields $\{\phi_i\}$. The gauge group is broken to $U(1)^r$, i.e. the subgroup of G which preserves each individual ϕ_i.

There is also a dual photon σ such that $d\sigma = \star dA$. We consider this scalar instead of the whole gauge field A_μ. It is the only component which is important in the IR limit.

In total, this gives us $4r$ scalars $\{\phi_1, \phi_2, \phi_3, \sigma\}$. In the infrared, the physics should be described by the sigma model to the vacua. Since we started with $N = 4$, it follows that this moduli of vacua should be hyperkähler with SU(2)-action. (This is a heuristic guess.)

Example 3.1. Let's consider $G = \mathrm{SU}(2)$. Then the scalar fields are
$$\phi_i = \begin{pmatrix} a_i & 0 \\ 0 & -a_i \end{pmatrix}.$$
The Weyl group is $a_i \leftrightarrow -a_i$. The dual photon σ under this action is also $\sigma \leftrightarrow -\sigma$. The classical moduli space is therefore
$$\mathcal{M} = (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2.$$
For $U(1)$, we have $\mathbb{R}^3 \times S^1$ without the $\mathbb{Z}_2$. To make them the same, let's postpone modding by $\mathbb{Z}_2$. The classical metric we can guess to be
$$ds^2 = \frac{1}{e^2} \sum d\phi_i^2 + e^2 d\sigma^2.$$

There should be quantum corrections to this classical metric. At ∞, the space $\mathbb{R}^3 \times S^1$ looks like $S^2 \times S^1$. After quantum correction, this should be violated and we should get some S^1-bundle over S^2. The ansatz we make is
$$ds^2 = \frac{1}{e^2} \sum d\phi_i^2 + e^2 (d\sigma + s B_i(\phi) d\phi_i)^2$$
where $s \in \mathbb{Z}$ and B_i is the **monopole field**.

What is a monopole? They are instantons of codimension 3. Take a 3d theory,
$$\int F_{\mu\nu} F^{\mu\nu} + \frac{1}{e^2} (D_\mu \phi)^2.$$
For example, in classical electrodynamics, if we take $B = g\vec{r}/4\pi r^3$, then it violates the monopole condition. At ∞, we can have non-trivial maps $S^2 \to \mathrm{SU}(N)/U(1)^{N-1}$, to the set of $\{\phi_i\}$. These non-trivial maps arise from monopole insertions. Such maps are classified by
$$\pi_2(\mathrm{SU}(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbb{Z}_{N-1}.$$
In this setting, such a thing is called a monopole because $D_\mu(\phi) \to 0$ as $1/r^2$. In this case, $\partial\phi \to 0$ as $1/r$, and therefore we should have $A \sim 1/r$. Hence $B \sim 1/r^2$.

How do we parametrize S^1-bundles over S^2? Over any base B, they are parametrized by the Euler class $H^2(B,\mathbb{Z})$. For $B = S^2$, we have

$$H^2(S^2,\mathbb{Z}) = \mathbb{Z}.$$

This is the $s \in \mathbb{Z}$ we had in the ansatz earlier. To explicitly construct this bundle, we introduce $u_1, u_2 \in \mathbb{C}$ such that $|u_1|^2 + |u_2|^2 = 1$, forming a sphere S^3. As $u_1, u_2 \mapsto [u_1 : u_2] \in \mathbb{CP}^1 = S^2$, we get the Hopf fibration corresponding to $s = 1$. In general, for $s \in \mathbb{Z}$, introduce a transformation as follows. Take $S^3 \times S^1$ with coordinates $((u_1, u_2), \psi)$. Consider $U(1)$ acting by

$$(u_1, u_2, \psi) \mapsto (e^{i\theta} u_1, e^{i\theta} u_2, \psi + s\theta).$$

Taking the quotient $(S^3 \times S^1)/U(1) = S^3/\mathbb{Z}_s =: L_s$, called the lens space. If we forget about S^1, we can project to S^2, giving the desired S^1-bundle for the given s. All these bundles have symmetry $\mathrm{SU}(2) \times U(1)$, acting on (u_1, u_2) and ψ.

Now we want to determine the s for our theory. What is the contribution of the monopole to the quantum correction of the metric? If I is the value of the action on the monopole, then we get $e^{-I+i\sigma}$, where we add $i\sigma$ to make sure the flux at infinity is correct. There are also Fermi 0 modes (which in localization is the contribution of the tangent bundle). Recall the symmetry group $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_E$. The monopole is invariant under $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_R$, but $\mathrm{SU}(2)_N$ is broken into $U(1)$ because we constructed the monopole from the scalars ϕ_1, ϕ_2, ϕ_3.

We showed earlier that in $N = 4$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_E$, supercharges decompose as $2 \otimes 2 \otimes 2$. Fact: BPS monopoles are invariant under half of the supersymmetries. The rest of the supersymmetries generate Fermi 0 modes. Under the broken symmetry group $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_E \times U(1)_N$, supercharges decompose as $(2,2)^{1/2} \oplus (2,2)^{-1/2}$, and BPS monopoles are invariant under half of the supersymmetries in $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_E \times U(1)_N$.

Then the unbroken symmetry is e.g. $(2,2)^{-1/2}$. (We can choose either one.) So fermion zero modes transform as $(2,2)^{1/2}$. So the instanton amplitude has the form

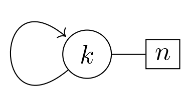
$$\int \psi\psi\psi\psi e^{-(I+i\sigma)}.$$

Here these ψ's have 1/2 charge each, and I is invariant. The transformation of σ is the action of the generator of $U(1)$ as $2\partial_\sigma$, to make the whole thing invariant.

Because the ψ's have charge 1/2, we deduce that the bundle at ∞ must be L_{-4}, i.e. $s = -4$. With hypermultiplets, $s = -4 + 2N_f$ where N_f is the number of fermions. (Here the computation to do is that each component of a fermion contributes 1 to s.) For $U(1)$ there are no monopoles, so $s = N_f$. So now we know the whole metric on the Coulomb branch.

Let's consider an example of duality between quivers. The content of a quiver gauge theory is encoded in a quiver. Given a quiver with labels n_i on vertices, the gauge group is $G = \prod U(n_i)$, and every edge is a hypermultiplet in $n_i^* \otimes n_j$. If we put a square instead of a vertex, we consider the gauge group there as a global symmetry, not as a gauge symmetry. Consider the following two theories.

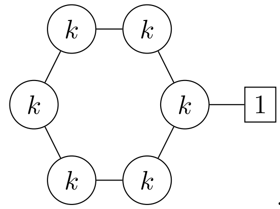
1. $U(k)$ gauge group with n hypermultiplets in fundamental representation and 1 hypermultiplet in the adjoint representation.

   .

   The Higgs moduli here is the moduli $\mathcal{M}_k(\mathrm{SU}(n))$ of k-instantons.

2. $\prod U(k)$ gauge group with 1 hypermultiplet in fundamental representation and n hypermultiplets in the adjoint representation.



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Let's provide some counting evidence for Coulomb branches and Higgs branch have the same dimension, and the number of mass and FI parameters are the same. FI parameters are constants ξ added to the action as

$$+\lambda|\mu-\xi|^2$$

where μ is the moment map. So the number of FI parameters is the number of $U(1)$ factors in the gauge group.

1. (A) $\operatorname{rank} G = k$ and

$$\dim \mathcal{M}_{\text{Higgs}} = \dim(\text{hypermultiplets}) - \dim(\text{gauge}) = (nk+k^2) - k^2 = nk,$$

   and $\dim \mathcal{M}_{\text{Coulomb}} = r$ (because we take hyperkähler reduction from dim $4r$). The number of FI terms is 1. The number of mass parameters is $n+1-1=n$. (Some mass terms can be gauged out, hence the -1.)

2. (B) $\operatorname{rank} G = kn$, and

$$\dim \mathcal{M}_{\text{Higgs}} = nk + k^2 - nk^2 = k,$$

   and $\dim \mathcal{M}_{\text{Coulomb}} = kn$. The number of FI terms is n, since in each $U(k)$ we have one $U(1)$ factor. The number of mass parameters is $n+1-n=1$.

Numerically, we see that perhaps the Higgs and Coulomb branches are swapped in these two different theories, and similarly we swap mass and FI parameters.

There is a string realization of these quiver gauge theories in IIB string theory. Consider

1. NS5 branes $X^0, X^1, X^2, X^3, X^4, X^5$;

2. D5 branes $X^0, X^1, X^2, X^7, X^8, X^9$;

3. D3 branes X^0, X^1, X^2, X^6

where the D3 branes are stretched between NS5 and NS5 branes, and D5 branes are along the orthogonal direction. If between the i-th and $(i+1)$-th NS5 branes there are k_i 3-branes, then we get a $U(k_i)$ gauge theory. If $\vec{\omega}_i$ is the position of NS5 in X^7, X^8, X^9, then FI terms are $\vec{\xi}_i = \vec{\omega}_i - \vec{\omega}_{i+1}$. If $\vec{m}_i$ are positions of the D5 branes in X^3, X^4, X^5, then they are the masses of hypermultiplets. If we compactify X^6, then strings don't have to start and end on the NS5. They can wrap around it along X^6. This way, we get the A-model above, with hypermultiplets between D3 and D5 branes. The $\mathrm{SL}(2,\mathbb{Z})$ symmetry for IIB strings preserves D3 branes (because $N=4$ SYM has this symmetry, called Montonen–Olive duality) and swaps D5 and NS5 branes. (D5 is charged under $\int B\wedge B\wedge B$ and NS5 is charged under $\int B^*$ such that $dB^* = \star dB$.)

4 Shuai (Feb 28): Koszul duality

The first step is the classical version of Koszul duality. Let's first recall something familiar. If we want to consider the representation theory of finite/compact groups G, we can think about the character group $\hat{G}$, which is on the dual side. Then $\mathsf{Rep}(G)$ is identified with functions on $\hat{G}$. We can also look at $L^2(G)$ vs $L^2(\hat{G})$. What do we mean by "duality"? In this context, it is realized by $G \mapsto \hat{G}$.

In our situation, we don't consider finite/compact groups, but instead Koszul algebras. The picture is still almost the same. On one hand, we have some Koszul algebra A, and on the other hand we have its dual algebra $A^!$. The classical example is

$$A = \wedge^* V \quad \leftrightarrow \quad A^! = (\wedge^* V)^! = \mathrm{Sym}^*(V^\vee).$$

Think about $\mathsf{Mod}(\wedge^* V)$ and $\mathsf{Mod}(\mathrm{Sym}^*(V^\vee))$. What is the analogue of "taking the character" in our situation? This operation is given by

$$A \mapsto \mathrm{Ext}^*_A(k,k).$$

In this context and many other contexts, $\mathrm{Ext}^*(-,-)$ serves as a "character".

Example 4.1. Let V be a vector space.

1. Let's first consider $V = k$. Then $\wedge^* V = k[\epsilon]/\epsilon^2$. The resolution is

   $$\cdots \xrightarrow{\epsilon} k[\epsilon] \xrightarrow{\epsilon} k[\epsilon] \xrightarrow{\epsilon} k \to 0.$$

   Now compute Ext by

   $$0 \to \mathrm{Hom}_{k[\epsilon]}(k[\epsilon],k) \xrightarrow{\epsilon} \mathrm{Hom}_{k[\epsilon]}(k[\epsilon],k) \xrightarrow{\epsilon} \cdots.$$

   So each term is a copy of k. Computing the pairing, we see the answer is $k[x]$.

2. In general, for $V = \mathbb{C}^n$, we have the usual Koszul resolution

   $$\cdots \to \mathrm{Sym}^2 V \otimes \wedge^* V \to \mathrm{Sym}^1 V \otimes \wedge^* V \to \wedge^* V \to k \to 0$$

   with the usual maps. Compute Ext by

   $$0 \to \mathrm{Hom}_A(\wedge^* V, k) \to \mathrm{Hom}_A(\mathrm{Sym}^1 V \otimes \wedge^* V, k) \to \cdots.$$

   So the i-th term is $\mathrm{Sym}^i V$, and the whole algebra is isomorphic to $\mathrm{Sym}^* V$. (The harder part is checking the algebra structure, but it is true.)

The conclusion is that $\mathrm{Ext}^*_{\wedge^* V}(k,k) = \mathrm{Sym}^*(V^\vee)$.

Remark. Since we are calling this a "duality", it should be that $\mathrm{Ext}_{A^!}(k,k) = A$. In the case of $A = \wedge^* V$, we can use the same Koszul resolution to show this.

Theorem 4.2. *There is an equivalence*

$$\begin{array}{ccc} D^b(\mathbb{P}^n) & \longleftarrow & D^b(\mathrm{Sym}^*)/\mathcal{F}_{fg} \\ & \searrow \quad \nearrow & \\ & D^b(\wedge^*)/\mathcal{F}_{free} & \end{array}$$

Here $\mathcal{F}_{fg}$ is the full subcategory generated by finitely generated modules, and similarly for $\mathcal{F}_{free}$ and free modules.

Remark. The right-most arrow is what we call Koszul duality. The horizontal arrow is actually a theorem due to Serre.

Proof sketch. For finite/compact groups, we go between one side and the dual by something like

$$\begin{aligned} L^2(G) &\leftrightarrow \ell^2(\hat{G}) \\ V &\to \sum \dim \operatorname{Hom}(R, V_i)\chi_i \\ \bigoplus V_i^{\langle f, \chi_i \rangle} &\leftarrow f. \end{aligned}$$

Similarly, in our case, we have

$$\begin{aligned} D^b(S) &\leftrightarrow D^b(\wedge) \\ (W, d) &\to (\operatorname{Hom}_k(\wedge, W), \partial) \quad \partial(v)\lambda \coloneqq -\sum x_i v(\xi_i \lambda) + d(v(\lambda)) \\ (S^*(V^\vee) \otimes V, d) &\leftarrow (V, \partial) \quad d(s \otimes v) \coloneqq \sum x_i s \otimes \xi_i v + s \otimes \partial V. \end{aligned}$$

□

We can gain some intuition about these equivalences via the equivalence to $D^b(\mathbb{P}^n)$. What is that equivalence? Let $\mathcal{P}$ be the class of free $\wedge^* V$-modules. Say $V_1 \sim_{\mathcal{P}} V_2$ are **$\mathcal{P}$-equivalent** if $V_1 \oplus P_1 \cong V_2 \oplus P_2$ for some $P_1, P_2 \in \mathcal{P}$. Given a module V over $\wedge^* V$, the equivalence is

$$V \mapsto L^\bullet, \quad L_j \coloneqq V_{-j} \otimes \mathcal{O}(j).$$

For example, $k \mapsto \mathcal{O}$ and $\wedge^* V \mapsto 0$.

Definition 4.3. A $\wedge$-module is called **faithful** if

$$H^i(L_\xi(V)) = 0, \quad \forall \xi \neq 0,\ i \neq 0.$$

Here $L_\xi(V)$ means the localization of the complex $L(V)$.

Remark. If the module is faithful, because all other H^i are zero, let $\Phi(V) \coloneqq H^0$ be some bundle over $\mathbb{P}^n$. It turns out that:

1. $\Phi(V) = \Phi(V')$ iff $V \sim_{\mathcal{P}} V'$;
2. the construction Φ commutes with $\otimes, S^*, \wedge^*, (-)^\vee$;
3. for given V, V', there is an exact sequence $0 \to V \to P \to V' \to 0$ where P is free, and then

$$\Phi(V'[1]) = \Phi(V);$$

Let $\omega \coloneqq \xi_0 \wedge \cdots \wedge \xi_n$ be the volume form. Then $\mathsf{Vect}(\mathbb{P}^n) \cong \mathsf{Mod}(\wedge/(\omega))$.

Definition 4.4. Now let's talk about parabolic-singular duality. Fix a semisimple Lie algebra $\mathfrak{g}$, with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $U\mathfrak{g}$ be the universal enveloping, with center Z. Let W be the Weyl group.

1. **Category $\mathcal{O}$** is the sub-category of $\mathsf{Mod}(U\mathfrak{g})$ of objects which are:
   (a) finitely generated;
   (b) locally finite over $\mathfrak{b}$, i.e. the sub-module generated by the $\mathfrak{b}$-action should be finite;
   (c) semisimple over $\mathfrak{h}$.
2. Fix $\lambda \in \mathfrak{h}^*$. **Verma modules** are $M(\lambda) \coloneqq U \otimes_{U\mathfrak{h}} \mathbb{C}_\lambda$. Its simple quotient is the highest-weight irrep $L(\lambda)$. It has a **projective cover** $P(\lambda) \to L(\lambda)$. (For a given $\mathfrak{g}$, the number of projective covers is finite. More precisely, it is $|W|$.) The analogue of the "regular representation" is

$$P \coloneqq \bigoplus_{i=1}^{|W|} P_i.$$

3. Define $L \coloneqq \bigoplus_{i=1}^{|W|} L_i$, where L_i are the simple highest-weight modules which have trivial infinitesimal character, i.e. $\mathrm{Ann}_Z(L) == \mathrm{Ann}_Z(\mathbb{C})$.

Theorem 4.5 (Parabolic-singular duality in $\lambda = 0$ case)**.** *There is an isomorphism of Koszul rings*

$$\mathrm{End}_{\mathcal{O}}(P) \cong \mathrm{Ext}^*_{\mathcal{O}}(L, L).$$

Definition 4.6. A **Koszul ring** is a $\mathbb{Z}_+$-graded ring

$$A = \bigoplus_{j \geq 0} A_j$$

such that:

1. A_0 is semisimple (as a module over A);
2. there is a resolution $\cdots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$ such that P^i is generated by only degree i elements, i.e. $P^i = AP^i_i$.

Its **dual** should be something like $A^! \coloneqq \mathrm{Ext}_A(A_0, A_0)$ (up to taking opposite categories).

Proof idea. Why is this a natural identification? In the case of finite/compact groups, one important result is how to decompose $\mathbb{C}[G]$. Let $\mathcal{A}$ be an abelian category with projective generator P. This means P has a surjection to all the simple objects $L \in \mathcal{A}$. (So P is really analogous to the regular representation.) By an abstract construction,

$$\mathcal{A} \cong \mathrm{End}_{\mathcal{A}}(P), \quad M \mapsto \mathrm{Hom}_{\mathcal{A}}(P, M).$$

For example, if $G = S^1$ or $\mathbb{Z}/n$, then this says $\mathsf{Rep}(G) \cong \mathsf{Mod}(\mathbb{C}[G])$. In some sense, the lhs of parabolic-singular duality can be thought of in this way. The rhs can be interpreted as some kind of "characters". □

Example 4.7. Let $\mathfrak{g} = \mathfrak{sl}_2$ with standard Borel. Then $G/B = \mathbb{P}^1$. The parabolic-singular duality allows us to take the category of perverse sheaves $\mathcal{P}(\mathbb{P}^1, W)$ (parametrized by the Weyl group) and identify it with category $\mathcal{O}$. To compute things like $\dim \mathrm{Hom}(\mathrm{IC}_x, \mathrm{IC}_y)$, we can push it to $\mathcal{O}$ and just compute something about Homs between modules.

Definition 4.8. A more general version of Koszul duality gives us a correspondence between categories $\mathcal{O}_\lambda$ and $\mathcal{O}^q$.

1. Define
$$\mathcal{O}_\lambda \coloneqq \{M \in \mathcal{O} : \mathrm{Ann}_Z(M) = \mathrm{Ann}_Z(L(\lambda))\}.$$
Define $s \cdot \lambda \coloneqq s(\lambda - \rho) - \lambda = \lambda$. In $\mathcal{O}_\lambda$, all the simples and projective covers are given by
$$\{L(x \cdot \lambda)\}_{x \in W^\lambda}, \quad \{P(x \cdot \lambda)\}_{x \in W^\lambda}.$$

2. Let q denote a parabolic subgroup. Define
$$\mathcal{O}^q \coloneqq \{M \in \mathcal{O}_0 : q\text{-locally finite}\}.$$
The simples and projective covers are given by
$$\{L^q_x \coloneqq L(x^{-1} w_0 \cdot 0)\}, \quad \{P^q_x \coloneqq P(x^{-1} w_0 \cdot 0)\}.$$

Theorem 4.9. *If $s_\lambda = s_q$, then*

$$\mathrm{Ext}_{\mathcal{O}_\lambda}(\bigoplus P(x \cdot \lambda)) \cong \mathrm{Ext}_{\mathcal{O}^q}(\bigoplus L^q_x \oplus L^q_x)$$
$$\mathrm{Ext}_{\mathcal{O}^q}(\bigoplus P^q_x) \cong \mathrm{Ext}_{\mathcal{O}_\lambda}(\bigoplus L(x \cdot \lambda) \oplus L(x \cdot \lambda)).$$

Remark. If $P = B$, then we recover the original parabolic-singular duality, which says $\mathcal{O}_0$ is self-dual.

5 Henry (Mar 07): Symplectic resolutions

If a singular variety Y comes from some representation theory problem, we want a resolution $\pi\colon X \to Y$ which is:

1. **semismall**, i.e. for every closed $Z \subset X$ we have $2\operatorname{codim} Z \geq \operatorname{codim} \pi(Z)$;
2. **equivariant** with respect to some group action on Y.

This turns out to be possible for Springer resolutions and quiver varieties. Interestingly, in these cases, X is holomorphic symplectic. Actually, this turns out to be important: that X is holomorphic symplectic ensures semismall-ness, cohomological purity of fibers, etc.!

Definition 5.1. A **symplectic singularity** is a normal irreducible algebraic variety Y with a nondegenerate symplectic form

$$\omega \in H^0(Y^{\mathrm{sm}}, \Omega^2)$$

which extends to a *possibly-degenerate* symplectic form on a smooth resolution $X \to Y$. (Later, for a symplectic resolution, we will require non-degeneracy.)

Remark. We want Y to be normal so that functions extend uniquely from Y^{sm} to Y. This is because normal is equivalent to:

1. (R_1) the singular locus is codimension 2, and
2. (S_2) functions extend uniquely over codimension 2.

Lemma 5.2. *Symplectic singularities are:*

1. **canonical**, *i.e.* $K_X = f^* K_Y + \sum a_i E_i$ *with* $a_i \geq 0$*;*
2. **rational**, *i.e.* $R^i f_* \mathcal{O}_X = 0$ *for* $i > 0$.

We will most often be concerned with affine Y. Since we assume Y to be normal, $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$, and it follows that $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$. Hence if we assume Y affine, we can forget about Y altogether!

Definition 5.3. A **symplectic resolution** is a smooth variety X with a closed *non-degenerate* 2-form ω such that the affinization map

$$X \to Y \coloneqq \operatorname{Spec} H^0(X, \mathcal{O}_X)$$

is a resolution of singularities, i.e. projective birational. If in addition $H^0(X, \mathcal{O}_X)$ has a grading, then Y has a $\mathbb{C}^\times$-action, and we call such a symplectic resolution **conical**.

Remark. Some people don't require non-degeneracy in symplectic resolutions, in which case a resolution of a symplectic singularity is symplectic iff it is crepant. We require non-degeneracy, so for us there is a difference between "symplectic" and "crepant".

Example 5.4 ($T^*(G/B)$)**.** Let G be a complex semisimple and simply-connected algebraic group.

1. Let $\mathcal{B}$ be its flag variety, parametrizing Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$.
2. Given a Borel $\mathfrak{b}$, let $\mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$ be its associated Cartan decomposition.

Define the **nilpotent cone**

$$\mathcal{N} \coloneqq \{x \in \mathfrak{g} : \mathrm{ad}_x \text{ is nilpotent}\}.$$

Its smooth locus consists of *regular* elements, i.e. those for which $\operatorname{charpoly}(\mathrm{ad}_x) = \operatorname{rank} \mathfrak{g}$. A resolution is given by

$$\tilde{\mathcal{N}} \coloneqq \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} : x \in \mathfrak{b}\}.$$

The projection $\tilde{\mathcal{N}} \to \mathcal{B}$ has fibers $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, which are exactly the fibers $T^*_{\mathfrak{b}}\mathcal{B}$. Hence

$$T^*\mathcal{B} \cong \tilde{\mathcal{N}}.$$

Fact: it is a resolution of singularities because any nilpotent x is contained in a unique Borel. This is the **Springer resolution**.

Remark (Demailly–Campana–Peternell). Conjecturally, every symplectic resolution of the form T^*M is actually of the form $T^*(G/P)$ for a semisimple algebraic group G and a parabolic $P \subset G$.

Example 5.5 (Hypertoric varieties)**.** We constructed hypertoric varieties

$$\mathcal{M}_{\theta,\zeta} \coloneqq \mu^{-1}(\zeta) /\!\!/_{\theta} (\mathbb{C}^*)^k$$

as hyperkähler quotients. General theory of GIT quotients gives us a projective birational map

$$\mathcal{M}_{\theta,\zeta} \to \mathcal{M}_{0,\zeta} = \operatorname{Spec} \mathbb{C}[\mu^{-1}(\zeta)]^{(\mathbb{C}^*)^k}.$$

Such a map is a symplectic resolution.

Example 5.6 (Quiver varieties)**.** A large class of symplectic resolutions is given by Nakajima quiver varieties, which are also given by hyperkähler reduction. The symplectic resolution is given by the projective GIT map

$$\mathcal{M}_{\theta,\zeta}(\vec{v}, \vec{w}) \to \mathcal{M}_{0,\zeta}(\vec{v}, \vec{w}) = \operatorname{Spec} \mathbb{C}[\mu^{-1}(\zeta)]^{G_{\vec{v}}}.$$

One of Kaledin's key insights toward the study of singular symplectic varieties is to apply the Poisson methods developed in the theory of symplectic manifolds. Specifically, look at the **Poisson structure** instead:

$$\{f, g\} \coloneqq \iota_{\Theta}(df \wedge dg) \in H^0(Y, \mathcal{O}_Y) \quad f, g \in H^0(Y, \mathcal{O}_Y)$$

where Θ is the 2-vector dual to the symplectic form. A priori this is only defined on Y^{sm}, but functions extend uniquely. We can now look at the algebraic equivalent of "symplectic leaves".

Theorem 5.7 (Kaledin)**.** *Let X be a symplectic variety. Then there exists a canonical stratification $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ such that:*

1. *X_{i+1} is the singular part of X_i;*
2. *the normalization of every irreducible component of X_i is a symplectic variety.*

Another of Kaledin's contributions is to use purely algebraic methods (e.g. Hodge theory) to study the geometry of symplectic resolutions.

Theorem 5.8 (Kaledin)**.** *Let $X \to Y \coloneqq \operatorname{Spec} H^0(X, \mathcal{O}_X)$ be a symplectic resolution. Then:*

1. *(semismall) $X \to Y$ is semismall, i.e. for any stratum Y_i of codim 2ℓ, its pre-image has codim $\geq \ell$;*
2. *(Hodge number vanishing) $H^q(X, \Omega^p_X) = 0$ for all $q > p$;*
3. *(cohomological purity) cohomology $H^*(X^{an}_y, \mathbb{C})$ of fibers is generated by algebraic cycles, i.e. no odd cohomology, and even cohomology is Hodge–Tate (generated by (p,p)-cycles);*
4. *(exactness of ω) in any formal neighborhood of a fiber, ω is exact.*

Proof of semismall-ness. Let X_i be the pre-image of Y_i and fix $y \in Y_i$. Fact: on fibers of $(X_i)^{\mathrm{sm}}$, the symplectic form ω restricts to zero. (This is a Hodge-theoretic calculation.) So

$$T_x(X_i)_y \subset \ker \omega|_x,$$

i.e. $(X_i)_y$ and X_i are orthogonal with respect to ω. But ω is non-degenerate on all of X,

$$\dim(X_i)_y + \dim X_i \le \dim X.$$

We have $\dim(X_i)_y = \dim X_i - \dim Y_i$, so

$$2\,\mathrm{codim}\, X_i = 2\dim X - 2\dim X_i \ge \dim X - \dim Y_i = \mathrm{codim}\, Y_i$$

since π is birational and $\dim X = \dim Y$. □

What is the deformation theory of a symplectic resolution X? We require deformations preserve the symplectic form ω, so first-order deformations are parametrized by $H^2_{\mathrm{dR}}(X)$. There is an analogue of Tian–Todorov unobstructedness here.

Theorem 5.9 (Kaledin–Verbitsky). *Let (X,ω) be a symplectic resolution. Then it admits a (topologically trivial) universal formal deformation $(\mathfrak{X},\omega_{\mathfrak{X}})$ whose base is the completion of $H^2_{dR}(X)$ at $[\omega]$.*

Proof sketch. The idea for deformations of a manifold X over a base S is standard:

1. (Kodaira–Spencer theory) to an n-th order deformation X_n/S_n, there is an associated Kodaira–Spencer class
$$\theta_n \in H^1(X_n, T_{X_n/S_n} \otimes \Omega^1(S_n));$$
2. (T_1-lifting property) to get an $(n+1)$-th order deformation, θ_n must lift to
$$\hat{\theta}_n \in H^1(X_n, T_{X_n/S_n} \otimes i_n^* \Omega^1(S_{n+1})).$$

For manifolds, Ran proves liftings exist via Hodge theory. For symplectic deformations, we replace $T_{X/S}$ by

$$K^\bullet_{X/S} := F^1(\Omega^\bullet_{X/S})[-1] = [\Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots].$$

Kaledin–Verbitsky prove that liftings exist in this general context. □

Example 5.10 ($T^*(G/B)$). Let G be a complex semisimple and simply-connected algebraic group. Note

$$H^2_{\mathrm{dR}}(T^*\mathcal{B}) = H^2(\mathcal{B};\mathbb{Z}) \otimes \mathbb{C} \cong \mathrm{Hom}(T,\mathbb{C}^*) \otimes \mathbb{C} \cong \mathfrak{t}^*$$
$$D_\lambda := c_1(\mathcal{O}(\lambda)) \mapsto \lambda,$$

so we expect the Springer resolution to fit into a family over $\mathfrak{t}^*$. This is the (more canonical) **Grothendieck–Springer resolution**, with total space

$$\tilde{X} := \{(x,\mathfrak{b}) : x \in [\mathfrak{b},\mathfrak{b}]^\perp\} \subset \mathfrak{g}^* \times \mathcal{B}$$

and map

$$\tilde{X} \ni (x,\mathfrak{b}) \xrightarrow{\phi} x|_{\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]} \in \mathfrak{t}^*.$$

1. (**Springer resolution**) The special fiber is
$$X := \phi^{-1}(0) \cong T^*\mathcal{B}$$
as follows. The condition $(x,\mathfrak{b}) \mapsto 0$ means $x|_{\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]} = 0$, i.e. that x is purely the data of $(\mathfrak{g}/\mathfrak{b})^*$. Hence $X \cong G \times_B (\mathfrak{g}/\mathfrak{b})^*$, which is exactly the cotangent bundle.
2. The generic fiber of ϕ is a coadjoint orbit of G with its canonical Kirillov–Kostant form ω.

This picture extends to $T^*(G/P)$ for parabolics P, and then to resolutions of (normalizations of closures of) nilpotent orbits in $\mathfrak{g}$.

Definition 5.11 (Weyl group)**.** Let $X \to Y$ be a conical symplectic resolution. The singular part Σ in Y is codimension 2. Let $\{\Sigma_j\}$ be the connected components of its smooth part.

1. (Normal slices) Let $x_j \in \Sigma_j$. Let N_j be a normal slice to Σ_j at x_j. Then it is a surface, with a singularity.
2. (ADE singularities) Symplectic singularities are canonical, so N_j has a canonical singularity. Such surface singularities are always ADE, i.e. rational double points, giving a Dynkin diagram D_j.
3. (Automorphisms) $\pi_1(\Sigma_j)$ acts on D_j by diagram automorphisms via the monodromy representation.

The **Namikawa Weyl group** is the product $W \coloneqq \prod W_j$ of *centralizers* of this $\pi_1(\Sigma_j)$ action in the Coxeter group associated to D_j.

Theorem 5.12 (Namikawa)**.** *Let $\mathfrak{X} \to H^2(X, \mathbb{C})$ be the universal deformation of X.*

1. *Let $\mathfrak{Y}$ be the universal (Poisson) deformation of Y. Then $\mathfrak{X} \to \mathfrak{Y}$ is locally a Galois covering with Galois group W, giving a diagram*

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ H^2(X,\mathbb{C}) & \longrightarrow & H^2(X,\mathbb{C})/W. \end{array}$$

2. *Put $\mathfrak{Y}' \coloneqq \mathfrak{Y} \times_{H^2(X,\mathbb{C})/W} H^2(X,\mathbb{C})$, with induced diagram*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\Pi} & \mathfrak{Y}' \\ \downarrow & & \downarrow \\ H^2(X,\mathbb{C}) & \xrightarrow{\mathrm{id}} & H^2(X,\mathbb{C}). \end{array}$$

   Then Π is a crepant projective resolution. There is a hyperplane arrangement in $H^2(X, \mathbb{Q})$ such that away from the hyperplanes, Π is an isomorphism (and hence produces affine fibers $\mathfrak{X}_p$).

Example 5.13. Take the A_{n-1} resolution

$$X \to Y \coloneqq \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^n = 0\}.$$

1. (Weyl group) $H^2_{\mathrm{dR}}(X)$ is an $(n-1)$-dimensional vector space generated by components $E_1, \dots, E_{n-1}$, with negative definite intersection pairing $(-,-)$. In it is the A_{n-1} root system, with Weyl group S_n.
2. (Deforming Y) The universal deformation of Y is

   $$\mathfrak{Y} \coloneqq \{(x, y, z, u_1, \dots, u_{n-1}) \in \mathbb{C}^{n+2} : x^2 + y^2 + z^n + u_1 z^{n-2} + \cdots + u_{n-2} z + u_{n-1} = 0\}$$

   where $(u_1, \dots, u_{n-1}) \in \mathbb{C}^{n-1}$ is the base. (By a change of variables, we can get rid of the z^{n-1} term.)
3. (Deforming X) The base for the universal deformation of X ought to be

   $$V \coloneqq \{(s_1, \dots, s_n) \in \mathbb{C}^n : \sum s_i = 0\} \cong H^2_{\mathrm{dR}}(X) \otimes \mathbb{C}.$$

   This is an S_n Galois cover of the base $\mathbb{C}^{n-1}$ for $\mathfrak{Y}$ via

   $$(s_1, \dots, s_n) \mapsto (\sigma_2, \dots, \sigma_n)$$

where σ_i is the i-th elementary symmetric function. Write $\mathfrak{Y}' \coloneqq \mathfrak{Y} \times_{V/S_n} V$, or more explicitly,

$$\mathfrak{Y}' \coloneqq \{(x, y, z, s_1, \ldots, s_n) \in \mathbb{C}^{n+3} : x^2 + y^2 + (z - s_1)(z - s_2)\cdots(z - s_n) = 0,\ \sum s_i = 0\}$$

This is singular along

$$\mathcal{H} \coloneqq \bigcup_{i<j} L_{ij}, \quad L_{ij} \coloneqq \{s_i = s_j\}.$$

Take a simultaneous resolution of all of $\mathfrak{Y}' \to V$ along $\mathcal{H}$ to get the universal deformation $\mathfrak{X} \to V$ of X.

For $p \notin \mathcal{H}$, note that $\mathfrak{X}_p \to \mathfrak{Y}'_p$ are isomorphisms, and therefore $\mathfrak{X}_p$ is affine. For p contained in k different L_{ij}, we require k blow-ups and therefore get k proper curves (resolving an A_{k-1} singularity). Those $\mathfrak{X}_p$ are not affine.

Remark. The general proof is not much harder than this. The key idea is to prove that the hyperplanes arise as codimension 1 faces of the closure of the ample cone for some crepant projective resolution $X' \to Y$. This we do by flopping around. In the process, the argument proves that Y can have only finitely many crepant projective resolutions.

We actually care about *non-commutative deformations* of X, and even in this case there is a nice unobstructedness theorem.

Definition 5.14 (Quantization)**.** Let $(X, \{-,-\})$ be a Poisson variety over k. A **quantization** of X is a sheaf $\mathcal{O}_\hbar$ of flat $k[[\hbar]]$-algebras on X, complete in the $\hbar$-adic topology, such that:

1. there is an isomorphism $\mathcal{O}_\hbar/\hbar \cong \mathcal{O}_X$ given by $f \mapsto \bar{f}$;
2. for any two local sections $f, g \in \mathcal{O}_\hbar$,

$$fg - gf = \hbar\{\bar{f}, \bar{g}\} \bmod \hbar^2.$$

Example 5.15 (Local model)**.** Let (V, ω) be a symplectic vector space. Then $\operatorname{Sym}^* V$ is a Poisson algebra with (degree -2) bracket determined by

$$\{u, v\} \coloneqq \omega(u, v), \quad u, v \in V.$$

Its quantization is the **Weyl algebra**

$$W(V) \coloneqq T(V)/\langle u \otimes v - v \otimes u - \omega(u, v)\rangle,$$

filtered by the monomial degree.

1. Explicitly, for something like $V = T^*\mathbb{A}^1_k$, we have

$$W(T^*\mathbb{A}^1_k) = k[x, \partial_x], \quad \partial_x x - x\partial x = 1.$$

   In general, a quantization of T^*X is therefore the sheaf $\mathcal{D}_X$ of differential operators.

2. To add in the necessary $\hbar$, modify the product structure by

$$v * f \coloneqq fv + \hbar v f, \quad u * v - v * u \coloneqq \hbar[u, v], \quad u, v \in \mathcal{T}_X,\ f \in \mathcal{O}_X,$$

   and then $\hbar$-adically complete it to get $\mathcal{D}_{\hbar,X}$.

Theorem 5.16 (Bezrukavnikov–Kaledin)**.** *Let X be a symplectic resolution with universal formal deformation $\mathfrak{X}/S$. Then there exists a canonical quantization $\tilde{\mathcal{O}}_\hbar$ of $\mathfrak{X}$ universal in the sense that for any quantization $\mathcal{O}_\hbar$ of X, there is a section*

$$s\colon \operatorname{Spec} k[[\hbar]] \to \operatorname{Spec} k[[\hbar]] \hat{\times} S$$

of the projection such that $s^*(\tilde{\mathcal{O}}_\hbar) \cong \mathcal{O}_\hbar$.

Remark. In other words, non-commutative deformations are classified by elements in $H^2_{\mathrm{dR}}(X)[[\hbar]]$ with leading-order term $[\omega]$. Let $\mathcal{Q}(X)$ be the isomorphism classes of quantizations of X. The map $\mathcal{Q}(X) \to H^2_{\mathrm{dR}}(X)[[\hbar]]$ is the **non-commutative period map**.

Proof sketch. The idea is to quantize the whole argument above.

1. (Local model) Prove a "quantum" Darboux theorem: locally, every quantization of $(T^*\mathbb{A}^n_k)^\wedge_\hbar$ is isomorphic to $D \coloneqq D_{\hbar,\mathbb{A}^n_k}$. If $\mathrm{Aut}_{\geq \ell}(D) \subset \mathrm{Aut}(D)$ denotes the subgroup of automorphisms which are the identity on $D/\hbar^\ell D$, then
$$1 \to k[[\hbar]]^* \to D^* \to \mathrm{Aut}_{\geq 1}(D) \to 1$$
is an exact sequence. (This is some Hochschild cohomology computation.)

2. Problem: $H^2_{\mathrm{dR}}(X)$ parametrizing deformations of symplectic structure and $k[[\hbar]]$ parametrizing deformations $\mathrm{Aut}_{\geq 1}(D)$ don't decouple and may be obstructed. Manually analyze the entire lifting problem to show that formally locally, it does decouple, and is unobstructed.

3. (Globalization) Express this all-order unobstructedness in the language of formal geometry. ($\mathrm{Aut}(D)$ becomes a torsor, etc.) ☐

6 Macky (Mar 14): Quantizations of conical symplectic resolutions

Definition 6.1. A **conical symplectic resolution** of weight $n \geq 1$ is a symplectic variety (M, ω) over $\mathbb{C}$, with an action of $\mathbb{S} \coloneqq \mathbb{C}^\times$ such that:

1. $z^*\omega = z^n\omega$;

2. $M \to M_0 \coloneqq \operatorname{Spec} \mathbb{C}[M]$ is a projective resolution of singularities;

3. $\mathbb{S}$ contracts M_0 to a cone point, i.e. the action of $\mathbb{S}$ on $\mathbb{C}[M]$ has positive weights on all generators.

This is a symplectic resolution, as we defined last time, with this additional $\mathbb{S}$ action.

Definition 6.2. Recall that a (formal) **quantization** of M is a sheaf $\mathcal{Q}$ of $\mathbb{C}[[\hbar]]$-algebras with an isomorphism
$$\mathcal{Q}/\hbar\mathcal{Q} \cong \mathcal{O}_M$$
such that the induced Poisson structure is the one induced by ω.

Theorem 6.3 (Bezrukavnikov–Kaledin)**.** *There is a* **period map** *giving an isomorphism*
$$\mathrm{Per}\colon \{\textit{quantizations of } (M,\omega)\} \xrightarrow{\sim} [\omega] + \hbar H^2(M,\mathbb{C})[[\hbar]].$$

Definition 6.4. We need to upgrade the quantization to take into account the $\mathbb{S}$ action. Recall that a G-equivariant sheaf on X is a pair
$$(\mathcal{F}, \alpha\colon a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F})$$
for $G \times X \rightrightarrows^a_p X$ where a is the action and p is the projection. We would like to define an **equivariant quantization** to be a sheaf $\mathcal{Q}$ on M such that
$$a^*\mathcal{Q} \cong a^{-1}\mathcal{Q} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{O}_{\mathbb{S}}[[\hbar]]$$
together with an isomorphism $a^*\omega \cong z^n p^*\omega$. But there can be no isomorphism between $a^*\mathcal{Q}$ and $p^*\mathcal{Q}$ just because they do not quantize the same symplectic form. We fix it by twisting the pullback a^*:
$$a^*_{\mathrm{tw}}\mathcal{Q} \coloneqq a^{-1}\mathcal{Q} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{O}_{\mathbb{S}}[[\hbar]]$$

where in the tensor product, we twist the embedding

$$\mathbb{C}[[\hbar]] \to \mathcal{O}_{\mathbb{S}}[[\hbar]], \quad \hbar \mapsto \hbar^n.$$

This tells us that $\hbar$ has weight n, the same as the symplectic form.

Theorem 6.5 (Losev)**.** *There is a period map*

$$\mathrm{Per}\colon \{\mathbb{S}\text{-equivariant quantizations of } (M,\omega)\} \xrightarrow{\sim} [\omega] + \hbar H^2(M,\mathbb{C}).$$

Definition 6.6. Fix a conical symplectic resolution M with $\mathbb{S}$-equivariant quantization $\mathcal{Q}$. Define sheaves

$$\mathcal{D} \coloneqq \mathcal{Q}[\hbar^{-1/n}], \quad \mathcal{D}(m) \coloneqq \hbar^{-m/n}\mathcal{Q}[\hbar^{1/n}].$$

So there is a chain of inclusions

$$\cdots \subset \mathcal{D}(1) \subset \mathcal{D}(0) \subset \mathcal{D}(-1) \subset \cdots \subset \mathcal{D}.$$

Define the **section rings**

$$A \coloneqq \Gamma_{\mathbb{S}}(\mathcal{D}), \quad A(m) \coloneqq \Gamma_{\mathbb{S}}(\mathcal{D}(m)).$$

Here $\Gamma_{\mathbb{S}}$ is (weight-0, as usual) equivariant sections. There is a filtration of A whose graded components are $A(m)$, and so there is an isomorphism

$$\operatorname{gr} A \xrightarrow{\sim} \mathbb{C}[M]A(m) \xrightarrow{\hbar^{m/n}} \Gamma(\mathcal{D}(0)) \qquad\qquad \to \Gamma(\mathcal{O}_M).$$

Example 6.7. Here are some examples of M with their corresponding A.

M	A
$T^*(G/B)$	central quotient of $U\mathfrak{g}$
hypertoric varieties	central quotient of hypertoric enveloping algebra
$\mathrm{Hilb}^m(\widetilde{\mathbb{C}^2/\Gamma})$	spherical symplectic reflection algebra of $S_m \wr \Gamma$.

These algebras were all of independent interest. If we had a localization relating the algebras on the rhs to D-modules on the lhs, then we would have a unified view of them.

Definition 6.8. The (equivariant) global sections functor is

$$\Gamma_{\mathbb{S}}\colon \mathcal{D}\text{-}\mathsf{Mod}_{\mathbb{S}} \to A\text{-}\mathsf{Mod}.$$

(Here subscript $\mathbb{S}$ means $\mathbb{S}$-equivariant.) The functor in the reverse direction is

$$\mathrm{Loc}\colon A\text{-}\mathsf{Mod} \to \mathcal{D}\text{-}\mathsf{Mod}_{\mathbb{S}}, \quad N \mapsto \mathcal{D} \otimes_A N,$$

called the **localization functor**. Just like in Beilinson–Drinfeld localization, in practice we restrict to certain subcategories:

$$\mathcal{D}\text{-}\mathsf{mod} \subset \mathcal{D}\text{-}\mathsf{Mod}, \quad A\text{-}\mathsf{mod} \coloneqq A\text{-}\mathsf{Mod}_{\mathrm{fg}} \subset A\text{-}\mathsf{Mod}.$$

The category $\mathcal{D}$-mod is of **good** $\mathcal{D}$-modules, which means it has a *coherent* $\mathcal{D}(0)$-lattice.

Definition 6.9. We say **localization holds for** $\mathcal{D}$ or **at** λ where $\lambda = \mathrm{Per}(\mathcal{D})$ if these two functors give an equivalence

$$\mathcal{D}\text{-}\mathsf{mod} \cong A\text{-}\mathsf{mod}.$$

We say **derived localization holds** if the derived functors $R\Gamma_{\mathbb{S}}$ and $L\,\mathrm{Loc}$ give an equivalence.

Theorem 6.10 (BPW)**.** *Let $\eta, \lambda \in H^2(M,\mathbb{C})$, with $\eta = c_1(\text{ample})$. Then:*

1. *derived localization holds at all but finitely many $\lambda + k\eta$ where $k \in \mathbb{C}$;*

2. localization holds at almost all $\lambda + k\eta$ where $k \in \mathbb{Z}$.

Example 6.11. For $T^*(G/B)$, the dominant chamber in $H^2(G/B, \mathbb{C}) \cong \mathfrak{h}^*$ is the ample cone. The theorem says that if we start with λ anywhere, adding enough copies of η gets us into the ample cone, and then localization holds.

Remark. The key ingredient is to take $\lambda \in H^2(M, \mathbb{C})$ and look at the **twistor deformation**, which is the 1-parameter family of deformations parametrized by the line λ. But there are multiple ways of taking a quantization on M_λ and restricting to a quantization of M. Note that the $\mathbb{S}$-action identifies all the non-zero fibers of the twistor deformation.

Theorem 6.12 (Kaledin)**.** *If $\lambda = c_1(\text{ample})$, then $M_\lambda(\infty)$ is affine.*

Definition 6.13 (Hypertoric case)**.** (Reference: the WARTHOG on symplectic duality in this case.) The setup is

$$K \subset T = (\mathbb{C}^\times)^n \curvearrowright T^*\mathbb{C}^n$$

with a choice of character $\eta\colon K \to \mathbb{C}^\times$. This gives a moment map

$$\mu\colon T^*\mathbb{C}^n \to \mathfrak{k}^*$$

with respect to which we can take algebraic symplectic reduction and get the **hypertoric variety**

$$M \coloneqq \mu^{-1}(0) /\!\!/_\eta K.$$

We assume M is smooth. (This is some combinatorial condition.) The torus $\mathbb{S}$ acts on $T^*\mathbb{C}^n$ by $s \cdot (z, w) \coloneqq (s^{-1}z, s^{-1}w)$, and commutes with the T-action, and guarantees that M_0 is contracted to the cone point, the image of $(0,0)$. This gives a conical symplectic resolution of *weight* 2.

Definition 6.14 (Quantization in hypertoric case)**.** Let's look at the **quantization** U_λ of $\mathbb{C}[M]$ for a character $\lambda\colon Z(U) \to \mathbb{C}$. Let $\mathbb{D}$ be the Weyl algebra, the canonical quantization of $T^*\mathbb{C}^n$. Schematic:

$$\begin{array}{ccc}
T \times \mathbb{S} \curvearrowright T^*\mathbb{C}^n & \xleftarrow[\text{quantizes}]{} & T \times \mathbb{S} \curvearrowright \mathbb{D} \\
\Big\downarrow {\scriptstyle \cdot/K} & & \Big\downarrow \\
 & & U \coloneqq \mathbb{D}^K \\
 & & \Big\downarrow \\
T/K \times \mathbb{S} \curvearrowright M & \xleftarrow[\text{quantizes}]{} & U_\lambda \coloneqq U/(\ker\lambda)U.
\end{array}$$

Here U is called the **hypertoric enveloping algebra**. Concretely, write

$$\mathbb{D} \coloneqq \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle / \sim$$

so that under the T-action we can write $\mathbb{D} = \bigoplus_{z \in \mathfrak{t}^*_\mathbb{Z}} \mathbb{D}_z$. The K-invariant parts $\mathbb{D}^K$ are

$$U = \mathbb{D}^K = \bigoplus_{z \in (\mathfrak{t}/\mathfrak{k})^*_\mathbb{Z}} \mathbb{D}_z.$$

Example 6.15. Let $M = T^*\mathbb{P}^1$. (This can be easily extended to $T^*\mathbb{P}^n$.) Here we have

$$1 \to K = \mathbb{C}^\times \xrightarrow{(1,1)} T = (\mathbb{C}^\times)^2 \to \mathbb{C}^\times \to 1.$$

Imagine $\mathbb{D}$ sitting on the lattice $\mathbb{Z}^2$. For example,

$$\mathbb{D}_{(0,0)} = \mathbb{C}[x_1\partial_1, x_2\partial_2].$$

The part $(\mathfrak{t}/\mathfrak{k})^*_{\mathbb{Z}}$ is the anti-diagonal line. So

$$\mathbb{D}_{(i,-i)} = \begin{cases} \mathbb{D}_0 \cdot (x_1\partial_2)^i & i \geq 0 \\ \mathbb{D}_0 \cdot (x_2\partial_1)^i & i \leq 0. \end{cases}$$

Hence in this case we have a triangular decomposition of U as

$$U = \mathbb{D}_0[x_1\partial_2] \oplus \mathbb{D}_0 \oplus \mathbb{D}_0[x_2\partial_1].$$

A calculation shows that

$$Z(U) = \mathbb{C}[x_1\partial_1 + x_2\partial_2] \subset \mathbb{D}_0.$$

(More generally, $Z(U) \cong \operatorname{Sym} \mathfrak{k}$.) So a central character $\lambda \colon Z(U) \to \mathbb{C}$ is specified by where we send $x_1\partial_1 + x_2\partial_2$. The quotient U_λ is therefore twisted differential operators on $T^*\mathbb{P}^1$.

Remark (Kirwan surjectivity). Earlier we said quantizations are indexed by H^2, but here we took a central character instead. In this case, claim: U_λ account for all quantizations of M. To make this plausible, we can check that

$$\{\text{central characters of } U\} \cong \mathfrak{k}^* \to H^2_K(T^*\mathbb{C}^n) \xrightarrow{\text{Kirwan map}} H^2(M, \mathbb{C})$$

is an isomorphism. More generally, if we start with a (conical) symplectic resolution X with Hamiltonian G-action, we can do quantum Hamiltonian reduction of the canonical quantization of X to get a quantization of the algebraic symplectic reduction $\mu^{-1}(0) /\!\!/_\eta G$, with period given using the Kirwan map. So if the Kirwan map is surjective, we produce all quantizations in this way.

Definition 6.16. We can now globalize to get $\mathcal{U}_\lambda$, which quantizes $\mathcal{O}_M$. Note that we could have arrived at U_λ in two ways.

$$\begin{array}{ccc} \mathbb{D} & \searrow & \\ \downarrow & & \\ U & & Y_\lambda \coloneqq \mathbb{D}/(\ker \lambda)\mathbb{D} \\ \downarrow & & \downarrow \\ U_\lambda & & U_\lambda \coloneqq \operatorname{End}_{\mathbb{D}}(Y_\lambda). \end{array}$$

The right hand side path sheafifies, by taking the canonical quantization $(\mathcal{D}, *)$ on $T^*\mathbb{C}^n$ where

$$\begin{aligned} \mathcal{D} &\coloneqq \mathcal{O}_{T^*\mathbb{C}^n}((\hbar^{1/2})) \\ f * g &\coloneqq \operatorname{mult}(e^{\hbar\chi/2}(f \otimes g)), \quad \chi \coloneqq \text{Poisson bivector } \textstyle\sum \partial_{x_i} \wedge \partial_{y_i}. \end{aligned}$$

Here $*$ is called the **Moyal product**. This sheafifies the Weyl algebra in the following way. Compute that

$$\begin{aligned} \hbar^{-1/2}x * \hbar^{-1/2}y &= h^{-1}xy + \frac{\hbar}{2}\hbar^{-1} \\ \hbar^{-1/2}y * \hbar^{-1/2}x &= h^{-1}xy - \frac{\hbar}{2}\hbar^{-1}. \end{aligned}$$

So these satisfy exactly the commutation relations for x_i and ∂_i. (We also see that the $1/2$ is very important, since x_i and ∂_i have weights ± 1.)

Definition 6.17 (Twisting functors)**.** Let $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ be BGG category $\mathcal{O}$. There is a decomposition of it into **blocks**

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\cdot)} \mathcal{O}_\lambda,$$

where $\mathcal{O}_\lambda$ is all modules which have the same central character as $L(\lambda)$. (The Weyl group action here is with a shift by ρ.) We define a few functors on category $\mathcal{O}$.

1. Choose weights $\lambda, \lambda' \in \mathfrak{h}^*$ such that $\lambda - \lambda'$ is integral. There is a unique w such that $w(\lambda - \lambda')$ is dominant integral, and
$$L(w(\lambda - \lambda'))$$
is finite-dimensional. Define the **translation functor**
$$T_\lambda^{\lambda'} : \mathcal{O}_\lambda \hookrightarrow \mathcal{O} \xrightarrow{\otimes L(w(\lambda-\lambda'))} \mathcal{O} \twoheadrightarrow \mathcal{O}_{\lambda'}.$$
This relates different blocks and has a bunch of nice properties.

2. If s is a simple reflection, then we can go from 0 to some integral weight which is on the s-wall, and come back. This composition is called the **wall-crossing functor**. In other words,
$$\theta_s \coloneqq T_\lambda^0 T_0^\lambda : \mathcal{O}_0 \to \mathcal{O}_0.$$

3. Let s be a simple reflection again. Define the **shuffling functor**
$$\mathrm{Sh}_s \coloneqq \mathrm{cone}(\mathrm{id} \to \theta_s) : D^b\mathcal{O}_0 \to D^b\mathcal{O}_0.$$

4. **Twisting functors**. Secretly, twisting functors will be symplectic dual to something defined in terms of translation functors.

7 Macky (Mar 28): Quantizations of conical symplectic resolutions

Today we will define categories $\mathcal{O}_g$ and $\mathcal{O}_a$, which are full subcategories of the two sides of the (derived) localization correspondence we saw last time. They will depend on additional geometric data.

Example 7.1 (B-B decomposition)**.** Let $B \subset G$ be a Borel in a connected complex reductive G. There is a Schubert stratification
$$G/B = \bigsqcup_{w \in W} BwB/B.$$
There is an alternate way of thinking about this. Choose a maximal torus $T \subset B$. Introduce another torus $\mathbb{T} \coloneqq \mathbb{C}^\times$ and take a generic cocharacter $\chi : \mathbb{T} \to T$ with the property that its adjoint action on B has all > 0 weights. This action has finitely many fixed points
$$(G/B)^{\mathbb{T}} = \{wB/B\}_{w \in W}$$
and gives a Bialynicki-Birula decomposition which coincides with the Schubert stratification, i.e.
$$BwB/B = \{p \in G/B : \lim_{t \to 0} t \cdot p = wB/B\}.$$
Pass to $\mathbb{T}$ acting on $T^*(G/B) \eqqcolon M$. Then $M^{\mathbb{T}}$ is still the same, and
$$X_w^\circ \coloneqq \{p \in M : \lim_{t \to 0} t \cdot p = wB/B\}$$
has closure $X_w \coloneqq \overline{X_w^\circ}$ which is the conormal to $\overline{BwB/B}$ in G/B. Let
$$M^+ \coloneqq \bigcup X_w.$$

Example 7.2 (BGG category $\mathcal{O}$)**.** Fix λ. Localization says
$$(\lambda\text{-twisted } D(G/B)\text{-modules}) \simeq (A_\lambda\text{-modules}).$$
Define associated categories $\mathcal{O}$ as follows.

1. On the D-module side, take the subcategory given by those modules which are regular and have singular support in M^+. This is the **geometric category** $\mathcal{O}_g$.
2. On the A_λ-module side, this corresponds the subcategory to $U(\mathfrak{b})$-locally finite modules. This is the **algebraic category** $\mathcal{O}_a$.

Soergel says that if λ is regular, then $\mathcal{O}_a \simeq \mathcal{O}$ via some non-trivial equivalence.

Remark. We started with the choice of a generic cocharacter, but the choice is really just a choice of standard Borel.

Definition 7.3. In general, the definition of $\mathcal{O}_g$ and $\mathcal{O}_a$ depends on a $\mathbb{T} := \mathbb{C}^\times$ Hamiltonian action on M which commutes with the $\mathbb{S}$-action. Assume that $M^{\mathbb{T}}$ is a finite set $\{p_\alpha\}_{\alpha\in I}$. Then we make the exact same definition as the BGG category $\mathcal{O}$ case.

1. (Geometric side) Define the attracting set

$$X_\alpha^\circ := \{p \in M : \lim_{t\to 0} t\cdot p = p_\alpha\}$$

with closure $X_\alpha := \overline{X_\alpha^\circ}$. Let

$$M^+ := \bigcup X_\alpha$$

be the **relative core**. The **geometric category** $\mathcal{O}_g$ is the full subcategory of D-modules N which are:

   (a) (supported on conormal) set-theoretically supported on M^+;
   (b) (regularity) there exists a $D(0)$-lattice $N(0) \subset N$ which are stable under an action of ξ.

2. (Algebraic side) The $\mathbb{T}$-action gives a grading $A = \bigoplus_{k\in\mathbb{Z}} A^k$ on the section ring. Define

$$A^+ := \bigoplus_{k\geq 0} A^k.$$

The **algebraic category** $\mathcal{O}_a$ is the full subcategory of A-modules which are A^+-locally finite.

Example 7.4 (Hypertoric review)**.** Let $K \subset T := (\mathbb{C}^\times)^n$ be a subtorus acting on $T^*\mathbb{C}^n$. Let $\mathbb{S}$ be a torus scaling the fibers. After choosing a character $\eta : K \to \mathbb{C}^\times$, we get a hypertoric variety

$$M := \mu^{-1}(0) /\!\!/_\eta K$$

with residual T/K and $\mathbb{S}$-actions, making it a conical symplectic reduction. Let's review some associated objects.

1. Let $U := \mathbb{D}^K$ be the **hypertoric enveloping algebra** discussed last time. For example,

$$Z(U) \simeq \mathrm{Sym}_{\mathbb{C}}(\mathfrak{k}),$$

which lets us view η as a central character of U. Then the section ring is

$$U_\eta := U/(\ker\eta)U.$$

Recall that $\mathfrak{k}^\times \cong H^2(M;\mathbb{C})$ via Kirwan surjectivity.

2. Fix some combinatorial data, namely the hyperplane arrangement. For example, for $M = T^*\mathbb{P}^n$ with $n = 1$,

$$K = \mathbb{C}^\times \xrightarrow{t\mapsto(t,t)} T = (\mathbb{C}^\times)^2$$

we have $W_{\mathbb{Z}} := \mathfrak{t}_{\mathbb{Z}}^* \xrightarrow{(1,1)} \mathfrak{k}_{\mathbb{Z}}^* \to 0$. A **hyperplane arrangement** is the data of

$$\Lambda_0 := (\ker W_{\mathbb{Z}} \to \mathfrak{k}_{\mathbb{Z}}^*)$$

along with a choice $\eta \in \mathfrak{k}_{\mathbb{Z}}^*$. This really is a hyperplane arrangement, where the hyperplanes are where the coordinate axes intersect the affine $\Lambda_0 + \eta$.

3. There is a T-action on

$$\mathbb{D} = \mathbb{C}\langle x_1, x_2, \partial_1, \partial_2\rangle / \sim$$

giving a decomposition $\bigoplus_{z\in W_{\mathbb{Z}}} \mathbb{D}_z$. We worked out last time that

$$\mathbb{D}_0 = \mathbb{C}[x_1\partial_1, x_2\partial_2],$$

and that there is a natural triangular decomposition

$$U = \mathbb{D}^K = \mathbb{C}[x_2\partial_1] \otimes_{\mathbb{C}} \mathbb{D}_0 \otimes_{\mathbb{D}} \mathbb{C}[x_1\partial_2].$$

Definition 7.5 (Hypertoric category $\mathcal{O}$)**.** Consider $\mathbb{D}$-modules (or U-modules or U_η-modules) M which are **weight modules** for $\mathbb{D}_0$, i.e.

$$M = \bigoplus_{w\in W_{\mathbb{Z}}} M_w$$

where M_w are generalized eigenspaces corresponding to w. (Really there is some ambiguity here, because the identification of $\mathbb{D}_0$ is not canonical; we had to choose some normal ordering. So really we have to talk about a W-torsor $W_{\mathbb{Z}}$ and "quantized" hyperplane arrangements.) It is a computation in the Weyl algebra that

$$\mathbb{D}_z \cdot M_w \subset M_{z+w}.$$

Example 7.6. Return to $T^*\mathbb{P}^1$ with $\eta = (3,0)$ for example. Here, $Z(U) = \mathbb{C}[x_1\partial_1 + x_2\partial_2]$. If M is a weight module for U_η, then

$$\operatorname{supp}(M) \subset \Lambda_0 + \eta.$$

Let's describe a few U_η-modules. There is a decomposition of the lattice points in $\Lambda_0 + \eta$ into chambers $\Delta_{++}, \Delta_{-+}, \Delta_{+-}$ where the two coordinates have the respective signs. For each **sign vector** $\alpha \in \{\pm 1\}^n$, define an associated $\mathbb{D}$-module L_α. For our example,

$$\begin{aligned} L_{++} &\coloneqq \mathbb{D}/\mathbb{D}\langle\partial_1, \partial_2\rangle = \mathbb{C}[x_1, x_2] \\ L_{--} &\coloneqq \mathbb{D}/\mathbb{D}\langle x_1, x_2\rangle = \mathbb{C}[\partial_1, \partial_2] \\ L_{+-} &\coloneqq \mathbb{D}/(\mathbb{D}\langle\partial_1\rangle + \mathbb{D}\langle x_2\rangle) \\ L_{-+} &\coloneqq \mathbb{D}/(\mathbb{D}\langle x_1\rangle + \mathbb{D}\langle\partial_2\rangle). \end{aligned}$$

Let L_α^η be the projection of L_α to $U_\eta - \mathsf{Mod}$, i.e. to only the weight spaces that matter. For example,

$$\begin{aligned} L_{++}^{\eta=3} &= \mathbb{C}\langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\rangle \\ L_{--}^{\eta=3} &= 0. \end{aligned}$$

Theorem 7.7. *There is a bijection*

$$\begin{aligned} \{\textit{simples in } U_\eta\text{-}\mathsf{Mod}\}/\sim &\leftrightarrow \mathcal{F}_\eta \\ L_\alpha^\eta &\leftrightarrow \{\alpha : \Delta_\alpha \neq \varnothing\} \end{aligned}$$

where $\mathcal{F}$ is the η-feasible set of sign vectors.

Example 7.8. Continuing, this means $\operatorname{supp}(L_\alpha^\eta) = \Delta_\alpha$. Using a cocharacter $\xi\colon \mathbb{T} \to T/K$, we get a notion of positivity U_η^+. The algebraic category $\mathcal{O}$ is therefore

$$\mathcal{O}_a \coloneqq \begin{pmatrix} \text{full subcategory of } U_\eta - \mathsf{Mod} \\ \mathbb{D}_0\text{-weight modules which are } U_\eta^+\text{-locally finite} \end{pmatrix}.$$

Combinatorially, a cocharacter ξ of T/K is the same data as

$$\xi \in (\mathfrak{t}/\mathfrak{k})_{\mathbb{Z}} \cong (\Lambda_0)^\vee,$$

which we called a **polarization** before. So specifying ξ determines which lattice points on $\Lambda_0 + \eta$ we call positive. The whole package

$$\mathbb{X} \coloneqq (\Lambda_0 \subset W_{\mathbb{Z}}, \eta, \xi)$$

we called a polarized hyperplane arrangement. Write $\mathcal{O}_a = \mathcal{O}(\mathbb{X})$.

Theorem 7.9. *Let $\mathcal{P}_{\eta,\xi}$ denote the η-feasible and ξ-bounded sign vectors. Then*

$$\{\{\text{simples in } \mathcal{O}(\mathbb{X})\}/\sim \leftrightarrow \mathcal{P}_{\eta,\xi}.$$

Remark. In terms of the hyperplane arrangement, we can actually describe the relative core M^+ as the union of some toric varieties given by the chambers Δ_α where $\alpha \in P_{\eta,\xi}$.

Example 7.10 (Symplectic duality for BGG $\mathcal{O}$)**.** Now that we have seen an extended example, we can discuss symplectic duality. Previously, we described the Koszul self-duality of BGG category $\mathcal{O}$, which decomposes into blocks

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/W\cdot} \mathcal{O}_\lambda.$$

Concentrate on the principal block $\mathcal{O}_0$, which has simples $\{L(w\cdot 0)\}_{w\in W}$, with projective covers $\{P(w\cdot 0)\}_{w\in W}$. Symplectic duality in this situation comes from the fact that there are two completely different ways of relating this to the geometry.

1. ("Higgs side") Use BB localization and Riemann–Hilbert:

   $$\mathcal{O}_0 \xrightarrow{\text{loc}} D(G/B) \xrightarrow{RH} \mathsf{Perv}_{(B)}(GB)$$

   where simples L correspond to simples IC.

2. ("Coulomb side", Soergel's $\mathbb{V}$) Define the **coinvariant algebra**

   $$C \coloneqq \mathrm{Sym}_{\mathbb{C}}(\mathfrak{h})/\,\mathrm{Sym}_{\mathbb{C}}(\mathfrak{h})^W_+.$$

   Fact: there is a natural isomorphism $C \cong H^*(G^\vee/B^\vee, \mathbb{C})$. For example, for SL_2, we have

   $$C = \mathbb{C}[x]/(x^2) \cong H^*(\mathbb{P}^1, \mathbb{C})$$

   with $\deg \mathfrak{h} = 2$. Then Soergel proved the following.

Theorem 7.11 (Soergel)**.** *1.* $\mathrm{End}_{\mathcal{O}_0}(P(w\cdot 0)) \cong C$ *canonically.*

2. There is a functor

$$\mathcal{O} \xrightarrow{\mathrm{Hom}(P(w\cdot 0),-)} \mathsf{mod} - C \cong C - \mathsf{mod}$$

sending P_w to a combinatorially-defined module D_w called a Soergel module. This functor is fully faithful on projectives.

3. There is a functor

$$D^b_{(B^\vee)}(G^\vee/B^\vee) \xrightarrow{H^*(-)} C - \mathsf{grmod}$$

which is fully faithful on semisimples.

Remark. Koszul self-duality combines these two:

$$\begin{array}{ccc} L_w \in \mathcal{O}_0(\mathfrak{g}) & & \mathcal{O}_0(\mathfrak{g}^\vee) \ni P_{ww_0} \\ \big\downarrow\sim & \nearrow & \\ \mathrm{IC}_w \in \mathsf{Perv}(G/B) & & \end{array}.$$

To formulate this Koszul duality more precisely, we get the following.

Theorem 7.12. *There is an isomorphism of ungraded rings*

$$\mathrm{End}_{\mathcal{O}_0}(\bigoplus P_w) \cong \mathrm{Ext}^\bullet_{\mathcal{O}_0}(\bigoplus L_w).$$

Remark. Via this isomorphism, $\mathrm{End}_{\mathcal{O}_0}(\oplus P_w)$ receives a natural grading. So we can define *graded* category $\tilde{\mathcal{O}}_0$, giving a grading shift $\langle 1\rangle$.

Theorem 7.13 (BGS)**.** *There is a triangulated equivalence*

$$(D^b\tilde{\mathcal{O}}_0, [1], \langle 1\rangle) \xrightarrow{\sim} (D^b\tilde{\mathcal{O}}_0, [1], \langle -1\rangle[1]).$$

Remark. This kind of behavior with two shifts is really the hallmark of Koszul duality. It means we really can't forget about the grading.

The symplectic duality BLPW conjectures says that associated to a physical theory M_C and M_H, and to each we associate a category $\mathcal{O}$. Their conjecture says that each one has a graded lift $\tilde{\mathcal{O}}_C$ and $\tilde{\mathcal{O}}_H$ and there is a derived equivalence between the two with exactly the BGS kind of triangulated auto-equivalence.

8 Ivan (Apr 04): The affine Grassmannian

Let $\mathcal{K} \coloneqq \mathbb{C}((t))$ and $\mathcal{O} \coloneqq \mathbb{C}[[t]]$. Think of $\operatorname{Spec}\mathcal{K} \eqqcolon \mathring{\mathcal{D}}$ as a formal punctured disk sitting inside $\operatorname{Spec}\mathcal{O} \eqqcolon \mathcal{D}$ which is a formal disk. This is a local picture, which embeds into a global picture that we will consider later.

Definition 8.1. Let G be a reductive group and P_0 be a trivial G-principal bundle over $\mathcal{D}$. The **affine Grassmannian** is

$$\mathrm{Gr} \coloneqq \{(P,\varphi) : P \in \mathsf{Bun}_G,\ \varphi\colon P_0|_{\mathring{\mathcal{D}}} \xrightarrow{\cong} P|_{\mathring{\mathcal{D}}}\}.$$

An equivalent description is

$$\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O}).$$

(We can think of $G(\mathcal{O})$ not as a Borel but as a parabolic. If we get rid of the negative parts at level-0, then we get the Iwahori subgroup, which behaves more like a Borel.)

Proposition 8.2. Gr *is an ind-scheme.*

1. *There is an infinite filtration by $G(\mathcal{O})$-stable subsets*

$$\mathrm{Gr}_1 \subset \mathrm{Gr}_2 \subset \cdots \subset \mathrm{Gr}.$$

2. *The inclusion $\mathrm{Gr}_i \hookrightarrow \mathrm{Gr}_j$ for $i < j$ is a projective embedding.*

Remark (Lusztig)*.* Define a map

$$gG(\mathcal{O}) \mapsto \mathrm{Ad}_g\, \mathfrak{g}(\mathcal{O}).$$

Then we have a description

$$\mathrm{Gr}_i = \{L \subset \mathfrak{g}(\mathcal{K}) : t^i\mathfrak{g}(\mathcal{O}) \subset L \subset t^{-i}\mathfrak{g}(\mathcal{O})\}.$$

It will turn out that the Gr_i have Poisson structures.

Definition 8.3. A **Manin triple** is $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ where:

1. $\mathfrak{g}$ is a Lie algebra with non-degenerate invariant pairing $(\cdot,\cdot)$;
2. $\mathfrak{g}_+, \mathfrak{g}_-$ are isotropic sublagebras under this pairing;
3. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$.

The pairing must induce an isomorphism $\mathfrak{g}_- \cong \mathfrak{g}_+^\vee$.

Definition 8.4. A **Lie bialgebra** $\mathfrak{g}$ is a Lie algebra equipped with a **cobracket**

$$\delta\colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$$

such that;

1. (cobracket) the dual $\delta^\vee\colon \wedge^2 \mathfrak{g}^\vee \to \mathfrak{g}^\vee$ is a Lie bracket;
2. (cocycle condition) $d\delta = 0$, or equivalently

$$\delta([a,b]) = (\mathrm{ad}_a \otimes 1 + 1 \otimes \mathrm{ad}_a)\delta(b) - (\mathrm{ad}_b \otimes 1 + 1 \otimes \mathrm{ad}_b)\delta(a).$$

Definition 8.5. Given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, we can construct a Lie bialgebra $\mathfrak{g}_+$. This is because the bracket on $\mathfrak{g}_-$ becomes a co-bracket on $\mathfrak{g}_+^\vee$. So we can think about Manin triples instead of Lie bialgebras.

Example 8.6. In our case, $\mathfrak{g}(\mathcal{K})$ naturally splits as

$$\mathfrak{g}(\mathcal{K}) = t^{-1}\mathfrak{g}[t^{-1}] + \mathfrak{g}(\mathcal{O}).$$

The pairing is the residue pairing

$$(f(t), g(t)) = \mathrm{Res}_{t=0}(K(f(t), g(t)))$$

where K is the Killing form on G. We want to focus on $t^{-1}\mathfrak{g}[t^{-1}]$, because it is the tangent space of $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ at the identity. So we can compute a cobracket on it. It is helpful to write it in terms of the **R-matrix**

$$r \coloneqq \sum_{n\geq 0} \sum_{i=1}^{\dim \mathfrak{g}} x_i t^{-n-1} \otimes x^i t^n$$

where $\{x_i\}$ and $\{x^i\}$ are dual bases in $\mathfrak{g}$. If we write $u = t \otimes 1$ and $v = 1 \otimes t$, then

$$r = \frac{\Omega}{u - v}$$

where Ω is the quadratic Casimir, expanded at $|u| > |v|$. In this notation,

$$\delta(a) = [a \otimes 1 + 1 \otimes a, r].$$

Definition 8.7 (Poisson structure)**.** We think of δ as the "derivative of the Poisson bivector"

$$P\colon G(\mathcal{K}) \to \wedge^2 G(\mathcal{K}).$$

So we want to exponentiate the above formula for δ, giving

$$P|_g = \mathrm{Ad}_g\, r - r.$$

Since the Casimir is invariant, we see that this descends to $G(\mathcal{K})/G(\mathcal{O})$. This gives a **Poisson structure** on Gr.

Definition 8.8. There is a $G(\mathcal{K})$-action on $G(\mathcal{K})/G(\mathcal{O})$, along with another $\mathbb{C}_s^\times$ scaling the loop parameter t. Given a coweight $\lambda^\vee\colon \mathbb{C}^\times \to T$, we can compose with the inclusion $\mathrm{Spec}(\mathcal{K}) \to \mathbb{C}^\times$ of the formal neighborhood of 0, to get

$$t^{\lambda^\vee} \in \mathrm{Hom}(\mathrm{Spec}\,\mathcal{K}, T) = T(\mathcal{K}) \subset G(\mathcal{K}).$$

We write $t^{\lambda^\vee} \in \mathrm{Gr}$ for the induced points.

Lemma 8.9. *The $t^{\lambda^\vee}$ are the only T-fixed points, and are also $\mathbb{C}^\times$-fixed points.*

Remark. This implies that the $G(\mathcal{O})$-orbits are

$$\mathrm{Gr}^{\lambda^\vee} := G(\mathcal{O}) \cdot t^{\lambda^\vee}.$$

Since $\mathrm{Gr}^{w\lambda^\vee} = \mathrm{Gr}^{\lambda^\vee}$, we can always pick $\lambda^\vee$ to be *dominant*.

Proposition 8.10. *1. There is a decomposition*

$$\mathrm{Gr} = \bigsqcup_{\lambda^\vee \text{ dominant}} \mathrm{Gr}^{\lambda^\vee}.$$

2. *There is a stratification*

$$\overline{\mathrm{Gr}^{\lambda^\vee}} = \bigsqcup_{\mu^\vee \le \lambda^\vee} \mathrm{Gr}^{\mu^\vee}.$$

Here $\mu^\vee \le \lambda^\vee$ means $\lambda^\vee - \mu^\vee \in R^\vee_+$ is a positive co-root.

3. $\mathrm{Gr}^{\lambda^\vee}$ *is a vector bundle over certain partial flag varieties G/P. It is contracted to the zero section by $\mathbb{C}^\times_s$.*

4. $\mathrm{Gr}^{\lambda^\vee}$ *is the smooth part of its closure. In particular, $\overline{\mathrm{Gr}^{\lambda^\vee}}$ is smooth iff $\lambda^\vee$ is minuscule.*

Definition 8.11. Whenever there is a stratification $\mathbb{S}$, we can talk about **perverse sheaves** with respect to it. They form a full subcategory

$$\mathsf{Perv}_{\mathbb{S}}(\mathrm{Gr}) \subset D^b_{\mathbb{S}}(\mathrm{Gr}, \mathbb{C})$$

of the category of $\mathbb{S}$**-constructible** $\mathbb{C}$-sheaves. This means sheaves which satisfy:

1. $H^k(X, \mathcal{F}) = 0$ if $k \neq 0$;
2. $H^k(\mathcal{F})|_S$ is a local system of finitely generated $\mathbb{C}$-modules, for any stratum $S \in \mathbb{S}$.

In other words, over each cell we have just a complex of constant sheaves (valued in $\mathbb{C}$-linear spaces). The subcategory we will take allows for non-trivial cohomological behavior in higher degrees.

Example 8.12. Take the constant sheaf on $\mathrm{Gr}^{\lambda^\vee}$ and extend by 0. This creates some "correction" on the boundary, and in general produces a *complex*

$$\mathrm{IC}_{\lambda^\vee} \in \mathsf{Perv}_{\mathbb{S}}(\mathrm{Gr})$$

called the **intersection cohomology (IC) sheaves**. Such IC sheaves are simple objects in $\mathsf{Perv}_{\mathbb{S}}(\mathrm{Gr})$. Now act by $G(\mathcal{O})$. Since these sheaves are *constant* on the open part, we get a trivial $G(\mathcal{O})$-equivariant structure there.

Proposition 8.13. *Let $\mathsf{Perv}_{G(\mathcal{O})}(\mathrm{Gr})$ be the category of $G(\mathcal{O})$-equivariant perverse sheaves. Then*

$$\mathsf{Perv}_{G(\mathcal{O})}(\mathrm{Gr}) \simeq \mathsf{Perv}_{\mathbb{S}}(\mathrm{Gr}).$$

Definition 8.14 (Lusztig)**.** In this category we have a **convolution** product. Start with $\mathrm{Gr} \times \mathrm{Gr}$ and consider the composition

$$\mathrm{Gr} \times \mathrm{Gr} \xleftarrow{\;p\;} G(\mathcal{K}) \times \mathrm{Gr} \overset{m}{\dashrightarrow} \mathrm{Gr}.$$

We want to do something like $m_*(p^* \mathcal{A}_1 \boxtimes^L \mathcal{A}_2)$. However the dashed arrow has infinite-dimensional fibers. We fix it by introducing an intermediate step

$$\begin{array}{ccccc} \mathrm{Gr} \times \mathrm{Gr} & \xleftarrow{\;p\;} & G(\mathcal{K}) \times \mathrm{Gr} & \overset{m}{\dashrightarrow} & \mathrm{Gr} \\ & & {\scriptstyle q}\searrow & & \nearrow{\scriptstyle m} \\ & & & G(\mathcal{K}) \times_{G(\mathcal{O})} \mathrm{Gr}. & \end{array}$$

Then we define

$$\mathcal{A}_1 * \mathcal{A}_2 \coloneqq Rm_* \tilde{\mathcal{A}}$$

for some perverse sheaf $\tilde{\mathcal{A}}$ on $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathrm{Gr}$ such that

$$q^* \tilde{\mathcal{A}} = p^*(\mathcal{A}_1 \boxtimes^L \mathcal{A}_2).$$

(Such a perverse sheaf $\tilde{\mathcal{A}}$ exists and is unique, by equivariant descent.) Hence we get

$$D^b_{G(\mathcal{O})}(\mathrm{Gr})_c \times D^b_{G(\mathcal{O})}(\mathrm{Gr}) \to D^b_{G(\mathcal{O})}(\mathrm{Gr}).$$

We need compact support for one factor.

Lemma 8.15. *1. If $\mathcal{A}_1, \mathcal{A}_2$ are perverse, then $\mathcal{A}_1 * \mathcal{A}_2$ is perverse.*

2. $\mathcal{A}_1 * \mathcal{A}_2 \simeq \mathcal{A}_2 * \mathcal{A}_1$, but non-canonically.

3. $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3 \simeq \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$ canonically.

Theorem 8.16 (Geometric Satake correspondence)**.** *There is an equivalence of Tannakian categories*

$$\begin{array}{ccc} \mathsf{Perv}_{\mathbb{S}}(\mathrm{Gr}_G) & \xrightarrow{\sim} & \mathsf{Rep}(G^\vee) \\ & {\scriptstyle H^*}\searrow \quad \swarrow{\scriptstyle \textit{forgetful}} & \\ & \mathsf{Vect} & \end{array}$$

where the downward arrows are fiber functors. The correspondence satisfies

$$\mathcal{A}_1 * \mathcal{A}_2 \leftrightarrow V_1 \otimes V_2$$
$$H^*(\mathcal{A}) \leftrightarrow V$$
$$\mathrm{IC}^{\lambda^\vee} \leftrightarrow V_{\lambda^\vee}\text{-}\textit{irreps}.$$

9 Gus (Apr 11): Introduction to BFN spaces

What are Braverman, Finkelberg and Nakajima trying to do? Take a complex reductive group G and a finite-dimensional rep V. Given this data, we can form the space

$$\mathcal{M}_H \coloneqq \mu^{-1}(0) /\!\!/ G.$$

When $G = (\mathbb{C}^\times)^n$, we get hypertoric varieties. When (G, V) comes from a quiver, we get Nakajima quiver varieties.

From $\mathcal{M}_H$ we can produce a quantization $\mathcal{A}_\hbar = \mathbb{C}_\hbar[\mathcal{M}_H]$, which is some associative algebra, and associated to it is some category $\mathcal{O}$, like the usual category $\mathcal{O}$ in Lie theory. We saw in the hypertoric case that there is an interesting combinatorial duality, called Gale duality, which allowed us to produce a *dual* hypertoric variety $\mathcal{M}^!$ and dual algebra $\mathcal{A}_\hbar[\mathcal{M}^!]$ and dual category $\mathcal{O}^!$. The upshot of Macky's talks was that $(\mathcal{O}, \mathcal{O}^!)$ were in some sort of Koszul duality.

Problem 9.1. Given a general pair (G, V), can we construct $\mathcal{M}^!, \mathbb{C}_\hbar[\mathcal{M}^!], \mathcal{O}^!$ dual to $\mathcal{M}, \mathbb{C}_\hbar[\mathcal{M}], \mathcal{O}$?

What BFN do in their sequence of paper is as follows. From (G, V), they produce an affine Poisson variety $\mathcal{M}_C$ which is a candidate for $\mathcal{M}^!$. The entirety of this talk will be to explain this recipe. (BFN do not formulate or prove any kind of Koszul duality between their $\mathcal{M}_C$ and $\mathcal{M}_H$, but there is a recent paper by Ben Webster which does.)

Example 9.1 (Basic pattern/analogy in rep theory)**.** Let G be a simple complex Lie group with $T \subset B \subset G$. Let

$$\mathcal{N} \coloneqq \{x \in \mathfrak{g} : x \text{ nilpotent}\}$$

be its nilpotent cone. Think of G/B as the variety of Borel subalgebras in $\mathfrak{g}$. The tangent space $T_{gB}(G/B)$ is $g\mathfrak{b}g^{-1}$ where $\mathfrak{b}$ is the standard Borel. Dually, using the Killing form, identify

$$T^*_{gB}(G/B) = (g\mathfrak{b}g^{-1})^{\perp} = g\mathfrak{n}g^{-1}$$

where $\mathfrak{n}$ is the unipotent radical. So set-wise we can think of $T^*(G/B)$ as

$$T^*(G/B) = \{(gB, x) \in G/B \times \mathcal{N} : x \in g\mathfrak{n}g^{-1}\}.$$

An equivalent way to say this is that

$$T^*(G/B) = G \times_B \mathfrak{n} \coloneqq (G \times \mathfrak{n})/B, \quad b \cdot (g, x) = (gb^{-1}, bxb^{-1}),$$

so that there is an isomorphism $[g, n] \mapsto (gB, gng^{-1})$. Using this idea, we can form the Steinberg variety

$$Z \coloneqq X \times_{\mathcal{N}} X = \{(g_1B, g_2B, x) \in G/B \times G/B \times \mathcal{N} : x \in g_1\mathfrak{n}g_1^{-1} \cap g_2\mathfrak{n}g_2^{-1}\},$$

also called a "variety of triples". We can identify Z as the union of conormal bundles to G-orbits in $G/B \times G/B$. There are three projections

$$p_{ij} \colon X \times_{\mathcal{N}} X \times_{\mathcal{N}} X \to Z$$

equipping H^{BM}_G or $K_G(Z)$ with convolution products.

Theorem 9.2 (Chriss–Ginzburg, Lusztig)**.** *With the convolution product,*

$$K_G(Z) \cong \mathbb{Z}[W_{\mathit{aff}}]$$

where $W_{\mathit{aff}} \coloneqq W_G \ltimes$ *weight lattice is the affine Weyl group. If* $\mathbb{C}^\times$ *scales cotangent fibers,*

$$K_{G \times \mathbb{C}^\times}(Z) \cong \mathcal{H}_W$$

where $\mathcal{H}_W$ *is the affine Hecke algebra of* W*.*

Definition 9.3. One way to think about the BFN theory is that it "affinizes" the above picture. First replace the flag variety G/B with the affine Grassmannian for G, which recall is

$$\mathrm{Gr}_G \coloneqq G(\mathcal{K})/G(\mathcal{O}), \quad \mathcal{K} \coloneqq \mathbb{C}((z)),\ \mathcal{O} \coloneqq \mathbb{C}[[z]].$$

Then replace the cotangent bundle with

$$\mathcal{T}_{G,V} \coloneqq G(\mathcal{K}) \times_{G(\mathcal{O})} V[[z]].$$

This embeds into $\mathrm{Gr}_G \times V((z))$, given by $[g, v] \mapsto ([g], gv)$. So set-wise,

$$\mathcal{T}_{G,V} = \{([g], x) : [g] \in \mathrm{Gr},\ x \in V((z)) \cap gV[[z]]\}.$$

Now we need an analogue of the Steinberg Z. The first try is

$$Z \coloneqq \mathcal{T}_{G,V} \times_{V((z))} \mathcal{T}_{G,V}$$

and study its $G(\mathcal{K})$-equivariant homology or K-theory. This turns out to be bad because Z is infinite-dimensional. But even in the finite-dimensional case, we can write

$$Z \cong G \times_B R, \quad R \coloneqq \{(gB, x) : x \in \mathfrak{n} \cap g\mathfrak{n}g^{-1}\}$$

where R is a union of conormals to Schubert cells in T^*G/B. Then we get an induction isomorphism

$$K_G(Z) = K_G(G \times_B R) = K_B(R).$$

It is better to work with $K_B(R)$ in the general BFN setup. The affine analogue of R is to take the map

$$a\colon \mathcal{T}_{G,V} = G(\mathcal{K}) \times_{G(\mathcal{O})} V[[z]] \to V((z))$$

and look at the pre-image

$$\mathcal{R}_{G,V} \coloneqq a^{-1}(V[[z]]) = \{([g], s) : s \in V[[z]] \cap g \cdot V[[z]]\}.$$

This is what BFN calls the **variety of triples**. The moduli-theoretic interpretation of it is as the moduli of (P, φ, s) where:

1. P is a principal G-bundle on the formal disk $\mathbb{D}$;
2. φ is a trivialization of P on the punctured formal disk $\mathbb{D}^\times$;
3. s is a section of $P \times_G V$ such that s is regular at zero under φ.

Example 9.4. If $V = 0$, then $R_{G,V} = \mathrm{Gr}_G$.

Example 9.5. If $G = \mathbb{C}^\times$ is abelian and $V = \mathbb{C}$, then

$$\mathrm{Gr}_G = \bigsqcup_{n \in \mathbb{Z}} \{z^n\}$$

is a collection of discrete points, one for each possible valuation of a Laurent series. Then

$$\mathcal{T} = \bigsqcup_{n \in \mathbb{Z}} \{z^n\} \times z^n \mathcal{O}$$

where $\mathcal{O} = \mathbb{C}[[z]]$. The space $\mathcal{R}$ is also some infinite-rank bundle over Gr:

$$\mathcal{R} = \bigsqcup_{n \in \mathbb{Z}} \{z^n\} \times (z^n \mathcal{O} \cap \mathcal{O}).$$

So $\mathcal{R} \subset \mathcal{T}$ as bundles over Gr_G, but with fibers of finite codimension.

Example 9.6. If G is simple and V is the adjoint $\mathfrak{g}$, then

$$\mathcal{R}_{G,V} = \{([g], x) \in \mathrm{Gr}_G \times \mathfrak{g}((z)) : x \in \mathfrak{g}[[z]] \cap g\mathfrak{g}[[z]]g^{-1}\}.$$

This is the "affine Grassmannian Steinberg variety". It is the union of conormals to $G(\mathcal{O})$-orbits in $T^* \mathrm{Gr}_G$.

Definition 9.7. Remember we want to study the affine analogue of $K_B(R)$, which for us is $K_{G(\mathcal{O})}(\mathcal{R}_{G,V})$, as convolution algebras. Then we take Spec to construct $\mathcal{M}^!$. Take the diagram

$$\mathrm{Gr} \times \mathrm{Gr} \xleftarrow{p} G(\mathcal{K}) \times \mathrm{Gr} \xrightarrow{q} G(\mathcal{K}) \times_{G(\mathcal{O})} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}$$

so by equivariant descent we get

$$H^{G(\mathcal{O}) \times G(\mathcal{O})}(G(\mathcal{K}) \times \mathrm{Gr}) = H^{G(\mathcal{O})}(G(\mathcal{K}) \times_{G(\mathcal{O})} \mathrm{Gr}).$$

Analogously, for $\mathcal{R}$, we would like to have the diagram

$$\begin{array}{ccccccc} \mathcal{T} \times \mathcal{R} & \xleftarrow{p} & G(\mathcal{K}) \times \mathcal{R} & \xrightarrow{q} & G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R} & \xrightarrow{m} & \mathcal{T} \\ ([g_1, g_2 v], [g_2, v]) & \longleftarrow & (g_1, [g_2, v]) & \longrightarrow & [g_1, [g_2, v]] & \longrightarrow & [g_1 g_2, v]. \end{array}$$

This diagram must be restricted to $\mathcal{R} \subset \mathcal{T}$:

$$\mathcal{R} \times \mathcal{R} \longleftarrow p^{-1}(\mathcal{R} \times \mathcal{R}) \longrightarrow qp^{-1}(\mathcal{R} \times \mathcal{R}) \xrightarrow{m} \mathcal{R}$$

in order for the multiplication map to land in $\mathcal{R}$. Again by equivariant descent,

$$H^{G(\mathcal{O}) \times G(\mathcal{O})}(p^{-1}(\mathcal{R} \times \mathcal{R})) \cong H^{G(\mathcal{O})}(qp^{-1}(\mathcal{R} \times \mathcal{R})).$$

Using the diagram, we therefore can define a convolution product on $H^{G(\mathcal{O})}(\mathcal{R}_{G,V})$:

$$c_1 * c_2 \coloneqq m_*(q^*)^{-1} p^!(c_1 \boxtimes c_2).$$

Theorem 9.8 (BFN)**.**
1. *This convolution defines a commutative and associative algebra structure on* $H^{G(\mathcal{O})}(\mathcal{R}_{G,V})$*.*
2. *If* $\mathbb{C}^\times_\hbar$ *acts by loop rotation, then*
$$H^{G(\mathcal{O}) \times \mathbb{C}^\times}(\mathcal{R}_{G,V})$$
is an associative algebra over $\mathbb{C}[\hbar]$ *which quantizes* $H^{G(\mathcal{O})}(\mathcal{R}_{G,V})$*.*

Definition 9.9. By (1) of the theorem, it makes sense to write

$$\mathcal{M}_C \coloneqq \operatorname{Spec}(H^{G(\mathcal{O})}(\mathcal{R}_{G,V})),$$

and by (2), it is an affine Poisson variety. The variety $\mathcal{M}_C$ is BFN's definition of the **Coulomb branch** for (G, V).

10 Gus (Apr 18): More on BFN spaces

Recall the setup from last time. The input to the BFN construction is a complex reductive group G together with a finite-dimensional representation V. Write

$$\mathcal{K} \coloneqq \mathbb{C}((z)), \quad \mathcal{O} \coloneqq \mathbb{C}[[z]] \subset \mathcal{K}.$$

The affine Grassmannian is $\mathrm{Gr}_G \coloneqq G(\mathcal{K})/G(\mathcal{O})$. We introduced a couple of important spaces last time:

$$\begin{aligned} \mathcal{T}_{G,V} &\coloneqq \{([g], s) \in \mathrm{Gr}_G \times V((z)) \cap gV[[z]]\} = G(\mathcal{K}) \times_{G(\mathcal{O})} V[[z]] \\ \mathcal{R}_{G,V} &\coloneqq \{([g], s) \in \mathrm{Gr}_G \times V[[z]] \cap gV[[z]]\} \subset \mathcal{T}_{G,V}. \end{aligned}$$

We called $\mathcal{R}_{G,V}$ the "variety of triples". We constructed a convolution product last time via the second row of the diagram

$$\begin{array}{ccccccc} \mathcal{T} \times \mathcal{R} & \xleftarrow{p} & G(\mathcal{K}) \times \mathcal{R} & \xrightarrow{q} & G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R} & \xrightarrow{m} & \mathcal{T} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{R} \times \mathcal{R} & \longleftarrow & p^{-1}(\mathcal{R} \times \mathcal{R}) & \longrightarrow & qp^{-1}(\mathcal{R} \times \mathcal{R}) & \xrightarrow{m} & \mathcal{R}. \end{array}$$

The convolution on $H^{G(\mathcal{O})}(\mathcal{R}_{G,V})$:

$$c_1 * c_2 \coloneqq m_*(q^*)^{-1} p^!(c_1 \boxtimes c_2).$$

Today we want to compute a few examples of this convolution, but to do that we will need to be more precise about what the pullback $p^!$. It is called the **refined Gysin map** or **pullback with support**.

Definition 10.1. Suppose we have a diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{p} & W. \end{array}$$

Let N be the pullback to Z of the normal bundle $N_{X/W}$. There is a specialization map

$$\sigma\colon H_*(Y) \to H_*(N), \quad [V] \mapsto [C_{(V\cap Z)/V}].$$

The **refined Gysin map** is the composition

$$p^!\colon H_*(Y) \to H_*(N) \to H_*(Z), \quad i_0^{-1} \circ \sigma$$

with pullback along the zero section.

Example 10.2 ($G = \mathbb{C}^\times$, $V = \mathbb{C}$)**.** In this case, $\mathrm{Gr}_G = \bigsqcup_{a\in\mathbb{Z}}[z^a]$. We have

$$\mathcal{T}_{G,V} = \bigsqcup_{a\in\mathbb{Z}}[z^a] \times z^a\mathcal{O}$$
$$\mathcal{R}_{G,V} = \bigsqcup_{a\in\mathbb{Z}}[z^a] \times (z^a\mathcal{O} \cap \mathcal{O}) =: \bigsqcup \mathcal{R}_a.$$

Write $\mathrm{Sym}(\mathrm{Lie}(G)^*) = \mathbb{C}[w]$. Then as a vector space, the equivariant Borel–Moore homology is

$$H_*^{G(\mathcal{O})}(\mathcal{R}_{G,V}) = \bigoplus_{a\in\mathbb{Z}} \mathbb{C}[w]\cdot[\mathcal{R}_a].$$

Let's compute $[\mathcal{R}_a] * [\mathcal{R}_b]$. We need to understand the square

$$\begin{array}{ccc} \mathcal{T}\times\mathcal{R} & \xleftarrow{p} & G(\mathcal{K})\times\mathcal{R} \\ \uparrow & & \uparrow \\ \mathcal{R}\times\mathcal{R} & \longleftarrow & p^{-1}(\mathcal{R}\times\mathcal{R}). \end{array}$$

The space $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R}$ is

$$G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R} = \bigsqcup_{a,b\in\mathbb{Z}} [z^a]\times[z^{a+b}]\times(z^{a+b}\mathcal{O}\cap z^a\mathcal{O})$$
$$[g_1,[g_2,v]] \mapsto ([g_1],[g_1g_2],g_1g_2v).$$

The fiber of the square over $[z^a]\times[z^{a+b}]$ is therefore

$$\begin{array}{ccc} (s, z^{-}as) & \longleftarrow & s \\ z^a\mathcal{O}\oplus(z^b\mathcal{O}\cap\mathcal{O}) & \longleftarrow & z^{a+b}\mathcal{O}\cap z^a\mathcal{O} \\ \uparrow & & \uparrow \\ (z^a\mathcal{O}\cap\mathcal{O})\oplus(z^b\mathcal{O}\cap\mathcal{O}) & \longleftarrow & \mathcal{O}\cap z^a\mathcal{O}\cap z^{a+b}\mathcal{O}. \end{array}$$

1. If $a = 1$ and $b = -1$, then we have

$$\begin{array}{ccc} z\mathcal{O}\oplus\mathcal{O} & \longleftarrow & z\mathcal{O} \\ \uparrow & & \uparrow \\ z\mathcal{O}\oplus\mathcal{O} & \longleftarrow & z\mathcal{O}. \end{array}$$

In the notation of the Gysin diagram, $Y = W$. So in this case, $p^! = p^*$, i.e.

$$p^!([\mathcal{R}_a] \boxtimes [\mathcal{R}_b]) = [z\mathcal{O}],$$

the fundamental class of the fiber in $G(\mathcal{K}) \times \mathcal{R}$. Now push forward to $H_*^{G(\mathcal{O})}(\mathcal{R}_{a+b})$. In our case $\mathcal{R}_{a+b} = \mathcal{O}$, and

$$m_*[z\mathcal{O}] = [z\mathcal{O}] = w[\mathcal{O}] = w[\mathcal{R}_0]$$

inside $\mathcal{O}$. (This class as cut out by a single linear section.) Hence

$$[\mathcal{R}_1] * [\mathcal{R}_{-1}] = w[\mathcal{R}_0].$$

2. If $a = -1$ and $b = 1$, then the diagram is

$$\begin{array}{ccc} W = z^{-1}\mathcal{O} \oplus z\mathcal{O} & \longleftarrow & \mathcal{O} = X \\ \uparrow & & \uparrow \\ Y = \mathcal{O} \oplus z\mathcal{O} & \longleftarrow & \mathcal{O} = Z. \end{array}$$

Note that $Y = X \oplus (0, z\mathcal{O})$ and $W = X \oplus \mathbb{C} \cdot (z^{-1}, 0) \oplus (0, z\mathcal{O})$. So Y is cut out in W by the linear condition that the $\mathbb{C} \cdot (z^{-1}, 0)$ vanishes, which is a weight $(w - \hbar)$ condition. Hence

$$p^!([\mathcal{R}_{-1}] \boxtimes [\mathcal{R}_1]) = (w - \hbar)[\mathcal{O}].$$

Pushing forward,

$$m_*((w - \hbar)[\mathcal{O}]) = (w - \hbar)[\mathcal{O}] = (w - \hbar)[\mathcal{R}_0].$$

Summary: we computed that

$$[\mathcal{R}_1] * [\mathcal{R}_{-1}] = w[\mathcal{R}_0], \quad [\mathcal{R}_{-1}] * [\mathcal{R}_1] = (w - \hbar)[\mathcal{R}_0].$$

So the commutator is

$$[[\mathcal{R}_1], [\mathcal{R}_{-1}]] = \hbar[\mathcal{R}_0].$$

In particular, if $\hbar = 0$, we get a commutative product, like we asserted last time. In general, if $r_a \coloneqq [\mathcal{R}_a]$, the multiplication (without the weight $\hbar$) is

$$r_a * r_b = w^{m_{a,b}} r_{a+b}, \quad m_{a,b} = \begin{cases} \min(|a|, |b|) & a, b \text{ different sign} \\ 0 & \text{otherwise.} \end{cases}$$

If we include $\hbar$, the $w^{m_{a,b}}$ becomes $(w - \hbar)(w - 2\hbar)\cdots$.

Example 10.3 ($G = \mathbb{C}^\times, V = \mathbb{C}^{\oplus N}$)**.** Let $x \coloneqq [\mathcal{R}_1]$ and $y \coloneqq [\mathcal{R}_{-1}]$, and w be the generator of $H_T^*(\mathrm{pt})$. Then they satisfy the following relations:

$$\begin{aligned} xy &= (w - \hbar)^N \quad [x, w] = 2\hbar x \\ yx &= w^N, \quad [y, w] = -2\hbar y. \end{aligned}$$

This is a quantization of the $\mathcal{A}_{N-1}$ surface. For example, when $N = 2$ we get the nilcone $xy = w^2$ of $\mathfrak{sl}_2$. The quantization is

$$U\mathfrak{sl}_2 / \langle \text{trivial central ideal} \rangle.$$

If we do this in K-theory, we get the quantization

$$U_q\mathfrak{sl}_2 / \langle \text{trivial central ideal} \rangle, \quad q = e^\hbar.$$

Remark. For general reductive G, use localization with respect to the maximal torus $T \subset G$. The first observation is that

$$(\mathrm{Gr}_G)^T = \mathrm{Gr}_T = \bigsqcup_{\lambda \in P_+^\vee} [z^\lambda].$$

Then in fact $(\mathcal{R}_{G,V})^T = \mathcal{R}_{T,V^T}$ where V^T is literally the T-invariants of V. The next step is to show

$$H_*^{T(\mathcal{O})}(\mathcal{R}_{G,V}) = H_*^{G(\mathcal{O})}(\mathcal{R}_{G,V}) \otimes H^*_{T(\mathcal{O})}(\mathrm{pt})$$

and $H^{G(\mathcal{O})}(\mathcal{R}_{G,V}) = H^{T(\mathcal{O})}(\mathcal{R}_{G,V})^W$. Now apply localization:

$$i_* \colon H^{T(\mathcal{O})}(\mathcal{R}_{T,V^T}) \to H^{T(\mathcal{O})}(\mathcal{R}_{G,V})$$

is an isomorphism after localizing at root hyperplanes. So

$$\mathrm{Frac}(H^{T(\mathcal{O})}(\mathcal{R}_{T,V^T})^W) \cong \mathrm{Frac}(H^{G(\mathcal{O})}(\mathcal{R}_{G,V})).$$

We can get the dimension of the Coulomb branch this way: in the abelian case, the dimension is $2\dim T$, so in general

$$\dim \mathcal{M}_C = 2\,\mathrm{rank}(G).$$

In fact, $\mathcal{R}_{G,V}$ are birational for different V.

Remark. The space given by $H^{T(\mathcal{O})}(\mathcal{R}_{T,V^T})^W$ is the "classical" Coulomb branch. The fraction fields are equal, but the inequality of actual rings demonstrates the physical quantum corrections.

Definition 10.4 (Monopole formula)**.** Look at the zero section embedding $z \colon \mathrm{Gr}_G \hookrightarrow \mathcal{R}_{G,V}$ and the pullback

$$z^* \colon H^{G(\mathcal{O})}(\mathcal{R}_{G,V}) \to H^{G(\mathcal{O})}(\mathrm{Gr}_G)$$

in order to help us study the Coulomb branch. Let's look at K-theory and consider

$$z^*(i_*)^{-1} \colon K^{G(\mathcal{O})}(\mathcal{R}_{G,V}) \to K^{T(\mathcal{O})}(\mathrm{Gr}_T).$$

For example, if $G = \mathrm{GL}_n$ and $V = \mathfrak{g}$ is the adjoint representation, then

$$K^{T(\mathcal{O})}(\mathrm{Gr}_T) = \mathbb{C}\langle w_i^\pm, T_i^\pm \rangle / (T_i w_j = q^{\delta_{ij}} w_j T_i).$$

We can ask what happens to the classes $[\mathcal{R}_{w_i}]$ under this homomorphism. Let $\mathbb{C}_t^\times$ scale V. Then

$$\begin{aligned} K^{G(\mathcal{O}) \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times}(\mathcal{R}_{G,V}) &\to K^T(\mathrm{Gr}_T) \\ [\mathcal{R}_{w_i}] &\mapsto \sum_{I \subset [\mathrm{rank}\, G]} \prod_{\substack{i \in I \\ j \notin J}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_i. \end{aligned}$$

These are Macdonald operators in spherical DAHA. Remember that $\mathcal{R}_{G,V}$ is supposed to be the affine Steinberg, so it makes sense that after affinization we got DAHA.

11 Andrei (Apr 25): q-difference equations, monodromy, and elliptic cohomology

We will talk today about q-difference equations

$$\Psi(qz) = M(z)\Psi(z).$$

In principle, $M(z)$ could also depend on q, but in general is a rational function of z. This equation is **regular at** 0 if 0 is not a pole, and if $M(0) \in \mathrm{GL}(N)$ is invertible. The situation of quantum difference equations is that $M(0)$ comes from a classical computation and doesn't depend on q at all. The local theory of solutions to such equations is the same as that of differential equations. Solutions form a sheaf over coordinate $z \in \mathbb{C}^\times/q^{\mathbb{Z}}$.

If we are interested in just the 1×1 case, we have constant coefficients, and we are solving $f(qz) = af(z)$. Already there is something to discuss. We can view solutions as meromorphic sections of some line bundle, e.g. of the form $\theta(azb)/\theta(a)\theta(zb)$, which have a sequence of poles of the form $b^{-1}q^N$. We can also have non-single valued solutions, of the form $\exp(\log a \log z/\log q)$.

The better approach is to take the theory of constant coefficient equations as something given, and consider something that conjugates the general equation to the constant coefficient case.

Definition 11.1. $U(z)$ is a **fundamental solution** if

$$M(z)U(z) = U(qz)M(0).$$

There exists a unique $U(z) = 1 + O(z)$, by the same argument as for differential equations. Plug in $1 + \sum z^n U_n$ to get

$$M(0)U_n = q^n U_n M(0) + \cdots.$$

Since we assumed $M(0)$ does not depend on q, then q^n is not an eigenvalue. So

$$(\mathrm{Ad}\, M(0) - q^n)U_n = \cdots$$

has a unique solution. In fact, this argument shows that $U(z)$ converges in some $|z| < \epsilon$.

A new feature: $U(z)$ is meromorphic in $\mathbb{C}$. This is like the gamma function $\Gamma(s+1) = s\Gamma(s)$: once we prove it exists for sufficiently large s, then it meromorphically continues to the rest of the plane. If we assume $|q| < 1$, then we can extend $|z| < \epsilon$ to e.g. $|z| < \epsilon q^{-1}$.

1. If we have a pole of M at z, then it will propagate to a sequence of poles $q^{-k}z$ of $U(z)$.
2. If we have a zero $\det M(z) = 0$ of M, then it will propagate to a sequence of poles of U^{-1}.

Then we can ask what happens with monodromy. If we assume the equation is **regular at** ∞, i.e. $M(\infty) \in \mathrm{GL}(N)$, then we have the exact same situation at ∞, with U_∞. The poles of U_∞^{-1} will be in the opposite geometric progression.

Definition 11.2. The **monodromy** is

$$\mathrm{Mon}(z;q) \coloneqq U_\infty^{-1}(z;q)U_0(z;q).$$

This is a meromorphic function, and we can ask what happens if we shift by q:

$$\mathrm{Mon}(qz;q) = M(\infty)\,\mathrm{Mon}\, M(0)^{-1}.$$

So $\mathrm{Mon}(z;q)$ is some section of a bundle over an elliptic curve, i.e. some elliptic function. For $q = 1+\epsilon$, we write $M(z) = 1 + \epsilon\bar{M}(z)$, and the equation converges to

$$z\frac{d}{dz}\Psi(z) = \bar{M}(z)\Psi(z),$$

which has actual monodromy. We can ask what happens to $\mathrm{Mon}(z;q)$ as $\epsilon \to 0$ (for a fixed z). It will be the ratio of two solutions: the transport of solutions from 0 to ∞ through z. On the elliptic base, we are shrinking one of the cycles. In the central fiber, we get a picture like in the Tate elliptic curve: countably many copies of $\mathbb{P}^1$. The point is that the asymptotics of elliptic functions in this limit is piecewise-constant on each copy of $\mathbb{P}^1$.

Example 11.3 (Elliptic function)**.** Speaking of elliptic functions, we should have an example to keep in mind. Look at the q-difference equation

$$f(qz) = \frac{1 - az}{1 - bz} f(z).$$

Then we have

$$f(z) = f(qz)\frac{1 - bz}{1 - az} = f(0)\prod_{n=0}^{\infty}\frac{1 - q^n bz}{1 - q^n az} = f(0)\frac{(bz)_\infty}{(az)_\infty}.$$

Here $M(0) = 1$ and $M(\infty) = a/b$. If we rewrite the same equation at ∞, we get

$$f_\infty(z) \propto \frac{(q/az)_\infty}{(q/bz)_\infty}.$$

The monodromy will be the ratio of the two:

$$\text{Mon} = \frac{(q/bz)_\infty (bz)_\infty}{(q/az)_\infty (az)_\infty}.$$

This starts looking good, because a theta function we would like to be

$$\theta(z) \coloneqq \text{“}\prod_{n\in\mathbb{Z}}(1 - q^n z)\text{''}.$$

(We can't actually do this, otherwise we would get a non-trivial regular function on a proper variety.) We salvage this by

$$\theta(z) \coloneqq (z)_\infty (q/z)_\infty.$$

So the monodromy is

$$\text{Mon} = \frac{\theta(bz)}{\theta(az)}.$$

In general, any 1×1 equation has q-difference operator of the form

$$\prod \frac{1 - a_i z}{1 - b_i z},$$

and in general the monodromy will be of the form

$$\text{Mon} = \prod \frac{\theta(b_i z)}{\theta(a_i z)}.$$

In general, writing monodromy like this is not necessarily helpful. It is a transcendental problem, just like with differential equations. However there are features of q-difference equations which can make the problem easier.

For a system of q-difference equations, a deep theorem of Deligne says the following. Take a variety X with some holonomic D-module on it, with singularities. (In the analytic world, this is just a system of differential equations.) The theorem says something about when it is *regular*. In one variable,

$$z\frac{d}{dz}\Psi(z) = \overline{M(z)}\Psi(z),$$

and an algebraic geometer would write

$$\frac{d}{dz}\Psi(z) = \frac{1}{z}\overline{M(z)}\Psi(z)$$

and say there is a first-order pole. An invariant way of saying this is that solutions grow as $\|\Psi(z)\| \leq |1/z|^C$, where C comes from the eigenvalues of the operator. Deligne's deep theorem says that it's enough to check regularity on generic points of each divisor.

However this is absolutely false for q-difference equations, even in the 1×1 case. Take $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates a and z, and look at the system of q-difference equations

$$\begin{aligned} f(qz, a) &= af(z, a) \\ f(z, qa) &= zf(z, a). \end{aligned}$$

We already looked at this; a solution is

$$f(z) \sim \exp \frac{\ln a \ln z}{\ln q}.$$

This grows at most polynomially as we approach a generic point of any divisor. But if we approach one of the intersections, e.g. $(0,0)$, then we get exponential growth. The problem is that the operator a and z are finite and invertible only away from these corners.

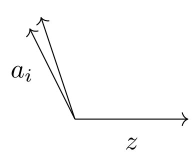
A more interesting example is q-hypergeometric functions, coming from the q-difference equation for $T^*\mathbb{P}^1$. In general, we can do it for $T^*\mathbb{P}^N$. The functions are

$$f_i(z) = \sum_{d=0}^{\infty} z^d \prod_{j=0}^{N} \frac{(\hbar a_j/a_i)_d}{(qa_j/a_i)_d}$$

where $(x)_d \coloneqq (1-x)\cdots(1-q^{d-1}x)$. The way to see this solves a nice q-difference equation is left as an exercise. For these to all solve the same equations, we need a prefactor

$$f_i(z) = \exp\left(\frac{\ln z \ln a_i}{\ln q}\right) \sum_{d=0}^{\infty} z^d \prod_{j=0}^{N} \frac{(\hbar a_j/a_i)_d}{(qa_j/a_i)_d}.$$

The picture is for the coordinates $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times$ and $a_i \in \mathrm{Pic}(X^\vee) \otimes \mathbb{C}^\times$ is



The equation is regular everywhere except the point $(z,a) = (0,0)$. This is because the spectrum of $M(0)$ is weights of line bundles on fixed points, and there is some bilinear expression

$$\text{fixed points} \to \mathrm{Pic}(X^\vee) \otimes A^\vee.$$

So we see why Deligne's theorem fails. In general, for

$$\begin{aligned} \Psi(qz, a) &= M(z, a)\Psi(z, a) \\ \Psi(z, qa) &= S(z, a)\Psi(z, a), \end{aligned}$$

the system is regular in each group of variables but **not** jointly. (It is not immediate that they *are* regular in each group of variables; some work is involved.) Poles in a accumulate near 0, but there is always a region $z \in [0, \epsilon]$ where there are no poles.

In the 3d mirror symmetry picture, we therefore have two kinds of solutions Ψ^z and Ψ^a, and they have different properties for poles in a vs poles in z. This is actually good, because then

$$P \coloneqq (\Psi^a)^{-1}\Psi^z$$

is a matrix of elliptic functions called the **pole subtraction matrix**. This is actually a computable matrix: it is triangular with respect to the order of growth of the bilinear pairing $e^{\log a_i \log z/\log q}$. So among all solutions, there will be one distinguished ordering giving a solution f_1 regular in all directions. For example, for the q-hypergeometric function, take $|a_1| \gg |a_2| \gg \cdots \gg |a_N|$. Now for f_2, we will get one part which has poles, and we can compute an elliptic function with those prescribed poles to cancel it. This is purely local and algorithmic.

How does this constrain the monodromy? The monodromy is the question of going from a solution $\Psi_{z=0}$ to $\Psi_{z=\infty}$. Instead, look at $\Psi_{a=0}$ and $\Psi_{a=\infty}$. But clearly we have a diagram

$$\begin{array}{ccc} \Psi_{z=0} & \xrightarrow{\text{Mon}} & \Psi_{z=\infty} \\ & {}_{P}\searrow \quad \nearrow_{P} & \\ & \Psi_{a=0}. & \end{array}$$

Lets look at where these things live. At ∞ we get some flop of X, because $z \in \operatorname{Pic}(X) \otimes \mathbb{C}^\times$. At $a = 0$, we get the fixed locus $K_T(X^a)$. Hence there is a commutative square

$$\begin{array}{ccc} K_T(X) & \xrightarrow{\text{Mon}} & K_T(X_{\text{flop}}) \\ {\scriptstyle P}\uparrow & & {\scriptstyle P}\uparrow \\ K_T(X^a) & \xrightarrow{\text{Mon}_{a=0}} & K_T(X_{\text{flop}}). \end{array}$$

If fixed points are isolated, the monodromy is a matrix of elliptic functions and we get a Gauss factorization of it. The map $P\colon K_T(X^a) \to K_T(X)$ should remind us of stable envelopes of some kind.

Theorem 11.4. *This P is the stable envelope in elliptic cohomology.*

Inside $X \times X^A$ is the important attracting correspondence

$$\text{Attr} \coloneqq \{(x,y) : y = \lim_{a \to \text{some infty}} a \cdot x\}.$$

Stable envelopes are an improved version

$$\overline{\text{Attr}} + \text{corrections}$$

acting on cohomology, K-theory, elliptic cohomology, etc. A cycle automatically acts on cohomology, and if we can put a sheaf on it then it acts on K-theory. With elliptic cohomology there is a whole other story. We declare all this data to be a morphism in some category where the objects are the respective cohomology groups. This has fairly rich structure. For example, it could be that

$$X^A = \bigsqcup X_i \times X_j,$$

in which case we get something like a tensor structure. (This is how, abstractly, we make these cohomology groups modules over quantum groups.)

Note that we don't just get one map from this construction: we get as many as there are infinities in A. For example, for $\mathbb{P}^1$, we get two maps, and the difference between them is some kind of braiding in the category. In this picture, the monodromy becomes a tensor isomorphism between the categories for X and X_{flop}.

What is elliptic cohomology? It sounds rather abstract but is actually quite concrete. Look at varieties without odd cohomology, Then the equivariant elliptic cohomology $\operatorname{Ell}_T(\cdot)$ is a functor taking values in schemes finite over

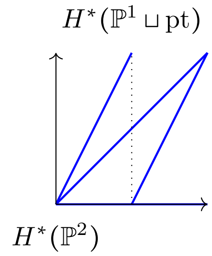
$$\operatorname{Ell}_T(\text{pt}) = T/q^{\text{cochar}}.$$

What does this scheme look like? The fiber over an element $t \in \operatorname{Ell}_T(\text{pt})$ is the ordinary cohomology

$$H^*(X^{\text{subgrp of } T})$$

where the subgroup is $\bigcap_{\chi(t)=1} \ker \chi$. Very concretely, it is $\operatorname{Spec} K_T(X)/q$.

Example 11.5. Let $X = \mathbb{P}^2$ with action $T \coloneqq \operatorname{diag}(1, a^{-1}, a^{-2})$. Then $\operatorname{Spec} K_T(X)$ has relation $(1-x)(1-a^{-1}x)(1-a^{-2}x)$, so in the fundamental region mod q we get



Then the stable envelope is the unique section of some sheaf

$$\operatorname{Ell}_T(X) \times \operatorname{Ell}_T(X^A)$$

defined by triangularity and automorphy.

12 Zijun (May 09): 3d mirror symmetry and elliptic stable envelopes

In usual GIT, we have $X /\!\!/ G = X^s/G$ (under mild assumptions). Abelianization is the comparison of this with $X /\!\!/ T = X^s/T$. Be careful: the stability conditions for G and T may be different. We therefore have a square:

$$\begin{array}{ccc} X^{G\text{-stable}}/T & \xrightarrow{j} & X^{T\text{-stable}}/T \\ \downarrow{\scriptstyle \pi} & & \\ X^{G\text{-stable}}/G & & \end{array}$$

In general,

$$H^*(X /\!\!/ G) = \frac{H^*(X /\!\!/ T)^W}{\operatorname{Ann}(e)}.$$

Given $\alpha \in H^*(X /\!\!/ G)$, an element $\gamma \in H^*(X /\!\!/ T)$ is called a **lift** if

$$\pi^*\alpha = j^*\gamma.$$

In good cases, the lift always exists, e.g. when X satisfies Kirwan surjectivity. There is a formula

$$\int_{X /\!\!/ G} \alpha = \int_{X /\!\!/ T} \gamma \cup \prod_{\alpha \text{ roots}} c_1(L_\alpha).$$

The factor comes from the G/T-fibration π,

This carries over to the holomorphic symplectic setting. Let (X, ω) have a G-action and $T \subset G$. Let $\mu: X \to \mathfrak{g}^*$ be the moment map for the G-action. The holomorphic symplectic quotient is

$$Y \coloneqq \mu_G^{-1}(0) /\!\!/_\theta G = \mu^{-1}(0)^{G\text{-stable}}/G.$$

The abelian quotient is

$$Y^{\text{ab}} \coloneqq \mu_T^{-1}(0)^{T\text{-stable}}/G.$$

The relation between μ_T and μ_G is the projection:

$$\mu_T: X \xrightarrow{\mu_G} \mathfrak{g}^* \to \mathfrak{t}^*.$$

So in this setting we have a similar square:

$$\begin{array}{ccccc} \mu_G^{-1}(0)^{G\text{-stable}}/T & \xrightarrow{\text{open}} & \mu_G^{-1}(0)^{T\text{-stable}}/T & \xrightarrow{\text{closed}} & \mu_T^{-1}(0)/T \\ \downarrow{\scriptstyle\pi} & & & & \\ \mu_G^{-1}(0)^{G\text{-stable}}/G & & & & \end{array}.$$

The fibration π is still a G/T fibration, so there will still be a factor $\prod_{\alpha \text{ roots}} c_1(L_\alpha)$. The open embedding will not contribute anything new, but the closed embedding will introduce an Euler class of a normal bundle. So in this case,

$$\int_Y \alpha = \int_{Y^{\text{ab}}} \gamma \cup \prod_{\alpha \text{ roots}} c_1(L_\alpha)(\hbar - c_1(L_\alpha)).$$

Note that the additional classes are symplectically dual. All this carries into K-theory as well.

This induced map between cohomologies of Y and Y^{ab} does not necessarily commute with stable envelopes. Aganagic–Okounkov have a different map. Work with the hyperkähler structure instead:

$$Y = \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\theta)/G_{\mathbb{R}}.$$

Then we have a square

$$\begin{array}{ccccc} \mu_{G,\mathbb{C}}^{-1}(0) \cap \mu_{G,\mathbb{R}}^{-1}(\theta)/T_{\mathbb{R}} & \xrightarrow{j_+} & \mu_{G,\mathbb{C}}^{-1}(\mathfrak{b}^\perp) /\!\!/_\theta T & \xrightarrow{j_-} & Y^{\text{ab}} \\ \downarrow{\scriptstyle\pi} & & & & \\ Y & & & & \end{array}.$$

This induces a map

$$H^*(Y^{\text{ab}}) \xrightarrow{\pi_* \circ j_+^* \circ (j_{-,*})^{-1}} H^*(Y).$$

Aganagic–Okounkov prove that *this* map commutes with stable envelopes. The difference between this diagram and the preceding one is in the intermediate terms. For example, π here is a $G_{\mathbb{R}}/T_{\mathbb{R}} \cong G/B$ fibration, but in the earlier diagram it was a G/T fibration.

Let's look at the sequence for the Grassmannian:

$$Y = T^* \operatorname{Gr}(k,n) = \{(i,j) : ij = 0, \operatorname{rank} j = k\}.$$

On the other hand,

$$Y^{\text{ab}} = (T^*\mathbb{P}^{n-1})^k = \{(i,j) : i_\ell j_\ell = 0,\, j_\ell \neq 0\ \forall 1 \le \ell \le k\}.$$

The diagram is

$$\begin{array}{ccccc} \{(i,j) : ij = 0 \ \operatorname{rank} j = k\}/T & \longrightarrow & \{(i,j) : ij = 0,\, j_\ell \neq 0\ \forall \ell\}/T & \longrightarrow & \{(i,j) : i_\ell j_\ell = 0,\, j_\ell \neq 0\ \forall \ell\} \\ \downarrow & & & & \\ \{(i,j) : ij = 0, \operatorname{rank} j = k\}/G. & & & & \end{array}$$

Recall our formula for the stable envelope on $T^* \operatorname{Gr}(k,n)$. If $p \in T^* \operatorname{Gr}(k,n)^T$, then

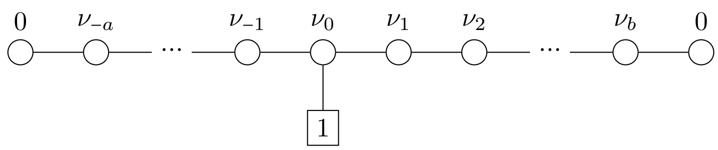
$$\operatorname{Stab}(p) = \operatorname{Sym}_{y_1,\dots,y_n} \left(\frac{\prod_{\ell=1}^k \boxed{\left(\prod_{i<p_\ell} \theta(y_\ell u_i \hbar^{-1})\right) \dfrac{\theta(y_\ell u_{p_i} z^{-1} \hbar^{(\cdots)})}{\theta(z^{-1}\hbar^{(\cdots)})} \prod_{i>p_\ell} \theta(y_\ell^{-1} u_i^{-1})}}{\prod_{1 \le i < j \le k} \theta(y_i/y_j)\theta(\hbar y_i/y_j)} \right).$$

The boxed part is actually the stable envelope for $T^*\mathbb{P}^{n-1}$. The theta function part can be viewed as the "Kähler part". The product $\prod_{\ell=1}^k$ comes from there being k copies of $T^*\mathbb{P}^{n-1}$. Finally, the denominator comes from the $\prod_{\alpha \text{ roots}} c_1(L_\alpha)(\hbar - c_1(L_\alpha))$ part of abelianization.

Abelianization in this case has the nice property that pre-images of fixed points are actually still points, not positive-dimensional varieties. Let's think about

$$X = \mathrm{Hilb}^n(\mathbb{C}^2).$$

Then fixed points have a quiver description

X^T = 

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The pre-image $\pi^{-1}(x)$ of this point x is therefore the abelianization of this quiver variety. Do a dimension count on

$$T^*\Big(\bigoplus_{i=-a}^{b-1} \mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(\mathbb{C}, V_0)\Big) /\!/\!/\!/ T = \prod_{i=-a}^{b} (\mathbb{C}^*)^{\nu_i}$$

and the gauge group to see that the dimension is in general > 0. What are the fixed points of a hypertoric variety? Write the quotient as $T^*M /\!/\!/\!/ (\mathbb{C}^*)^{|\nu|}$. Then

$$M \cong \mathbb{C}^{2N} \ni (x_1, \dots, x_N, y_1, \dots, y_N)$$

and fixed points must have some non-zero values in the first N coordinates. In other words, we have exactly N maps between the nodes, and all others are zero. If we draw a tree in our partition, each edge of the tree will represent a map. A tree has $N-1$ edges. Along with the single framing map $\mathbb{C} \to V_0$, in total there are N edges, specifying N non-zero maps in T^*M.

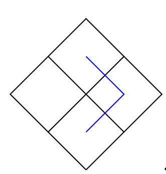
The compatibility of stable envelopes with abelianization gives

$$\begin{array}{ccc} H^*(x) & \xleftarrow{\pi_* j_+^* (j_{-,*})^{-1}} & H^*(\pi^{-1}(x)) \\ {\scriptstyle \mathrm{Stab}}\downarrow & & {\scriptstyle \mathrm{Stab}}\downarrow \\ H^*(\mathrm{Hilb}) & \longleftarrow & H^*(\mathrm{Hilb}^{\mathrm{ab}}). \end{array}$$

The $(\mathbb{C}^*)^2$ acting on Hilb is not the largest torus. In particular, it fixes the *entire* abelianized fixed loci $\pi^{-1}(x)$. The goal is to find the Stab on the lhs. The first step is to lift $1 \in H^*(x)$ to $H^*(\pi^{-1}(x))$, and this produces a sum over trees, of the form

$$\sum_{p \text{ trees}} \mathrm{Stab}(p).$$

This is not a sum over *all* trees, because we have to choose some lift of 1. For example, in the Grassmannian case, we had a choice of $k!$ lifts. In addition, when we lift from x to $\pi^{-1}(x)$, we have another freedom to choose a $(\mathbb{C}^*)^M$-fixed point on $\pi^{-1}(x)$, where $(\mathbb{C}^*)^M$ is an enlargement of $(\mathbb{C}^*)^2$. This is encoded in the condition on trees that we cannot have corners like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