0.1

Fix integers $k < n$ and an $(n - k) \times k$ integer matrix $C$. Let a torus $T^k$ act on $V \cong \mathbb{C}^n$ with weights

$$t_1, \ldots, t_k ; \prod_i t_i^{C_{1i}}, \ldots, \prod_i t_i^{C_{n-k,i}}$$

where $(t_1, \ldots, t_k) \in T^k \cong (\mathbb{C}^\times)^k$. Let

$$X = T^*V/\!\!/T^k$$

be the algebraic symplectic reduction of $T^*V$ by $T^k$ with respect to some fixed stability parameter $\theta \in \mathbb{R}^k$.

On $X$, there is a natural action of

$$T^{n-k} \times C^\times_h \cong (\mathbb{C}^\times)^{n-k} \times C^\times \ni (a_1, \ldots, a_{n-k}, h)$$

where $a_i$ scales the last $(n - k)$ coordinates of $\mathbb{C}^n$ while $h$ scales the cotangent directions. The point

$$x = (1^k, 0^{n-k}) \in V \subset T^*V$$

projects onto a torus-fixed point of $X$ which we will denote by the same symbol. We begin by writing the 1-leg vertex corresponding to the point $x \in X$ as a multiple hypergeometric series.

0.2

Consider the space of quasi-maps to $X$ from $B = \mathbb{P}^1$. By definition [], quasi-map from $B$ to a quotient by a group $G$, here $G = T^k$, is a principal $G$-bundle $P$ over $B$ together with a section $f$ of the induced bundle

$$T^*V \longrightarrow P \times_G T^*V.$$  \hspace{1cm} (1)

In our case, because we work with a symplectic reduction, we will need to restrict to the zero locus of the moment map.

Let $d_1, \ldots, d_k$ be the degrees of $P$. Then

$$P \times_G T^*V = \bigoplus_i \mathcal{O}(d_i) \oplus \bigoplus_i \mathcal{O}(-d_i) \oplus h \oplus \bigoplus_j \mathcal{O}(\sum C_{ji} d_i) \otimes a_i \oplus \bigoplus_j \mathcal{O}(-\sum C_{ji} d_i) \otimes h/a_i.$$  \hspace{1cm} (2)

as an equivariant bundle. An additional torus $\mathbb{C}^\times_q$ acts on quasimaps scaling the source $B$ with weight $q \in \mathbb{C}^\times_q$ at the fixed point $0 \in \mathbb{P}^1$.

A torus fixed quasimap $f$ that evaluates to $x$ at infinity of $B$ is necessarily given by nonzero sections of $\bigoplus_i \mathcal{O}(d_i)$ with a $d_i$-fold zero at $0 \in B$. Its existence requires $d_i \geq 0$. We denote such unique torus-fixed quasimap of given degrees $d_i$ by $f_d$. 1
0.3

The deformations of the map \( f_d \) are the following. First, each of the sections of \( \mathcal{O}(d_i) \) may be deformed modulo an overall scaling (automorphism of the source principal bundle), which gives the characters

\[
-1 + \text{char} \, H^*(\mathcal{O}(d_i)) = q + q^2 + \cdots + q^{d_i}, \quad i = 1, \ldots, k.
\]

The dual obstruction weights come from

\[
-h + \text{char} \, H^*(\mathcal{O}(-d_i) \otimes h) = -h(1 + q^{-1} + \cdots + q^{1-d_i}),
\]

where the \(-h\) term comes from the moment map equation. Analogously, subtracting the tangent weights at \( x \in X \), we get

\[
-a_i + \text{char} \, H^*(\mathcal{O}(\sum C_{ji}d_i) \otimes a_i) = a_i \frac{q^{\sum C_{ji}d_i} - 1}{1 - q^{d_i}}
\]

and similarly for the dual term. Note that the character of the virtual tangent space at \( f_d \) is self-dual

\[
\text{char Obs}(f_d) = \kappa \otimes \text{char Def}(f_d), \quad \kappa = \hbar q.
\]

Since the product of 5 equivariant parameters in 5-dimensions equals 1, the duality works as follows

\[
(h, \kappa) \xrightarrow{\text{duality}} (1/\kappa, 1/h),
\]

which fixes the ratio \( q = \kappa/\hbar \).

0.4

Together with the square roots of the virtual canonical bundle, the weight of the deformation theory of \( f_d \) equals

\[
\prod_w \frac{(\kappa/w)^{1/2} - (w/\kappa)^{1/2}}{w^{1/2} - w^{-1/2}} = \prod_w (-\kappa^{1/2}) \frac{1 - w/\kappa}{1 - w},
\]

where the product is over all weights in the deformation space \( \text{Def}(f_d) \) of \( f_d \).

We shift the Kähler parameters so that the contribution of degree \( d \) is weighted by \( \prod_i ((-\kappa^{-1/2})^{1 + \sum C_{ji}Q_i})^{d_i} \), this is a generalization of the shift by \( i\pi c_1(\mathcal{X}_\mathcal{V}) \) in \( H^2(X, \mathbb{C}) \). Then we get the following formula for the vertex

\[
V = \sum_{d_1, \ldots, d_k \geq 0} Q^d \prod_{i=1}^k (1/h)_{d_i} \prod_{j=1}^{n-k} (a_j/\kappa)_{(Cd)_j},
\]

where

\[
(z)_k = \frac{(z)_\infty}{(q^k z)_\infty}, \quad (z)_\infty = \phi(z) = \prod_{n \geq 0} (1 - q^n z)
\]

and where \( (Cd)_j = \sum C_{ji}d_i \) is the \( j \)th entry of the matrix product \( Cd \).
0.5

It is better to replace the vertex $V$ by the following series

$$\tilde{V} = \prod_{j=1}^{n-k} \frac{(a_j/\kappa)}{(a_j)_{\infty}} V$$

$$= \sum_{d_1,\ldots,d_k \geq 0} Q^d \prod_{i=1}^{k} \frac{(1/\hbar)_{d_i}}{(q)_{d_i}} \prod_{j=1}^{n-k} \frac{(a_j q^{(Cd)_{j}})}{(a_j q^{(Cd)_{j}}/\kappa)_{\infty}}.$$  (5)

The prefactor in (4) should have a natural interpretation in terms of replacing $B = \mathbb{P}^1$ by $B = \mathbb{C}$ and regularizing the corresponding infinite product.

0.6

Now recall the $q$-binomial theorem that says

$$\frac{(bz)_{\infty}}{(z)_{\infty}} = \sum_m \frac{(b)_m}{(q)_m} z^m.$$

It implies

$$\frac{(a_j q^{(Cd)_{j}})}{(a_j q^{(Cd)_{j}}/\kappa)_{\infty}} = \sum_m \frac{(\kappa)_m}{(q)_m} (a_j/\kappa)^m q^{m(Cd)},$$

and therefore

$$\tilde{V} = \sum_{d,m} Q^d (a/\kappa)^m q^{m \Delta} \prod_{i=1}^{k} \frac{(1/\hbar)_{d_i}}{(q)_{d_i}} \prod_{j=1}^{n-k} \frac{(\kappa)_m}{(q)_m}.$$  (4)

This is manifestly symmetric under (3) and moreover, this is a result of applying an operator which looks like an abelian $R$-matrix to a fully factored expression

$$\tilde{V} = q^2 \frac{
abla^k}{i=1} \frac{(Q_i h)_{\infty}}{(Q_i)_{\infty}} \prod_{j=1}^{n-k} \frac{(a_j)_{\infty}}{(a_j/\kappa)_{\infty}}$$

where

$$\Delta = \sum_{ij} C_{ij} a_j \frac{\partial}{\partial a_j} \otimes Q_i \frac{\partial}{\partial Q_i},$$

where the $\otimes$ sign is put in purely for emphasis.