Q: When do we have $[M(\lambda) : L(\mu)] \neq 0$?

Last time (Verma): It suffices to put $\mu = s_w \cdot \lambda$ s.t. $\lambda \geq \mu$ because then $H(\mu) \subseteq H(\lambda)$.

Cor: if we have $\mu = s_w \ldots s_{w_0} \cdot \lambda \leq s_{w_0} \ldots s_{w_1} \cdot \lambda \leq \ldots \leq s_{w_0} \cdot \lambda \leq \lambda$ (*)
we get $H(\mu) \subseteq H(\lambda)$ also.

When (*) is satisfied we say $\mu$ is strongly linked to $\lambda$ and we write $\mu \triangleright \lambda$.

BGG Theorem: If $[M(\mu) : L(\mu)] \neq 0$, then $\mu \triangleright \lambda$.
We will prove this shortly.

Recall that $\Theta$ is split into categories $\Theta_\lambda$, indexed by $\mu$- antidominant weights $\lambda$. Assume $\lambda$ is integral and dot-regular, so in particular $\lambda$ is the unique $\mu$- antidominant weight in $W \cdot \lambda$. Then we can rephrase the condition $w' \cdot \lambda \triangleright w \cdot \lambda$ exclusively in terms of the Weyl group.

Prop: if $\lambda$ is integral, dot-regular and $\mu$- antidominant, then $w' \cdot \lambda \triangleright w \cdot \lambda \iff w' \leq w$.

Proof: It suffices to prove $s_w \cdot w' \cdot \lambda \triangleright s_w \cdot w \cdot \lambda$.

These are equivalent since

$s_w (w' \cdot \lambda) < w \cdot \lambda + s_w$
$
\iff s_w (\lambda + s_w) < w (\lambda + s_w)$
$
\iff \lambda + s_w < w \cdot \alpha > 0$
$
\iff w' \cdot \alpha > 0$

So

$w' \cdot \lambda \triangleright w \cdot \lambda$

Red: The condition $w \leq w'$ is independent of $\lambda$! (So we may only look at $\Theta_0$)

Summary: $[H(\mu) : L(\mu)] \neq 0 \iff w' \cdot \lambda \triangleright w \cdot \lambda \iff w' \leq w$.

(If $\lambda$ dot-regular, $\mu$- antidominant, integral)
The BGG Theorem is a corollary of:

**Theorem (Jantzen Filtration):** Let $\lambda \in \mathfrak{h}^*$ arbitrary. Then $M(\lambda)$ has a filtration $N(\lambda) = N(\lambda)^0 \supset N(\lambda)^1 \supset \ldots$ (\(M(\lambda)^0 = 0\)) such that

(a) Each nonzero quotient $N(\lambda)^i / N(\lambda)^{i+1}$ has a nondegenerate contravariant form

(b) $N(\lambda)^i / N(\lambda)^{i+1} = L(\lambda)$ (i.e. $N(\lambda)^{\lambda} = M(\lambda)$)

(c) $\sum_{\alpha \neq 0, \alpha \cdot \lambda < \lambda} \dim N(\lambda)^{\alpha} = \sum_{\alpha \neq 0, \alpha \cdot \lambda < \lambda} \dim M(\lambda(\alpha))$ ("Jantzen sum formula")

**Corollary (BGG Theorem):** By induction on $k = \# \{ \mu \in W \cdot \lambda | \mu \leq \lambda \}$. If $k = 1$ then $\lambda$ is minimal so $P(\lambda)$ is simple. Otherwise, assume $[M(\lambda): L(\mu)] > 0, \mu < \lambda$. Then $[M(\lambda)^2: L(\mu)] > 0$ so $[M(\lambda^\mu): L(\mu)] > 0$ for some $\mu > 0$ s.t. $\mu \cdot \lambda < \lambda$, so $\mu^\prime \cdot \lambda \cap \lambda$. But by induction hypothesis $\mu \cap \lambda \cdot \mu^\prime$, so $\lambda = \lambda^\prime$.

**Application of Jantzen’s Filtration**

For the next application of the theorem, we need a result that was mentioned last time:

(1.10)

**Prop:** Let $\lambda$ be integral, dot-regular, $\gamma$-antidominant. Then for any $\omega \cdot \lambda \in W \cdot \lambda$, $M(\lambda) = L(\lambda)$ is the unique simple submodule of $M(\omega \cdot \lambda)$. Furthermore, $[M(\omega \cdot \lambda): L(\lambda)] = 1$.

**Proof:** We saw the first part last time.

For the second part, it suffices to prove, by BGG Reciprocity, that $[P(\lambda): M(\omega \cdot \lambda)] = 1$.

Now, recall that $-\gamma$ is $\gamma$-dominant $\Rightarrow H(-\gamma)$ is projective. Also, $\omega \cdot \lambda$ is $\gamma$-dominant so $\omega \cdot \lambda + \gamma \in \Lambda^\gamma$. Thus $L(\omega \cdot \lambda + \gamma)$ is $\gamma$-dim, hence $M(\omega \cdot \lambda + \gamma) \otimes L(\omega \cdot \lambda + \gamma)$ is projective.
Claim: \( M \) has a standard filtration with \( M(\omega \cdot \lambda) \) appearing at most once and \( M(\lambda) \) appearing only at the top.

Using the claim: since \( M \to M(\lambda) \) we have \( P(\lambda) \subseteq T \) and so \( P(\lambda) \) has a std filtration with \( M(\omega \cdot \lambda) \) appearing at most once, as desired.

Proof of the claim: let \( v_1, \ldots, v_n \) be a basis of weight vectors for \( L(\omega_0 \cdot \lambda + f) \) with weights \( \mu_1, \ldots, \mu_n \). Reorder the basis so that \( \mu_i \leq \mu_j \iff i \leq j \) and so that \( v_1 \) has weight \( \lambda \) (which is minimal since \( \omega_0 (\omega_0 \cdot \lambda + f) = \lambda + f \)).

Now tensor the filtration of \( U(\mathfrak{b}) \)-modules

\[
C_j \otimes \text{Span}_{U(\mathfrak{b})}(v_n) \subseteq C_j \otimes \text{Span}_{U(\mathfrak{b})}(v_n,v_{n-1}) \subseteq \ldots \subseteq C_j \otimes L(\omega_0 \cdot \lambda + f) \quad (\text{with quotients } C_j \otimes C_{\mu_i})
\]

with \( U(\mathfrak{g}) \otimes U(\mathfrak{y}) \) (which is exact for all \( \mathfrak{y} \)-modules). We get

\[
U(\mathfrak{g}) \otimes U(\mathfrak{b}) (C_j \otimes \text{Span}_{U(\mathfrak{b})}(v_n)) \subseteq \ldots \subseteq U(\mathfrak{g}) \otimes U(\mathfrak{b}) (C_j \otimes L(\omega_0 \cdot \lambda + f)) \]

\[
( U(\mathfrak{g}) \otimes U(\mathfrak{b}) C_j ) \otimes L(\omega_0 \cdot \lambda + f)
\]

and each quotient is isomorphic to \( U(\mathfrak{g}) \otimes U(\mathfrak{b}) (C_j \otimes C_{\mu_i}) = M(\mu_i - f) \), in particular \( M(\mu_2 - f) = M\lambda \), as desired.

Break?
Finding the composition factors of $H(\lambda)$ (\lambda \text{ integral, dot-reg}) for $sl_3 C$

Recall that knowing $\chi H(\omega \cdot \lambda)_{\omega \in W}$ in terms of $\chi L(\omega \cdot \lambda)_{\omega \in W}$ (or vice-versa).

\[ g = g_3 C, \lambda \text{ integral, dot-regular, } \gamma \text{-antidominant (e.g. } \lambda = -2g) \]

So far we have $\chi H(\lambda) = \chi L(\lambda)$ (since $\gamma$-antidom, $H(\lambda)$ simple).

Next, $[H(s_i \cdot \lambda) : L(s_i \cdot \lambda)] = 1$

and these are all the candidates, so

\[ \chi H(s_i \cdot \lambda) = \chi L(\lambda) + \chi L(s_i \cdot \lambda) \]

Next, for $s_i s_j \gamma$ we know

\[ [H(s_i s_j \gamma) : L(s_i s_j \gamma)] = 1 = [H(s_i s_j \gamma) : L(\lambda)] \]

but we need

\[ [H(s_i s_j \gamma) : L(s_i \cdot \lambda)] = [H(s_i \gamma) : L(s_i \cdot \lambda)] \]

Jantzen's sum formula:

\[ \sum_{i > 0} \chi H(s_i s_j \gamma) = \chi H(s_i \gamma) + \chi H(s_j \gamma) \]

\[ = \chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda) + 2 \chi L(\lambda) \]

Since $[H(s_i s_j \gamma) : L(\lambda)] = 1$, we deduce $H(s_i s_j \gamma)^2 = L(\lambda)$ and

\[ \chi H(s_i s_j \gamma)^2 = \chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda) + \chi L(\lambda) \]

Therefore,

\[ [H(s_i \gamma) : L(\lambda)] = 1 = [H(s_i s_j \gamma) : L(s_i \cdot \lambda)] \]

Finally, for $s_i s_j s_i s_j \gamma = w_0 \cdot \lambda$, we have

\[ \sum_{i > 0} \chi H(w_0 \cdot \lambda) = \chi H(s_i s_i \gamma) + \chi H(s_j s_i \gamma) + \chi L(\lambda) \]

\[ = \chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda) + 2(\chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda)) + 3 \chi L(\lambda) \]

Again, since $[H(w_0 \cdot \lambda) : L(\lambda)] = 1$, we deduce $H(w_0 \cdot \lambda)^3 = L(\lambda)$

Two possibilities remain:

\[ \chi H(w_0 \cdot \lambda)^2 = \chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda) + \chi L(\lambda) \]

\[ \chi H(w_0 \cdot \lambda)^3 = \chi L(s_i s_j \cdot \lambda) + \chi L(s_j s_i \cdot \lambda) + \chi L(s_i \cdot \lambda) + \chi L(s_j \cdot \lambda) + \chi L(\lambda) \]

or

\[
\begin{array}{cccccc}
2 & 0 & 1 & 0 & 0 & 0
\end{array}
\]
Humphreys claims this case can be dealt with using the same tools as in the previous cases, but I don’t see how. Instead, here is an ad hoc argument: the weight space \( M(w_0 \cdot \lambda) \) has dimension 2 and

\[
\begin{align*}
\text{ch } M(w_0 \cdot \lambda)^2 &= \text{ch } L(\lambda) \\
\text{ch } M(w_0 \cdot \lambda)^4 &= \text{ch } L(g_1 g_2 \cdot \lambda) + 2 \text{ch } L(g_2 \cdot \lambda) + 2 \text{ch } L(g_1 \cdot \lambda) + \text{ch } L(\lambda)
\end{align*}
\]

(\#)

so \( [M(w_0 \cdot \lambda) : L(\lambda)] \leq 1 \) and we know \( \text{ch } M(g_1 g_2 \cdot \lambda) = 2 \) and we know \( \text{ch } M(g_1 \cdot \lambda) \) and \( \text{ch } M(g_2 \cdot \lambda) \)

Hence \( \text{ch } M(w_0 \cdot \lambda) = 2 \) and we thus discard \( (\#) \) and conclude

\[ [M(w_0 \cdot \lambda) : L(w \cdot \lambda)] = 1 \quad \forall w \in W \]

Incidentally, this provides an example of the question we discussed last week:

\[
\text{soc } (M(0) / L(-2g)) = \frac{M(0)^2 / L(-2g)}{L(-2g)}
\]

Skip: (Proof of \( \oplus \): let \( \psi \in \text{Out}(S_{\psi 0}) \) corresponding to \( \overline{a_{\alpha_1}, a_{\alpha_2}} \). Then \( M = M(0) / L(-2g) \), set \( M' = M \), action twisted by \( \psi \). This is isomorphic to \( M \) since \( \psi(0) = 0 \). However, \( L(-g \cdot \alpha_1) \) is \( L(-g \cdot \alpha_2) \) so one cannot be on top of the other.)
Proof of Jantzen’s Filtration Theorem

"Key Lemma"  
Let $A$ be a PID and $M = A^r$ with a nondegenerate bilinear form $(,)$ with determinant $D \neq 0$. Let $p \in A$ be prime. Define a filtration $H = M(0) > M(1) > \ldots$ by $H(n) = \{m \in M : (m, M) \subseteq p^n A\}$. Write also $\overline{A} = A/pA^r$, $\overline{M} = H/pH$, etc.

(a) $\nu_p(0) = \sum_{n=0}^\infty \dim_A H(n)$  
(b) $(,)_n = p^{-n}()$ induces a nondeg form on $H(n)/H(n+1)$

Skip? Proof omitted. Example instead: $A = \mathbb{C}[T]$, $M = AX \otimes AY$, bilinear form $(2T \ 0 \ 0 \ T^3)$

Then $\mathcal{V}_p(D) = \sum_{n=0}^\infty \dim M(n)/(\dim M(n+1))$ has form (2).

Theorem (Jantzen Filtration): Let $\lambda \in \mathfrak{h}^*$ arbitrary. Then $M(\lambda)$ has a filtration $M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset \ldots \supset M(\lambda)^N = 0$

such that

(a) Each nonzero quotient $M(\lambda)/M(\lambda)^i$ has a nondegenerate contravariant form

(b) $M(\lambda)/M(\lambda)^1 = L(\lambda)$  
(c) $\sum_{\alpha \in \Phi} \chi M(\lambda)^i = \sum_{\alpha \in \Phi} \chi M(\lambda - \lambda)$  

("Jantzen sum formula")

We need to define $M(\lambda)^i$. The idea is to introduce a free variable $T$ by setting $A = \mathbb{C}[T]$, $K = \mathbb{C}(T)$ and $\overline{g}_X = K \otimes \mathbb{C} g$, $\overline{g}_A = A \otimes \mathbb{C} g$. Let $\lambda_T := \lambda + T \overline{g}_T \in \mathfrak{h}_K^*$.

The theory over $K$ is the same as over $\mathbb{C}$, and since $\langle \lambda_T + \overline{g}_T, \alpha^* \rangle \in \mathbb{Z} \forall \alpha \in \Phi$, $\lambda_T + \overline{g}_T$ is antiderminant and therefore $M(\lambda_T)$ is simple, so the contravariant form on it is nondegenerate. Also, the weight space are $M(\lambda_T, \lambda_{T-\nu})$ for $\nu \in \mathbb{R}^+$.  

We also have an $A$-form $M(\lambda_T)_A \subset M(\lambda_T)$. Write $M(\lambda - \nu) = M(\lambda_T, \lambda_{T-\nu}) \cap M(\lambda_T)_A$

Finally, set $M(\lambda_T)_A = \sum_{\nu \in \mathbb{R}^+} M(\lambda - \nu)$  

To get the filtration for $M(\lambda)$, set $T = 0$, i.e. $M(\lambda)^i = \sum_{\nu \in \mathbb{R}^+} M(\lambda - \nu)$  

The Key lemma tells us $T^{-i}()$ induces a nondeg contravariant form on $M(\lambda)^i/M(\lambda)^{i+1}$. Since $M(\lambda)/M(\lambda)^i$ is also a $h$-module, it must be simple. This proves (a) and (b).
Example of this construction: $s_{\beta} c, \lambda = -\lambda_2$. Then $\lambda_T = -\lambda_2 + T_g$.

We look at a single weight space $M_{\lambda_T - \alpha_i - \alpha_k}$. Now $M_{\lambda_T - \alpha_i - \alpha_k}$ has two basis vectors: $a = \delta_{\alpha_i} \omega^+ \delta_{\alpha_k}$ and $b = \delta_{\alpha_i} \omega^+ \delta_{\alpha_k}^{-T}$.

The contravariant form wrt this basis is

$$
\begin{pmatrix}
T^2 & -T \\
-T & 2T - 1
\end{pmatrix}
$$

For instance, $(f_{\alpha_i} \omega^+ v^+, f_{\alpha_i} \omega^+ v^+) = (v^+, e_{\alpha_i} e_{\alpha_i} f_{\alpha_i} \omega^+ v^+) = e_{\alpha_i} h_{\alpha_k} \omega^+ v^+ + e_{\alpha_i} \omega^+ \omega^+ v^+ = (\lambda + h_{\alpha_k}) h_{\alpha_k} v^+$

$$
= (\lambda + h_{\alpha_k}) (-\lambda_1 + T_g)(\alpha_k)v^+ = T(\lambda + (-\lambda_1 + T_g)(\alpha_k))v^+ = T^2 v^+
$$

Therefore $M_{\lambda_T - \alpha_i - \alpha_k}(\ell) = \begin{cases} A_a \circ A_b & i = 0, \ell = \lambda_1 - \alpha_k \\
A_a \circ (\ell) b & i = 1, \ell = \lambda_1 - \alpha_k \\
A_a \circ (\ell) b & i = 2, \ell = \lambda_1 - \alpha_k \\
(\ell) a \circ (\ell) b & i = 3, \ell = \lambda_1 - \alpha_k
\end{cases}
$$

In order to verify (c) $\sum_{\ell i} \dim H(\lambda) = \sum_{\ell i} \dim H(s_\ell \lambda)$, we need to compute $\sum_{\ell i} \dim H(\lambda)_{\lambda - \nu}$. The Key Lemma tells us that this is $\nu_T(D_\nu(\lambda_T))$ (Here $D_\nu(\lambda_T)$ is the determinant of the form on $M_{\lambda_T - \nu} = M(\lambda_1) \cap M(\lambda_T)_{\lambda - \nu}$)

Claim: (Shapovalov) $D_\nu(\lambda_T) = \prod_{\nu, \nu \leq \lambda} (\lambda_1 + \nu_+ \alpha^+) - \nu$ (Here $\nu$ is a sum of positive roots)

Using the claim, note $(\lambda_1 + \nu_+ \alpha^+) - \nu = (\lambda + \nu_+ \alpha^+) - \nu + T$ is a multiple of $T$ if and only if $\nu_+ \alpha^+ \lambda$. So $\nu_T(D_\nu(\lambda_T)) = P(\nu - \langle \lambda_1 + \nu_+ \alpha^+ \rangle \alpha)$ and therefore

$$
\sum_{\ell i} \dim H(\lambda) = \sum_{\nu \in \mathbb{R}^+} P(\nu - \langle \lambda_1 + \nu_+ \alpha^+ \rangle \alpha) e^{\lambda - \nu}
$$

$$
= \sum_{\nu \in \mathbb{R}^+, \nu \leq \lambda} P(\nu - \langle \lambda_1 + \nu_+ \alpha^+ \rangle \alpha) e^{\lambda - \nu}
$$

$$
P(\nu - \langle \lambda_1 + \nu_+ \alpha^+ \rangle \alpha) = 0
$$

On the other hand, $\dim H(s_\ell \lambda) = \sum_{\nu \in \mathbb{R}^+} P(\nu) e^{s_{\alpha_k} \lambda - \nu}$, and (c) follows.
In order to get the bilinear form on $H_{\lambda, \nu}$ above, we computed the contravariant form by simplifying the PBW monomials as follows:

\[(g_i^*, g_j^*) = (v^*, \mathcal{Z}(g_i^*) g_j^* v^*) = (v^*, t_i v^*), \text{ where } t_i \in U H_i.\]

Then $t_i v^* = \lambda_i (t_i) v^*$. The determinant of the form is then $\det (\lambda_i(t_{ij}))$, but we can also get this by

\[\det (t_{ij}) v^* = \det (\lambda_i(t_{ij})) v^* .\]

Ultimately, knowing $\det (t_{ij})$ will tell us $D_{\nu}(\lambda_i)$ for all $\lambda$. This is Shapovalov's formula:

\[D_{\nu} = \det (t_{ij}) = \prod_{a \geq 0} \prod_{\alpha \neq 0} \left( h_{\alpha} + \langle \gamma, \alpha^* \rangle - r \right) \prod (v - r \alpha) .\]

Continuing the previous example, we would have $t_{ij} = \left( \begin{array}{cc} h_p (h_{\alpha} + 1) & -h_p \\ -h_p & h_{\alpha} + h_p \end{array} \right)$, whose determinant is $h_p h_{\alpha} (h_{\alpha} + h_p + 1)$.

Comment on the proof strategy?

Comment on what happens if we replace $g$ by something else?