1 Contravariant Forms

Recall that there is an anti-automorphism $\tau$ of $U(g)$ given by $x_\alpha \mapsto y_\alpha, y_\alpha \mapsto x_\alpha, h_\alpha \mapsto h_\alpha$ for all simple roots $\alpha$. Using this transpose map we define:

Definition 1.1. A symmetric bilinear form $(v, v')_M$ on a $U(g)$-module $M$ is called contravariant if

$$(u \cdot v, v')_M = (v, \tau(u) \cdot v') \forall u \in U(g), v, v' \in M.$$  

1.1 Basic Properties

Proposition 1.2. Suppose that $U(g)$-modules $M, M_1, M_2$ have contravariant forms. Then

(a) Distinct weight spaces $M_\lambda$ and $M_\mu$ of $M$ are orthogonal.

(b) If $M = U(g) \cdot v^+$ is a highest weight module generated by a maximal vector $v^+$ of weight $\lambda$, then a nonzero contravariant form on $M$ is uniquely determined up to a scalar multiple by the nonzero value $(v^+, v^+)_M$. The radical of this form is the unique maximal submodule $N$ of $M$.

(c) The tensor product $M_1 \otimes M_2$ also has a contravariant form, given by $(v \otimes w, v' \otimes w') := (v, v')_{M_1}(w, w')_{M_2}$. If the forms on $M_i$ are nondegenerate, so is the product form.

(d) For any submodule $N \subset M$, its orthogonal space $N^\perp := \{v \in M \mid (v, v_0)_M = 0 \forall v_0 \in N\}$ is also a submodule.

(e) If $M \in \mathcal{O}$, then the summands $M^\chi$ for distinct central characters $\chi$ are orthogonal.

We will prove part (b). Assuming (a), it’s enough to look at the form on a weight space $M_\mu$. Vectors $v, v' \in M_\mu$ can be written as $v = u \cdot v^+$ and $v' = u' \cdot v^+$ for some $u, u' \in U(n^-)$. Then

$$(v, v')_M = (u \cdot v^+, v')_M = (v^+, \tau(u)v')_M$$

Since $u$ maps $M_\lambda$ into $M_\mu$, it’s transpose $\tau(u)$ takes $M_\mu$ to $M_\lambda$, which is a one-dimensional space spanned by $v^+$. So, $\tau(u) \cdot v'$ is a scalar multiple of $v^+$ and $(v, v')_M$ is a scalar multiple of $(v^+, v^+)_M$ determined by the action of $U(n^-)$ on $M$. 

Since \( N \) is a weight module which does not have \( \lambda \) as one of its weights, \((v^+, N)_M = 0\). Then for any \( u \in U(\mathfrak{g}) \), we have \((u \cdot v^+, N)_M = (v^+, \tau(u))N = 0\). This means that \( N \) is contained in the radical of the form. On the other hand, the radical of a nonzero form is a proper submodule of \( M \) and must be contained in \( N \).

1.2 Example - Verma modules of \( \mathfrak{sl}_2(C) \)

Recall that \( M(\lambda) \) has weights \( \lambda, \lambda - 2, \lambda - 4, \ldots \) and we may choose a basis of corresponding weight vectors \( v_0, v_1, v_2, \ldots \) such that
\[
\begin{align*}
x \cdot v_i &= (\lambda - i + 1)v_{i-1} \\
y \cdot v_i &= (i + 1)v_{i+1}
\end{align*}
\]

For a contravariant form on \( M(\lambda) \), we must have \((v_i, v_j) = 0\) for \( i \neq j \). Also, for every \( i > 0 \), we get
\[
(v_i, v_i) = \frac{1}{\lambda - i + 1}(v_{i-1}, v_{i+1})
\]

Induction shows that
\[
(v_i, v_i) = \frac{(\lambda - i + 1)(\lambda - i + 2)\ldots(\lambda)}{i!}(v_0,v_0)
\]

(It is easy to verify that a form defined using this formula is, in fact, a contravariant form.)

Note that since distinct \( v_i \) are orthogonal to each other, the form is non-degenerate if \((v_i, v_i)\) is nonzero for all \( i \geq 0 \) iff \( \lambda \notin \mathbb{Z} > 0 \) iff \( M(\lambda) \) is simple. On the other hand, if \( \lambda \in \mathbb{Z} > 0 \), then \((v_i, v_i) = 0\) for \( i \geq \lambda + 1 \) (vectors of weights \( \leq -\lambda - 2 \)) and the radical of the form is \( M(-\lambda - 2) \).

1.3 Universal Construction

Our goal is to construct contravariant forms on highest weight modules. We start by constructing a form on \( U(\mathfrak{g}) \). Let \( \varepsilon^+: U(\mathfrak{n}) \to \mathbb{C} \) and \( \varepsilon^-: U(\mathfrak{n}^-) \to \mathbb{C} \) be the maps sending all nonconstant PBW basis elements to 0. Use the PBW theorem to define the linear map \( \varphi := \varepsilon^- \otimes id \otimes \varepsilon^+: U(\mathfrak{g}) \equiv U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \to U(\mathfrak{h}) \). This gives us a symmetric bilinear form on \( U(\mathfrak{g}) \)
\[
C(u, u') := \varphi(\tau(u)u').
\]

Since \( \tau \) is an anti-automorphism, we have
\[
C(u_0u, u') = C(u, \tau(u_0)u')
\]
for all \( u_0, u, u' \in U(\mathfrak{g}) \).

For a weight \( \lambda \), let \( \varphi_\lambda = \lambda \circ \varphi \) and define a form on \( U(\mathfrak{g}) \) by
\[
C^\lambda(u, u') := \varphi_\lambda(\tau(u)u').
\]
Now consider a highest weight module $M$ generated by maximal vector $v^+$ of weight $\lambda$. Suppose that $u_1, u_2 \in U(g)$ satisfy $u_1 \cdot v^+ = u_2 \cdot v^+$. By writing $u_i$ in the PBW basis and comparing the the components of $u_i \cdot v^+$ of weight $\lambda$, we obtain $\varphi_\lambda(u_1) = \varphi_\lambda(u_2)$. For any $u \in U(g)$, we get $uu_1 \cdot v^+ = uu_2 \cdot v^+$ and therefore $\varphi_\lambda(uu_1) = \varphi_\lambda(uu_2)$. Thus, $\varphi_\lambda(U(g)(u_1 - u_2)$ is zero and $(u_1 - u_2)$ lies in the radical of $C_\lambda$. This allows us to define a form on $M$ by

$$(v, v')_M := C_\lambda(u, u')$$

where $v = u \cdot v^+$ and $v' = u' \cdot v^+$ for $u, u' \in U(n)$. It is easy to check that this is a nonzero contravariant form. Thus we have

**Theorem 1.3.** If $M$ is a highest weight module of weight $\lambda$, there exists a (nonzero) contravariant form $(v, v')_M$ on $M$. The form is unique (up to scalar multiples) and completely determined by $(v^+, v^+)_M$. Its radical is the unique maximal submodule of $M$. In particular, the form is nondegenerate if and only if $M$ is the simple module $L(\lambda)$. \hfill $\square$

### 2 Simple Submodules of Verma Modules

**Proposition 2.1.** $M(\lambda)$ has a unique simple submodule.

**Proof.** Recall that as $U(n^)$-modules, $M(\lambda)$ and $U(n^-)$ are isomorphic. Under such an isomorphism, we may identify nonzero submodules of $M(\lambda)$ with nonzero left ideals of $U(n^-)$. Since $U(n^-)$ is left noetherian and does not have any zero divisors, any two nonzero left ideals of intersect non-trivially. Thus, any two nonzero submodules of $M(\lambda)$ must intersect non-trivially. This is impossible for distinct simple submodules. \hfill $\square$

**Example 2.2.** In case of $\mathfrak{sl}_2(\mathbb{C})$, if $\lambda \in \mathbb{Z}^{>0}$, then the unique maximal submodule $M(-\lambda - 2) \subset M(\lambda)$ is simple. Otherwise, $M(\lambda)$ itself is simple.

### 3 Homomorphisms between Verma Modules

**Theorem 3.1.** Let $\lambda, \mu \in \mathfrak{h}^*$. Then

(a) Any nonzero homomorphism $\varphi : M(\mu) \to M(\lambda)$ is injective.

(b) In all cases, $\dim \text{Hom}_O(M(\mu), M(\lambda)) \leq 1$.

(c) The unique simple submodule of $M(\lambda)$ is a Verma module.

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Proof. (a) Let $v^+_\mu$ and $v^+\lambda$ be maximal vectors in $M(\mu)$ and $M(\lambda)$, respectively. Let $u \in U(n^-)$ be such that $\varphi(v^+_\mu) = u \cdot v^+\lambda$. As left $U(n^-)$-modules, $M(\mu) = U(n^-)v^+_\mu \equiv U(n^-) \equiv U(n^-)v^+\lambda = M(\lambda)$ so that $\varphi$ corresponds to the map on $U(n^-)$ given by $u' \mapsto u' u$. Since $U(n^-)$ does not have zero divisors, $\varphi$ must be injective.

(b) Note that any nonzero homomorphism $M(\mu) \to M(\lambda)$ must descend to an isomorphism between the unique simple submodules of $M(\mu)$ and $M(\lambda)$. Thus, if $\varphi_1, \varphi_2$ are two such homomorphisms, there exists a scalar $c \in \mathbb{C}$ such that $\varphi_1 - c\varphi_2$ kills $L$. By part (a), we conclude $\varphi_1 - c\varphi_2 = 0$.

(c) Suppose that $M(\mu) \to M(\mu) \to M(\lambda)$ gives a nonzero homomorphism between Verma modules. By part (a), this is injective and $M(\mu) \subset M(\lambda)$.

Remark 3.2. Whenever there is a nonzero homomorphism $M(\mu) \to M(\lambda)$, we may write $M(\mu) \subset M(\lambda)$.

4 Simplicity Criterion and Embeddings

Theorem 4.1. Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda) = L(\lambda)$ if and only if $\lambda$ is $\rho$-antidominant, i.e., $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^+ \text{ for all positive roots } \alpha$.

Proof. We begin with integral weights.

Part (1) Suppose that $M(\lambda)$ is simple. Since $\lambda$ is integral, it is $\rho$-antidominant iff $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^+$ for all simple roots $\alpha$. If this fails for some simple root, then we have $s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha < \lambda$. This means that there is a nonzero homomorphism $M(s_\alpha \cdot \lambda) \to M(\lambda)$. However, such a morphism must be an embedding, which is impossible if $M(\lambda)$ is simple.

Part (2) We know that the highest weights of composition factors of $M(\lambda)$ must be of the form $w \cdot \lambda \leq \lambda$ with $w \in W$. If $\lambda$ is $\rho$-antidominant, then the only weight satisfying this constraint is $\lambda$ and only $L(\lambda)$ can occur as a composition factor. Since $\operatorname{dim} M_\lambda = 1$, we see that it occurs only once and therefore $M(\lambda) = L(\lambda)$.

To extend the first part of the proof to the general case, we need embeddings of the form $M(s_\alpha \cdot \lambda) \to M(\lambda)$ for arbitrary positive roots. It turns out that such embeddings exist as long as $s_\alpha \cdot \lambda \leq \mu$:

Theorem 4.2. Let $\lambda \in \mathfrak{h}^*$ and $\alpha > 0$. If $\mu = s_\alpha \cdot \lambda \leq \lambda$, then there exists an embedding $M(\mu) \subset M(\lambda)$.

The second part of the proof can be generalized by replacing $W$ by the reflection subgroup $W_\lambda$.\hfill\Box
Example 4.3. For $\mathfrak{sl}_2(\mathbb{C})$, there is a unique positive root $2$ and $\rho = 1$. So, $\lambda$ is $\rho$-antidominant iff $\langle \lambda + 1, 1 \rangle = \lambda + 1 \notin \mathbb{Z}^{>0}$ iff $\lambda \notin \mathbb{Z}^{\geq 0}$. We already know that these are precisely the weights for which $M(\lambda)$ is simple.

Corollary 4.4. If $\lambda$ is $\rho$-antidominant, then $L(\lambda)$ is the unique simple submodule and therefore a composition factor of $M(w \cdot \lambda)$ for all $w \in W[\lambda]$.

Proof. The unique simple submodule is a Verma module whose highest weight is in the orbit $W[\lambda] \cdot \lambda$. We know that $\lambda$ is the only $\rho$-antidominant weight in this orbit. □

5 Block Decomposition of Category $O$

Theorem 5.1. For a $\rho$-antidominant $\lambda$, let $O_\lambda$ be the subcategory of modules whose composition factors all have highest weights linked to $\lambda$ by $W[\lambda]$. Such $O_\lambda$ are precisely the blocks of $O$.

Proof. Consider a Verma module $M(\mu)$ and let $L(\lambda) = M(\lambda)$ be its unique simple submodule. By the simplicity criterion, $\lambda$ is $\rho$-antidominant. Then the composition factors of $M(\mu)$, including $L(\lambda)$ and $L(\mu)$ lie in the same block. The highest weights of these factors all must be in the orbit $W[\lambda] \cdot \lambda$. On the other hand, we have already shown that any Verma module with highest weight in the orbit $W[\lambda] \cdot \lambda$ has $L(\lambda)$ as its unique submodule. □

6 Error in Verma’s Thesis

Warning: Section contains false results.

Verma believed that he had proved the following Lemma:

Lemma 6.1. Let $M$ be the submodule generated by a weight vector $v_\mu$ of $M(\lambda)$. Then the submodule $M'$ of $M$ generated by vectors $x_\alpha \cdot v_\mu$ is either $0$ or $M$.

However, there is gap in the proof of this Lemma and it leads to some interesting results -

Theorem 6.2. Every submodule $M$ of $M(\lambda)$ are generated by the maximal vectors in $M$. 

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Proof. Since $M$ is a weight module, it’s enough to prove the theorem when $M$ is generated by a single weight vector $v_\mu$. If $\mu = \lambda$, then $M = M(\lambda)$ and we are done. Assume that the result holds for all weights $\mu' > \mu$. Then the module $M'$ in the above Lemma is generated by maximal weights. Either $M = M'$ or $M' = 0$, which means that $v_\mu$ is itself a maximal vector.

Consider a composition factor $M_i/M_{i-1}$ of $M(\lambda)$. By the above Theorem, we may assume that $M_i$ is generated by $M_{i-1}$ along with one maximal vector of weight $\mu$. This weight vector generates a copy of $M(\mu)$ in $M(\lambda)$ and we have $M_i/M_{i-1} = L(\mu)$. Thus, every composition factor of $M(\lambda)$ comes from an embedding of a Verma module. This means that every composition factor appears with multiplicity 1.