KL theory

- ref: Maki...
Introduction to
Soergel bimodules.
(chap 3, 13)

Coxeter system
(W, S)

$$\begin{tikzcd}
(W,S) \arrow[d, squiggly] \arrow[rd, squiggly] & & & \\
\boxed{\mathbb{Z}[W]}\text{-}\overset{\mathbb{Z}}{\text{alg}} & \mathbb{Z}[v^{\pm}]\text{-alg} \arrow[l, "v \to 1"] \arrow[rd, squiggly] & & \\
& & \underline{\mathcal{H}(W)} \arrow[ld, "\text{KL basis}"'] \arrow[rd, "\text{standard basis}"] & \\
& \{b_x\}_{x \in \underline{W}} \arrow[rr, leftrightarrow, "h(v)"] & & \{\delta_x\}_{x \in \underline{W}}
\end{tikzcd}$$

Motivation

M_λ ... L_λ $|W|$

$$[M_\lambda : L_\mu] = \underline{h_{\mu\nu}}(1)$$

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standard basis

To each $x \in W = s_1 s_2 s_3 \cdots s_n$, $s_n \in S$ — reduced expression

$$\delta_x = \delta_{s_1} \cdot \delta_{s_2} \cdots \delta_{s_n}$$

$$\mathcal{H}(W) = \left\langle \delta_s, s \in S \;\middle|\; \boxed{\begin{array}{l} (\delta_s - v^{-1})(\delta_s + v) = 0 \\ \underbrace{\delta_s \delta_t \delta_s \cdots}_{m_{st}} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{m_{st}} \end{array}} \right\rangle$$

$\{\delta_x\}$ is well defined by Matsumoto theorem [M, Thm 1.3]

$\{\delta_x\}$ is a basis

spans $\mathcal{H}(W)$: $\delta_x \delta_s = \begin{cases} \_\,\delta_{xs} + \_\,\delta_x \\ \_\,\delta_{sx} + \_\,\delta_x \end{cases}$

$(\forall x \in X\ \forall s \in S)$

linear independence : $E :=$ $\mathbb{Z}[v^{\pm 1}]$-module $\langle e_x \mid x \in W \rangle$

$\mathcal{H}(W) \curvearrowright E$

$\rightsquigarrow$ $\mathcal{H}(W) \longrightarrow E$

$\delta_x \longmapsto e_x$

Inversion in $\mathcal{H}(W)$

has a formula

quadratic eq^n $\Rightarrow$ $\exists\, \delta_s^{-1}$ $\Rightarrow$ $\exists\, \delta_x^{-1}$

$(s \in W)$ $(x \in W)$

Formula [M. Lemma 3.1]

$$\delta_{w^{-1}}^{-1} = \delta_w + \sum_{x<w} \underset{\mathbb{Z}[v^{\pm}]}{\underset{\cap}{a_x}}\, \delta_x$$

$\boxed{\begin{array}{c}\text{(anti-)}\\ \text{Involution}\end{array}}$ Let $\overline{(\ )}: \mathcal{H}(W) \longrightarrow \mathcal{H}(W)$

be a $\mathbb{Z}$-linear map

$$\delta_s \longmapsto \delta_s^{-1}$$

$$v \longmapsto v^{-1}$$

$\xrightarrow{\text{extend}}$ to an alg homomorphism, $\overline{(\ )}: \mathcal{H}(W) \to \mathcal{H}(W)$

$\xrightarrow{\text{can also extend}}$ to an alg antihomo, denoted by ω

Formula:

$$\left\{\begin{array}{ll} \overline{\delta_x} = \delta_{x^{-1}}^{-1} & (3.17) \\ \omega(\delta_x) = \delta_x^{-1} & (\forall x \in W) \end{array}\right.$$

Kazhdan-Lusztig basis $\{b_w / w \in W\}$

We call a $\mathbb{Z}[v^{\pm}]$-linear basis for $\mathcal{H}(W)$ to

be KL if ① (Self dual)

$$\overline{b_w} = b_w$$

② (Respects the Bruhat order)
(wrt std basis)

$$b_w = \delta_w + \sum_{z<w} \underset{\underset{\boxed{v\mathbb{Z}[v]}}{\cap}}{\textcircled{( )}} \delta_z$$

$v\mathbb{N}_{\geq 0}[v]$

Translation from one to another.

$$b_w = \delta_w + \sum_{z<w} \underline{\underline{h_{zw}(v)}}\,\delta_z$$

Def KL-polynomial.

$$\delta_x^2 = (q^2-1)\delta_x + \dots$$

$$\delta_x^2 = (v-v^{-1})\delta_x + 1$$

KL-conjectured (proved)

In $K(\mathcal{O}_0)$,

$$[M_{w\cdot 0}] = \sum_{x\leq w} h_{xw}(1)\,[L_{\underset{x}{\cancel{w}}\cdot 0}]$$

KL poly

$$W = S_3 = \langle \underset{(12)}{\overset{}{\underset{\shortparallel}{s}}}, \underset{(23)}{\underset{\shortparallel}{t}} \mid t^2 = s^2,\ sts = tst \rangle$$

Bruhat order

$$\begin{tikzcd}
& s \arrow[r, phantom, "\leq"] \arrow[ddr, dash, "\leq"] & st \arrow[dr, phantom, "\leq"] & \\
e \arrow[ur, phantom, "\leq"] \arrow[dr, phantom, "\leq"] & & & sts=tst \\
& t \arrow[r, phantom, "\leq"] \arrow[uur, dash, "\leq"] & ts \arrow[ur, phantom, "\leq"] &
\end{tikzcd}$$

$$\text{std basis} = \{\delta_e, \delta_s, \delta_t, \delta_{st} = \delta_s\delta_t, \delta_{sts}\}$$

$$\delta_{ts} = \delta_t\delta_s$$

KL basis

$$b_e = \delta_e$$

$$b_s = \delta_s + \overset{\in v\mathbb{Z}[v]}{\underline{v}}\,\delta_e$$

$$b_{st} = \delta_{st} + A(v)\delta_s + B(v)\delta_t + C(v)\delta_e$$

$$\underline{\ \vdots\ }\ ))$$

$$b_{st} \rightsquigarrow A(v) = B(v) = v \Rightarrow C(v) = v^2.$$

$$\begin{cases} h_{s,st}(1) = h_{t,st}(1) = 1 \\ h_{e,st}(1) = 1 \end{cases}$$

KL conjecture

Sketch of proof

great stuff.

-ref [S] Yi Sun's Perverse sheaves and the KL conj.

(version: Dec 7, 2011).

$$M_y \overset{h_{xy}(1)}{\sim} L_x$$

$$\delta_y \underset{h_{xy}(v)}{\sim} b_x$$

Thm [S, thm 3.13]

$\mathrm{Cat}\,\mathcal{O} \ni V$

- $\mathfrak{n}$ & $\mathfrak{h} \curvearrowright V$ loc. fin.
- $\mathfrak{h} \curvearrowright V$ semisimply
- finitely gen

B-equiv

coherent

$$\left(\mathrm{Cat}\,\mathcal{O}_{(0)}\right) \xrightarrow{\text{BB loc.}} \left(\begin{matrix}\text{regular holonomic}\\ \checkmark\ \text{B-equivariant}\end{matrix}\right)\text{-}\underline{\text{D-modules}}\text{ over } G/B =: X \xleftarrow{[S, prop 3.11]}$$

$\mathrm{Cat}\,\mathcal{O}_{(0)} \ni L_w$, M_w

$$\left(\begin{matrix}\text{B-equiv}\\ \text{perverse sheaves}\\ \text{over } X\end{matrix}\right) \xrightarrow{h} \mathcal{H}(W)$$

$$\xrightarrow{dR}$$

$$L_w \mapsto \mathcal{L}_w = IC(X_w) \in \left(\text{B-equiv perverse sheaves over } X\right)$$

$$M_w \mapsto \underline{\mathbb{C}}_{X_w^\circ}[\dim X_w] =: \mathcal{M}_w \in$$

CMX

BBDG decomposition

Def [S, p.20] : Let χ be the map from

$$K(\mathrm{Perv}^B(X)) \text{ to } \mathbb{Z}[W].$$

$$\chi : [E] \longmapsto \sum_{w \in W} \overbrace{\sum_{i \in \mathbb{Z}} (-1)^i h^i\left(E_{\boxed{wB/B}}\right)}^{\chi\left(E_{wB/B}\right)} \cdot w$$

Prop χ is an isomorphism (b/c $\chi(\mathcal{M}_w) \mapsto (-1)^{\ell(w)} \cdot w$)

TODO.

so ~~it suffices to~~ prove

$$\chi(\mathcal{L}_w) = \sum_{v \leq w} (-1)^{\ell(v)-\ell(w)} P_{v,w}(-1)\, \chi(\mathcal{M}_x)$$

$P_{v,w} \neq h_{v,w} \overset{?}{=} P_{w_0 \cdot v,\, w_0 \cdot w}$

Main point Compute $\chi(\mathcal{L}_w)$ explicitly.

$$\chi(\mathcal{L}_w) = \sum_{x \in W} \sum_{i \in \mathbb{Z}} (-1)^i h^i\left((\mathcal{L}_w)_{xB/B}\right) \cdot x$$

Thm [S, Cor 4.15]

Key computation

Define h: $D^b_c(X) \to \mathcal{H}(W)$

$$F \mapsto \sum_{x \in W} \sum_{i \in \mathbb{Z}} h^i(F_{xB/B}) \cdot q^i \cdot \delta_x$$

Then. $h(\mathcal{L}_w) = b_w$ $\quad \forall w \in W.$

$$\begin{tikzcd} \mathcal{Z}_w \arrow[d] & \\ D^b_c(\mathcal{X}_w) \times D^b_c(\mathcal{X}_w) \arrow[r] & D^b_c(\mathcal{X}_w) \end{tikzcd}$$

$\underline{w} = (s_1, s_2, \ldots, s_n)$

$$\begin{tikzcd} \mathcal{L}_w \arrow[d, maps to] & \mathcal{L}_w \arrow[d, maps to] \\ X_w \arrow[r, squiggly] & \mathcal{X}_w \end{tikzcd}$$