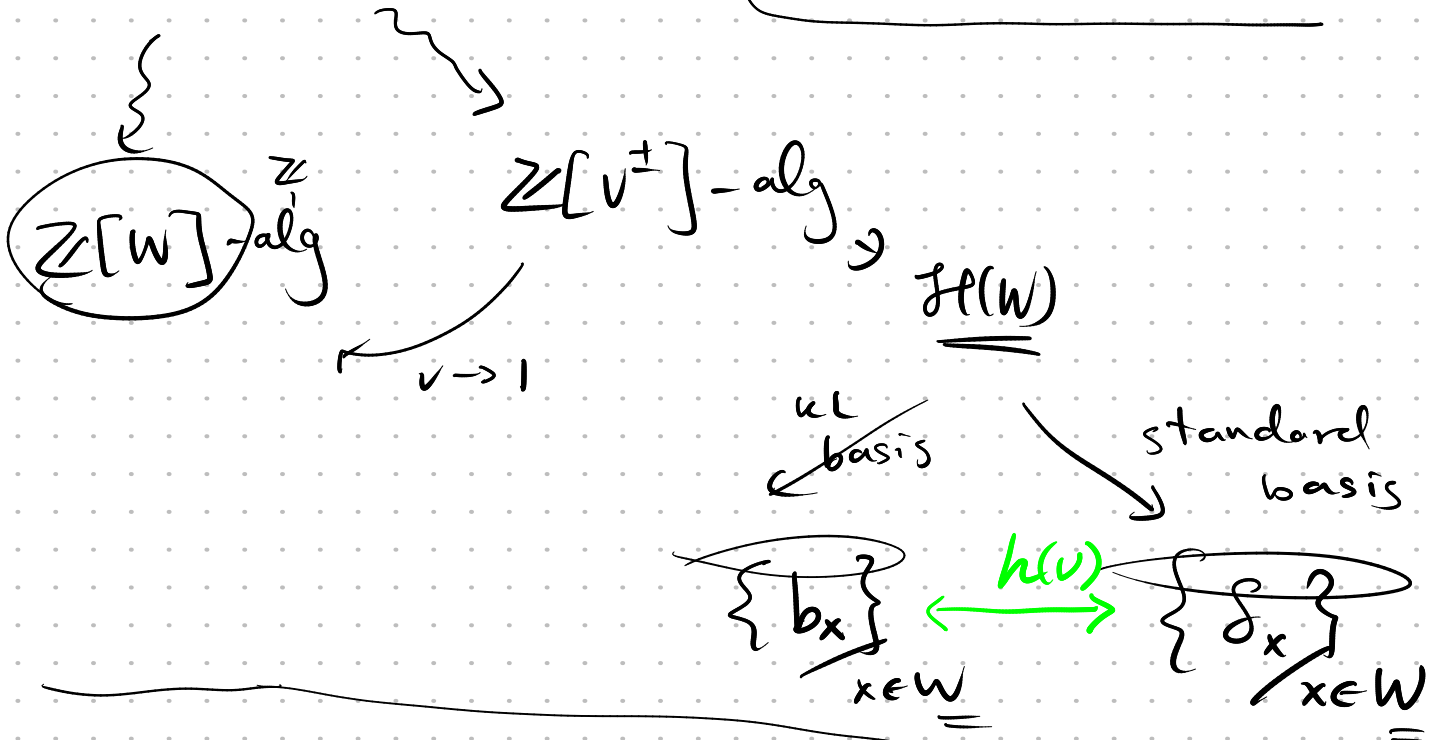


KL theory

- ref: Maki...
Introduction to
Soergel bimodules.
(chap 3, 13)

Coxeter system
(W, S)



Motivation

$$M_\lambda \cdots L_\lambda \quad |W|$$

$$[M_\lambda : L_\mu] = \underline{h_{\mu\nu}(1)}$$

Standard basis

To each $x \in W$
" $s_1 s_2 s_3 \cdots s_n \in S$
reduced expression

$$\delta_x = \delta_{s_1} \delta_{s_2} \cdots \delta_{s_n}$$

$$\mathbb{H}(W) = \langle \delta_s, s \in S \mid \underbrace{(\delta_s - v^{-1})(\delta_s + v)}_{\delta_s \delta_t + \delta_s \cdots = \delta_t \delta_s \delta_t \cdots} = 0 \quad m_{st} \rangle$$

$\{\delta_x\}$ is well defined by Matsumoto
Theorem [M, Thm 1.3]

$\{\delta_x\}$ is a basis

spans $\mathcal{H}(W)$: $\delta_x \delta_s = \begin{cases} -\delta_{xs^t} - \delta_x \\ -\delta_{sx^t} - \delta_x \end{cases}$
($\forall x \in X \forall s \in S$)

Linear independence : $E := \mathbb{Z}\langle V^{\pm 1} \rangle$ -module
 $\langle e_x \mid x \in W \rangle$

$$\mathcal{H}(W) \cong E$$

$$\begin{array}{ccc} \mathcal{H}(W) & \longrightarrow & E \\ \delta_x & \longmapsto & e_x \end{array}$$

Inversion in $\mathcal{H}(W)$

quadratic eqn' \Rightarrow $\exists \delta_s^{-1} \Rightarrow \exists \delta_x^{-1}$
($s \in W$) ($x \in W$)

has a formula

Formula [M, Lemma 3.1]

$$\delta_w^{-1} = \delta_w + \sum_{x \in W} \underbrace{a_x}_{\in \mathbb{Z}\langle V^{\pm 1} \rangle} \delta_x$$

(anti-) Involutions

Let $(\cdot) : \mathfrak{H}(W) \rightarrow \mathfrak{H}(W)$
 be a \mathbb{C} -linear map

$$\begin{aligned} \delta_s &\mapsto \delta_s^{-1} \\ v &\mapsto v^{-1} \end{aligned}$$

extend to an alg homomorphism, $(\cdot) : \mathfrak{H}(W) \rightarrow \mathfrak{H}(W)$

can also extend to an alg antihomo, denoted by ω

Formula : $\overline{\delta_x} = \delta_{x^{-1}}^{-1} \quad (3.17)$

$$\omega(\delta_x) = \delta_x^{-1} \quad (\forall x \in W)$$

Kazhdan-Lusztig basis $\{b_w / w \in W\}$

We call a linear basis $\{b_w\}$ for $\mathfrak{H}(W)$ to

be KL if \mathcal{D} (self dual)

$$\overline{b_w} = b_w$$

② (Respects the Bruhat order)
 (wrt std basis)

$$b_w = \delta_w + \sum_{z < w} \delta_z$$

$\forall w \in W$
 ≥ 0

δ_z
 $\forall z \in W$

Translation from one to another.

$$b_w = \delta_w + \sum_{z < w} h_{zw}(v) \delta_z$$

Def

KL-polynomial.

KL-conjectured (proved)

$$\delta_x^2 = (q-1)\delta_x + \dots$$

$$\delta_x^2 = (v-v^{-1})\delta_x + 1$$

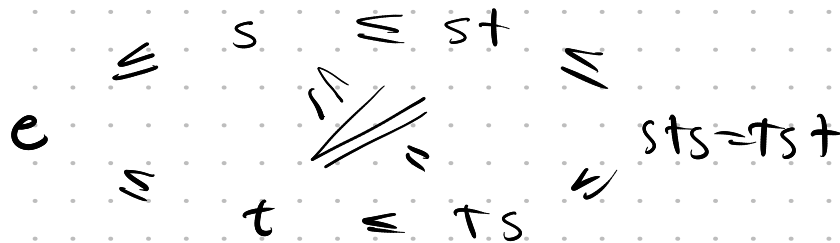
In $k(\mathbb{Q}_0)$,

$$[M_{w \cdot 0}] = \sum_{x \leq w} h_{xw}(1) [L_{x \cdot 0}]$$

KL poly $W = S_3 = \langle s, t \mid \begin{matrix} t^2 = s^2 \\ sts = tst \end{matrix} \rangle$

(12) (23)

Braid order



std basis = $\{ \delta_e, \delta_s, \delta_t, \delta_{st} = \delta_s \delta_t, \delta_{sts} \}$

$\delta_{ts} = \delta_t \delta_s$

KL basis

$$b_e = \delta_e \quad \text{in } \mathbb{Z}[v]$$

$$b_s = \delta_s + \underline{v} \delta_e$$

$$b_{st} = \delta_{st} + A(v) \delta_s + B(v) \delta_t + C(v) \delta_e$$

\vdots)

$$b_{st} \rightsquigarrow A(v) = B(v) = v \Rightarrow C(v) = v^2$$

$$\begin{cases} h_{s,st}(1) = h_{t,st}(1) = 1 \\ h_{e,st}(1) = 1 \end{cases}$$

KL conjecture

Sketch of proof

-ref ^{great stuff!} Yi Sun's Perverse sheaves and the KL conj (version: Dec 7, 2011)

$$M_Y \xrightarrow{h_{xy(1)}} L_X$$

$$\delta_Y \xrightarrow{h_{xy(v)}} b_X$$

Thm [S, thm 3.13]

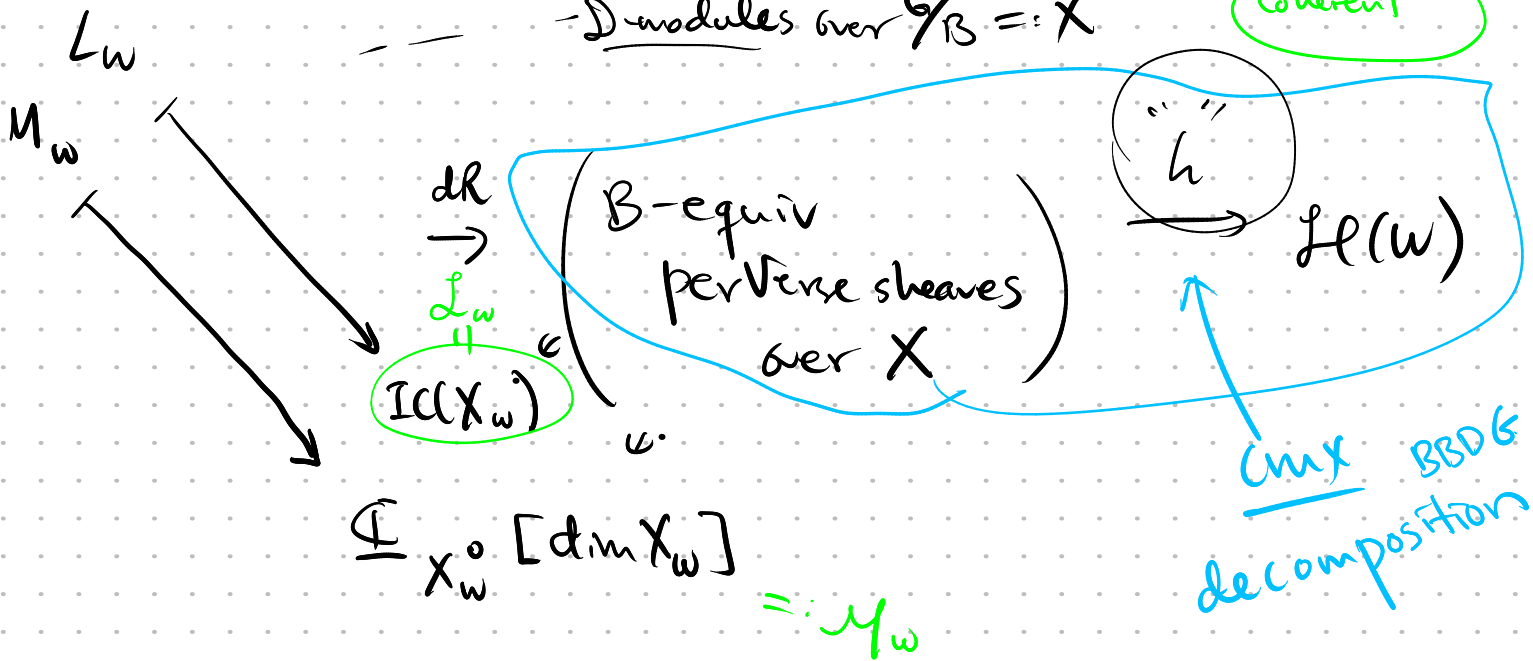
(B/B loc.)

regular holonomic

B-equivariant

D-modules over $G/B =: X$

- Cat $\mathcal{O} \ni V$
- $\uparrow \& h \ni V$ loc. fin.
 - $h \ni V$ semisimple
 - finitely gen.
 - coherent



Def [S, p.20]: let χ be the map from $K(\text{Per}^B(X))$ to $\mathbb{Z}[w]$.

$$\chi: [E] \mapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} (-1)^i h^i(E_{w/B}) \cdot w$$

Prop χ is an isomorphism (b/c $\chi(M_w) \mapsto (-1)^{\dim(w)} \cdot w$)

TODO:
 so it suffices to prove

$$\chi(L_w) = \sum_{v \leq w} (-1)^{\text{lev}(L_w) - \text{lev}(L_v)} p_{v,w}(-1) \chi(M_x)$$

$\underbrace{\hspace{10em}}_{\text{?}}$
 $\underbrace{\hspace{10em}}_{h_{v,w}}$
 $\underbrace{\hspace{10em}}_{P_{w_0 \cdot v, w_0 \cdot w}}$

Main point Compute $\chi(L_w)$ explicitly.

$$\sum_{x \in W} \sum_{i \in \mathbb{Z}} (-1)^i h^i((L_w)_{x \cdot B / B}) \cdot x$$

Thm [S, Cor 4.15]
~~key computation~~

Define $h: D_c^b(X) \rightarrow \mathbb{Z}(W)$
 $F \mapsto \sum_{x \in W} \sum_{i \in \mathbb{Z}} h^i(F_{x \cdot B / B}) \cdot q^i \cdot \delta_x$

Then $h(L_w) = b_w$

