Morita Theory for ~~R-mo~~ mod-R

Ref

{Maki -- Intro to Soergel Bimod §25}

$$\left\{ \frac{(\text{mod-}R)}{R:\ \text{unital ring}} \right\} \subseteq (\mathcal{A}b.) \supseteq \mathcal{A}$$

Q1 When is $\mathcal{A}$ equivalent to some mod-R ?

Q2 When $(\text{mod-}R_1) \simeq (\text{mod-}R_2)$?

Obverse Let P : projective obj in $\underline{\underline{\mathcal{A}}}$

each obj is of finite length

$$\langle P \rangle_{\mathcal{A}} \xrightarrow[\sim]{\mathrm{Hom}_{\mathcal{A}}(P,-)} \text{mod-}\mathrm{End}_{\mathcal{A}}(P)$$

Sketch:

P: proj $\Rightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P,-)$: fully faithful

$\mathcal{A}$: f.l. $\Rightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P,-)$: essentially surj

cf. [Bass's Alg K-theory 1.3].

#

Rmk

1) $\langle P \rangle \simeq \mathcal{A}$ call P: proj generator.

$\mathcal{A} \simeq$ mod-$\underbrace{\mathrm{End}_{\mathcal{A}}(P)}_{R}$

2) $\exists$ other proj gens in mod-R P'

s.t. $\mathrm{End}_{\boxed{R}}(P') \not\simeq R$!

In fact, the conv is true:

$(\text{mod-}R) \simeq (\text{mod-}R') \Rightarrow R' \simeq \mathrm{End}_R(P)$.

[Meyer, Morita Equiv in alg & geo 1.1]

3) R and $\mathrm{End}_R(P)$ have isomorphic centers.

3.1) If R: comm, then $R \underset{M}{\sim} R' \Rightarrow R \simeq R'$

3.2) Similar holds in the context of monoidal cats. [EGNO's Tensor categories] eg (finite dim'l alg). Let R = fd' alg. ($\mathbb{K}$ field) $1 = e_1 \oplus \cdots \oplus e_n$

Krull-Schmidt $\Rightarrow$ $\left(\begin{array}{l}\text{If } P \text{ is an indecomp proj module} \\ \text{then } P \in R\end{array}\right)$

Hence $P = \bigoplus_{\text{"}i\text{"}} P_i$ is a projective generator

<u>Claim</u> [Maki P515]: all simple mods over $\mathrm{End}_R(P)$ are 1-dim'l, and thus are isomorphic to the path alg of some quiver / ~.

(dg-Morita theory)

Given an abelian cat $\mathcal{A}$, (X)

$C^\circ(\mathcal{A})$: cats cpx in $\mathcal{A}$ $\cdots \to X_{-1} \to X_0 \to X_1 \to X_2 \to$

$K^\circ(\mathcal{A})$: $C^\circ(\mathcal{A})$ but with chain homotopic maps identified

$D^\circ(\mathcal{A})$: $K^\circ(\mathcal{A})$ with quasi-isoms inverted.

aside

triangulated cat

$\mathcal{A}$: additive cat $\circlearrowleft [1]$

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$

"mimicking short exact sequences"

$$\begin{cases} X \xrightarrow{f} Y \to \mathrm{Cone}(f) \xrightarrow{\delta} X[1] \\ Y \xrightarrow{g} Z \to \mathrm{Cone}(g) \xrightarrow{\delta} Y[1] \\ X \xrightarrow{gf} Z \to \mathrm{Cone}(gf) \xrightarrow{\delta} X[1] \end{cases}$$

<u>Def</u> $(\mathrm{End}(X))$

$\mathcal{A}$: additive $X \in \mathrm{Obj}(C(\mathcal{A}))$

$$\mathrm{End}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{End}^i(X)$$

$$:= \bigoplus_{i \in \mathbb{Z}} \prod_{k \in \mathbb{Z}} \underline{\mathrm{Hom}(X^k, X^{k+i})}$$

<u>Rmk</u> 1) $\mathrm{End}(X)$ is a unital algebra .... but more!

1.1) It is graded

1.2) It has a differential that RESPECTS the grading.

$$\mathrm{End}^i(X) \xrightarrow{d} \mathrm{End}^{i+1}(X)$$

$$(f^k)_{k \in \mathbb{Z}} \longmapsto (df^k - (-1)^i f^{k+1} \circ d)_{k \in \mathbb{Z}}$$

1.3) $d^2 = 0$ and $d(ab) = (da)b + (-1)^{|a|} a(db)$

2) We call such alg a dga ← alg

3) ← mod

dg-module

eg $\mathcal{H}om^\bullet(X^\bullet, Y^\bullet) \circlearrowleft End(X^\bullet)$

dg-module.

Def (notation) Given a dga A:

dg-$C(A)$: cat of right dg-mods over A

dg-$K(A)$: dg-$C(A)$ w/ homotopic maps identified

dg-$D(A)$: —— " —— q.i.s inverted

Claims ① dg-$K(A)$ dg-$D(A)$ ... triangulated

② $A \xrightarrow{q.i.} A'$

$$\text{dg-}D(A) \xrightarrow[\sim]{} \text{dg-}D(A')$$

$\triangle$-cat

(Caveat: that this statement has some caveat.)

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notes below are lost!