

Morita Theory for $\mathcal{A} \text{ mod } R$

Ref

[Maki - Intro to Soergel
Bimod §25]

$$\left\{ \begin{array}{l} (\text{mod-}R) \\ \hline \mathcal{A} \text{ unital ring} \end{array} \right\} \subseteq (\mathcal{A}b.)$$

Q1 When is \mathcal{A} equivalent to some $\text{mod-}R$?

Q2 When $(\text{mod-}R_1) \simeq (\text{mod-}R_2)$?

Obverse Let P = projective obj in \mathcal{A} .

each obj is of finite length

$$\langle P \rangle_{\mathcal{A}} \xrightarrow[\sim]{\text{Hom}_{\mathcal{A}}(P, -)} \text{mod-End}_{\mathcal{A}}(P)$$

Sketch:

P : proj $\Rightarrow \text{Hom}_{\mathcal{A}}(P, -)$ = fully faithful

\mathcal{A} : f.l. $\Rightarrow \text{Hom}_{\mathcal{A}}(P, -)$ = essentially surj
cf. [Bass's Alg K-theory 1:3] #

Rank

1)

$$\langle P \rangle \simeq A$$

s.t.

$$\text{mod-End}_R(P)$$

call $P =$

proj generator.

R .

2)

\exists other proj gens in mod- R

P'

$$\text{s.t. } \text{End}_R(P') \not\cong R !$$

(In fact, the conv is true.:

$$(\text{mod-}R) \simeq (\text{mod-}R') \Rightarrow R' \simeq \text{End}_R(P).$$

[Meyer, Morita Equiv alg & geo 1.1]

3)

R and $\text{End}_R(P)$ have isomorphic centers.

3.1) If R is comm, then $R \sim_M R' \Rightarrow R \simeq R'$

3.2) ~~Similar~~ Similar holds in the context of monoidal cats. [EGNO's Tensor Categories]

eg (finite dim'l alg) - \mathbb{K} field Let $R = \text{fd alg}$. $\mathbb{K} = e_1 \oplus \dots \oplus e_n$

Krull-Schmidt \Rightarrow $\left(\begin{array}{l} \text{If } P \text{ is an indecomp proj module} \\ \text{then } P \in R \end{array} \right)$

Hence $P = \bigoplus_{i=1}^n P_i$ is a projective generator

Claim [Maki P515]: all simple mods over $\text{End}_R(P)$ are 1-dim'l, and thus are isomorphic to the path alg of some quiver (\sim) .

(dg-Morita theory)

Given an abelian cat \mathcal{A} ,

(X)

$C^*(\mathcal{A}) = \text{cats } \text{cp} \times \mathcal{A}$



$K^*(\mathcal{A})$: $C^*(\mathcal{A})$ but with chain homotopy maps identified

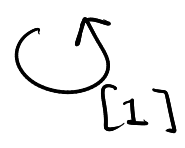
$D^*(\mathcal{A})$: $K^*(\mathcal{A})$ with quasi-isoms inverted.

aside

triangulated cat

\mathcal{A} = additive cat

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$



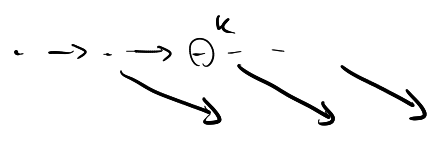
"mirroring short exact sequences"

$$\left\{ \begin{array}{l} X \xrightarrow{f} Y \rightarrow \text{Cone}(f) \xrightarrow{\delta} X[1] \\ Y \xrightarrow{g} Z \rightarrow \text{Cone}(g) \xrightarrow{\delta} Y[1] \\ X \xrightarrow{gf} Z \rightarrow \text{Cone}(gf) \xrightarrow{\delta} X[1] \end{array} \right.$$

Def $(\text{End}(X))$

\mathcal{A} = additive $X \in \text{Obj}(\mathcal{C}(\mathcal{A}))$

$$\text{End}(X) := \bigoplus_{i \in \mathbb{Z}} \text{End}^i(X)$$



$$:= \bigoplus_{i \in \mathbb{Z}} \prod_{k \in \mathbb{Z}} \text{Hom}(X^k, X^{k+i})$$

Remark 1) $\text{End}(X)$ is a unital algebra ... but more!

1.1) It is graded

1.2) It has a differential that RESPECTS the grading.

$$\text{End}^i(X) \xrightarrow{d} \text{End}^{i+1}(X)$$

$$\left(f^k \right)_{k \in \mathbb{Z}} \longmapsto \left(df^k - (-1)^i f^{k+1} \circ d \right)_{k \in \mathbb{Z}}$$

1.3) $d^2 = 0$ and $d(ab) = (da)b + (-1)^{|a|} a(db)$

2) We call such alg a dger \leftarrow alg

3) $\underbrace{\text{alg-module}}_{\text{mod}}$

eg $\text{Hom}(X, Y) \hookrightarrow \text{End}(X)$
 dg-module.

Def (notation) Given a dga A :

dg- $\text{C}(A)$: cat of right dg-mods over A

dg- $\text{K}(A)$: dg- $\text{C}(A)$ w/ homotopiz maps identified

dg- $\text{D}(A)$: $\text{---} \text{---} \text{---}$ f.i.s. inverted

Claims ① dg- $\text{K}(A)$ dg- $\text{D}(A)$... triangulated

② $A \xrightarrow{f.i.} A'$

dg- $\text{D}(A) \xrightarrow{\sim} \text{dg-}\text{D}(A')$
 Δ -cat

(Caveat: that this statement has some caveat.)

Notes below are lost!