(1) Evaluate the following definite integrals using any method.

(a) \[ \int_{0}^{2} (2x - x^2) \, dx \]

**Solution.** Using the fundamental theorem of calculus,
\[ \int_{0}^{2} (2x - x^2) \, dx = \left[ x^2 - \frac{x^3}{3} \right]_{x=2} - \left[ x^2 - \frac{x^3}{3} \right]_{x=0} = 4 - \frac{8}{3} = \frac{4}{3}. \]

(b) \[ \int_{-2}^{2} (1 + \sqrt{4 - x^2}) \, dx \]

**Solution.** First split up the integral:
\[ \int_{-2}^{2} 1 \, dx + \int_{-2}^{2} \sqrt{4 - x^2} \, dx. \]

The first integral is equal to 4. As for the second, it is hard to find an antiderivative for \( \sqrt{4 - x^2} \). Instead, interpret the integral as the area under a semicircle of radius 2:
\[ \int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi \cdot 2^2 = 2\pi. \]

So the final answer is \( 4 + 2\pi \).

(c) \[ \int_{0}^{\pi} \cos(\theta) \, d\theta \]

**Solution.** The antiderivative of \( \cos \) is \( \sin \), so using the fundamental theorem of calculus,
\[
\int_0^\pi \cos(\theta) \, d\theta = \sin(\pi) - \sin(0) = 0.
\]
(This makes sense because exactly half of the desired area is the negative of the other half, and they cancel.)

(d) \[
\int_3^3 \sin(x)^3\sqrt{x^7+1} \, dx
\]

**Solution.** It is hard to find an antiderivative for the integrand. But we don’t need to, because the limits of integration leave no area under the curve. So the answer is 0.

(e) \[
\int_1^6 (3f(x) - 4g(x)) \, dx
\]
if \(\int_1^8 f(x) \, dx = 2\) and \(\int_6^8 f(x) \, dx = 1\) and \(\int_6^1 g(x) \, dx = 3\).

**Solution.** Using properties of integrals,
\[
\int_1^6 (3f(x) - 4g(x)) \, dx = 3 \left( \int_1^8 f(x) \, dx - \int_6^8 f(x) \, dx \right) - 4 \left( -\int_6^1 g(x) \, dx \right).
\]
Now we just plug in the given values, to get \(3(2 - 1) - 4(-3) = 15\).

(f) \[
\int_0^1 (u + 2)(u - 1)\sqrt{u} \, du
\]

**Solution.** Expand everything in the integrand:
\[
(u + 2)(u - 1)\sqrt{u} = (u^2 + u - 2)\sqrt{u} = u^{5/2} + u^{3/2} - 2u^{1/2}.
\]
Its antiderivative is
\[
\frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}.
\]
Using the fundamental theorem of calculus, the integral is \(\frac{2}{7} + \frac{2}{5} - \frac{4}{3} = -68/105\).

(g) \[
\int_{-\pi}^\pi |\sin(\theta)| \, d\theta
\]

**Solution.** The best way to do this integral is to note that \(|\sin(-\theta)| = |\sin(\theta)| = |\sin(\theta)|\), so the area on \([-\pi, 0]\) is the same as the area on \([0, \pi]\). So the whole integral is just
\[
\int_{-\pi}^\pi |\sin(\theta)| \, d\theta = 2 \int_0^\pi |\sin(\theta)| \, d\theta = 2 \int_0^\pi \sin(\theta) \, d\theta.
\]
(Alternatively, you can manually compute the piece on \([-\pi, 0]\). Because of the absolute value, it is necessary to split the integral into two pieces.) The remaining integral can be computed using the fundamental theorem of calculus:

\[
\int_0^\pi \sin(\theta) \, d\theta = -\cos(\pi) - (-\cos(0)) = -(1) - (-(-1)) = 2.
\]

So the final answer is \(4\).

(2) Express the limit as a definite integral, and then evaluate it.

(a) 
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2}
\]

**Solution.** The guess is that we split up an interval into \(n\) pieces of width \(1/n\) each, because of the overall factor of \(1/n\) and the sum from \(k = 1\) to \(n\). Since we see a term \(k/n\), this suggests our pieces are \([0, 1/n], [1/n, 2/n], \ldots, [(n - 1)/n, 1]\]. Hence we have the Riemann sum for

\[
\int_0^1 \frac{1}{1 + x^2} \, dx.
\]

(Check this by writing down its right Riemann sum, if you are not convinced.)

Now recognize the integrand as the derivative of \(\arctan\). So by the fundamental theorem of calculus,

\[
\int_0^1 \frac{1}{1 + x^2} \, dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}.
\]

(b) 
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{1+k/n}
\]

**Solution.** We are still splitting an interval into \(n\) pieces of width \(1/n\). But now we have two different choices for exactly what the pieces are:

(i) \([0, 1/n], [1/n, 2/n], \ldots, [(n - 1)/n, 1]\], with integrand \(e^{1+x}\);

(ii) \([1, 1 + 1/n], [1 + 1/n, 1 + 2/n], \ldots, [1 + (n - 1)/n, 2]\], with integrand \(e^x\).

So the corresponding integral is either

\[
\int_0^1 e^{1+x} \, dx \quad \text{or} \quad \int_1^2 e^x \, dx.
\]

Both evaluate to \(e^2 - e\).

(3) Let \(f(x) = \sqrt{1 + x^4}\).

(a) Show that \(1 \leq f(x) \leq 1 + x^4\) for \(x \geq 0\).

**Solution.** If \(x \geq 0\) then \(1 + x^4 \geq 1\). But for \(z \geq 1\) we know \(\sqrt{z} \leq z\). So it follows that

\[
\sqrt{1 + x^4} \leq 1 + x^4.
\]
For the lower bound, take the square root of $1 \leq 1 + x^4$. 

(b) Show that $1 \leq \int_0^1 f(x) \, dx \leq 1.2$. (Hint: use (a).)

**Solution.** By properties of integrals, using (a),

$$1 = \int_0^1 1 \, dx \leq \int_0^1 \sqrt{1 + x^4} \, dx \leq \int_0^1 (1 + x^4) \, dx.$$ 

The last integral evaluates to

$$\int_0^1 (1 + x^4) \, dx = (1 - 0) + \left( \frac{1^5}{5} - \frac{0^5}{5} \right) = 1 + \frac{1}{5} = 1.2.$$ 

So we get the desired bounds.

(4) Find the derivative $f'(x)$.

(a) 

$$f(x) = \int_1^x \sin^3(\theta) \cos^4(\theta) \, d\theta$$

**Solution.** Using the fundamental theorem of calculus,

$$f'(x) = \frac{d}{dx} \int_1^x \sin^3(\theta) \cos^4(\theta) \, d\theta = \sin^3(x) \cos^4(x).$$

(b) 

$$f(x) = \int_0^{x^2+3} (u - 1)^{u-1} \, du.$$ 

**Solution.** Use the fundamental theorem of calculus, but keep the chain rule in mind:

$$f'(x) = (x^2 + 2)^{x^2+2} \cdot \left( \frac{d}{dx} (x^2 + 3) \right) = 2x(x^2 + 2)^{x^2+2}.$$ 

(If it helps conceptually, write $f(x) = A(x^2 + 3)$, so that $f'(x) = A'(x^2 + 3) \cdot 2x$.)

(5) Annoyed by your calculus homework, you crumple it into a ball and launch it into an infinitely deep hole using the Spring Launcher Technology\textsuperscript{TM} from Homework 5.
Your new and improved measurements show that at time \( t \) (in milliseconds), the end of the spring is at depth (in centimeters)
\[ x(t) = -5 - \int_0^t \frac{10 \sin x}{x} \, dx. \]
(The integral is a special function called the sine integral. It is important in electrical engineering.)

(a) There are infinitely many times \( t \) where the spring will be fully extended (and about to retract back). Find all such \( t \).

**Solution.** Such times \( t \) are local minimums of \( x(t) \). So we must first find critical points \( x'(t) = 0 \). Compute
\[ x'(t) = -\frac{d}{dt} \int_0^t \frac{10 \sin x}{x} \, dx = -\frac{10 \sin t}{t}. \]
This is zero exactly when the numerator is zero, i.e. \( \sin t = 0 \). So \( t \) can be any positive multiple of \( \pi \), i.e. \( n\pi \) for any positive integer \( n \). (We never consider negative \( t \).) To check which are local maxs vs local mins, use the second derivative test. Compute
\[ x''(t) = -10 \cdot \frac{\cos(t) - \sin(t)}{t^2}. \]
Note that \( \sin(n\pi) = 0 \) for any integer \( n \), but
(i) if \( n \) is odd, then \( \cos(n\pi) = -1 \);
(ii) if \( n \) is even, then \( \cos(n\pi) = 1 \).
So \( x''(n\pi) \) is positive only when \( n \) is odd. Hence the local minimums are where \( t \) is an odd positive multiple of \( 2\pi \).

(b) When is the first time \( t \) that the end of the spring changes from accelerating downward (i.e. extending) to accelerating upward (i.e. retracting)? You do not need to find an exact value for \( t \); just give an equation that \( t \) must satisfy. For example: “\( t \) is the only solution to \( e^{-t} = \sin(t) \) in the interval \((3, 4)\)”. (Hint: look back at Homework 5.)

**Solution.** Such times \( t \) are inflection points, i.e. \( x''(t) = 0 \). From the formula for \( x''(t) \) above, this means we want to solve
\[ \cos(t) - \sin(t) = 0. \]
The first inflection point must be in between the first and second critical points, at \( t = \pi \) and \( t = 2\pi \). This is because the spring starts off retracting, but at some point must start expanding again to “turn around”. You can check this mathematically using the intermediate value theorem:
\[ \cos(\pi)\pi - \sin(\pi) = -\pi, \quad \cos(2\pi)(2\pi) - \sin(2\pi) = 2\pi, \]
so somewhere in between we must have \( t \) such that \( \cos(t) - \sin(t) = 0 \). Hence \( t \) is the only solution to \( \cos(t) - \sin(t) = 0 \) in \((\pi, 2\pi)\).