Equivariant K-theory: Lecture 1

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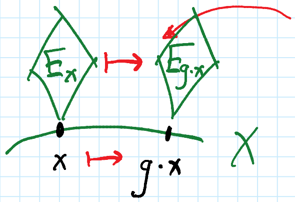
(All schemes / varieties / stacks are over $\mathbb{C}$, everything algebraic.)

Equivariant sheaves: setting: $\underset{\text{group}}{G} \curvearrowright \underset{\text{scheme}}{X}$

giving a map act: $G \times X \to X$

e.g. a G-equivariant vector bundle E:



some linear map

$\Phi: G \times E \to E$

(lifting act. fiberwise.

e.g. $GL(V) \curvearrowright \mathbb{P}(V)$, $\mathcal{O}_{\mathbb{P}(V)}(-1)$ has a $GL(V)$-equivariant structure:

$$g \cdot \underbrace{(x, \overset{V}{\overset{\in}{v}})}_{\in\, \mathcal{O}(-1)} = (g\cdot \underset{\mathbb{P}(V)}{\underset{\cap}{x}}, g\cdot v)$$

$\Rightarrow$ its tensors & duals (i.e. all $\mathcal{O}(n)$) do too.

Def: A G-equivariant structure on $\mathcal{F}$ ($\mathcal{O}_X$-mod.) is an iso

$$\begin{tikzcd} G \times X \arrow[r, shift left, "\mathrm{act}"] \arrow[r, shift right, "p"'] & X \end{tikzcd}$$

$(g,x) \in$ projection

$$\mathrm{act}^*\mathcal{F} \simeq p^*\mathcal{F}$$

(e.g. $\mathcal{F}_{g\cdot x} \simeq \mathcal{F}_x$ is stalk @ $(g,x) \in G \times X$)

compatible with group structure on G. (Exercise: look up this.)

E.g. $X = \mathrm{pt}$. sheaves $\cong$ vector spaces V

$\circlearrowright$ G $\quad$ $\circlearrowright$ G $\Big\}$ a representation of G.

If $\dim V = 1$, $g \cdot v = \underbrace{\rho(g)}\, v$

$\rho: G \to \mathbb{C}^\times$ is a character

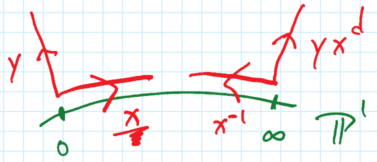
$\Rightarrow$ we say V has weight ρ.

write this as $\mathbb{C}_\rho$, or just ρ.

E.g. on $X = \mathbb{P}^1$, line bundles are

$\mathcal{L} = \mathcal{O}(d)$ $\quad y\lambda \quad \lambda y x^d$

$$\mathcal{L} = \mathcal{O}(d)$$



Let $\mathbb{C}^* \curvearrowright \mathbb{P}^1$ with $t \cdot [x:y] = [\rho(t)x : y]$

If $\mathcal{L}$ is $\mathbb{C}^*$-equivariant,

$$\underset{z}{\underline{\rho^{-d}}} \cdot \mathrm{wt}\ \underset{z}{\underline{\mathcal{L}|_0}} = \mathrm{wt}\ \underset{z}{\underline{\mathcal{L}|_\infty}}$$

$x \in$ function on $\mathbb{P}^1$

G acts dually on functions

i.e. $(g \cdot f)(z) = f(g^{-1}z)$.

maybe $t \cdot y = \underset{z}{\sigma(t)} \cdot y$

$\mathbb{C}^*$-representations

Note: may exist many equivariant structures on a non-equivariant sheaf
↑ or none. (Exercise: find one.)

Equivariant K-theory:

Def: $K_G(X) =$ Grothendieck group of $\mathrm{Coh}_G(X)$ (G-equivariant coherent sheaves)

generators: $\mathcal{F} \in \mathrm{Coh}_G(X)$

relations: $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$

$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$

everything in sight (esp. morphisms) must be G-equivariant.

(= Grothendieck group of $D^b\mathrm{Coh}_G(X)$)

Many many flavors of K-th:

$K_G^{\mathrm{Vect}}(X) =$ Grothendieck group of $\mathrm{Vect}_G(X)$ (G-eq. vect. bunds.) $\subset K_G(X)$

$K_G^{\mathrm{Top}}(X) =$ Grothendieck group of topological vector bundles

etc.

e.g. if X is smooth, (maybe quasi-proj.) $K_G(X) = K_G^{\mathrm{Vect}}(X)$

every coherent sheaf has finite resolution by vector bundles.
G- G- G-

(maybe quasi-proj.)

every G coherent sheaf has finite G resolution by G vector bundles.

e.g. $K(\mathbb{P}^1) = K^{\mathrm{Vect}}(\mathbb{P}^1) = \mathbb{Z}[\mathcal{L}^{\pm}] \Big/ \text{relations}$ $\quad \mathcal{L} = \mathcal{O}(1)$

exercise: find one

Simplest equivariant example: $X = pt.$

representation ring of G.

$$K_G(pt) = R(G)$$

e.g. $K_{\mathbb{C}^*}(pt) = \mathbb{Z}[t^{\pm}]$

1-dim rep of weight 1.

$$K_T(pt) = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}] \quad (\mathbb{C}^* \xrightarrow{id} \mathbb{C}^*).$$

n-dim torus $(\mathbb{C}^*)^n$

$$K_{GL(n)}(pt) = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]^{S_n}$$

← sym. polys. in $K_T(pt)$.

Functoriality: for $f: X \to Y$, to preserve exactness, must have

$$f_*[\mathcal{F}] \neq [f_*\mathcal{F}]$$
$$= \sum_i (-1)^i [R^i f_* \mathcal{F}]$$
$$f^*[\mathcal{G}] \neq [f^*\mathcal{G}]$$
$$= \sum_i (-1)^i [L^i f^* \mathcal{G}]$$

well-defined only if sums are finite.

Pushforward: $R^{>\dim Y} f_* = 0$ (Grothendieck vanishing)

BUT $R^i f_*$ may not preserve coherence.

e.g. $f: \mathbb{A}^1_{\mathbb{C}} \to pt \Rightarrow f_*\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}} = \mathbb{C}[x]$ is not a fin. dim $\mathbb{C}$-vector sp.

Require: f is proper. (Exercise: check.)

(can be relaxed, later.)

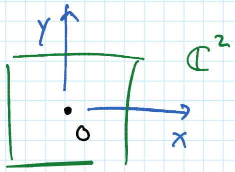
To compute, factor $f: X \underset{\text{graph}}{\hookrightarrow} X \times Y \underset{\text{projection}}{\twoheadrightarrow} Y$

e.g. ($\underset{\text{embedding}}{\overset{\text{closed}}{\hookrightarrow}}$) $\quad 0 \to \mathcal{O}_{\mathbb{C}^2} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}_{\mathbb{C}^2}^{\oplus 2} \xrightarrow{(x \ \ y)} \mathcal{O}_{\mathbb{C}^2} \to \iota_* \mathcal{O}_0 \to 0$

wt t_1^{-1}

wt t_2^{-1}

embedding $\{0\} \hookrightarrow \mathbb{C}^2$



For $T=(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^2$: this is T-equivariant,

scaling with weights t_1, t_2

$$\iota_*[\mathcal{O}_0] = 1 - (t_1 + t_2) + t_1 t_2$$
$$= (1-t_1)(1-t_2) \quad \in K_T(\mathbb{C}^2)$$

More generally, $\iota : Z \hookrightarrow X$ has a Koszul resolution.

(regular embeddings)