Equivariant K-theory: Lecture 12

June-02-22
4:48 PM

Recall: want some bigraded S_n-module D_λ s.t. $F_{q,t}(D_\lambda) = \tilde{H}_\lambda$ ← Haiman-normalized Macdonald polynomials.

Use: $K_T(\mathrm{Hilb}(\mathbb{C}^2)) \otimes \mathbb{Q}(q,t) \xrightarrow{\sim} \Lambda_{q,t}$ ← (q,t) sym. fncs.

Def: G-Hilbert scheme: $\mathrm{Hilb}_G(X)$ (G finite group) of "scheme-theoretic G-orbits" in X.

0-dim G-invariant $Z \subset X$ s.t. $H^0(Z, \mathcal{O}_Z) \simeq \mathbb{C}[G]$ is the regular rep. — length$(Z) = |G|$

∃ Hilbert-Chow morphism: $\mathrm{Hilb}_G(X) \xrightarrow[\pi]{} X/G$.

Thm: Let $Y = \mathrm{Hilb}_{S_n}((\mathbb{C}^2)^n)$ ← $S_n \curvearrowright$ by permutation

1. [Haiman] ∃ iso.

$$\mathrm{Hilb}_n(\mathbb{C}^2) \xrightarrow{\sim} Y$$
$$I \longmapsto S_n\text{-orbit of } \mathrm{supp}(\mathcal{O}_{\mathbb{C}^2}/I)$$

2. [Bridgeland-King-Reid]

$$D^b\mathrm{Coh}_T(Y) \xrightarrow{\sim} D^b\mathrm{Coh}_T([(\mathbb{C}^2)^n/S_n])$$

↑ Fourier-Mukai transform with kernel

"universal incidence correspondence".

$$Z_n := \{(I, p_1, p_2, \dots, p_n) : \pi(I) = \{p_1, p_2, \dots, p_n\}\} \subset Y \times (\mathbb{C}^2)^n$$

Idea of 1. (& Haiman's main contribution):

pretty hard. { The "isospectral Hilbert scheme"

$$X_n := \{(I, p_1, \dots, p_n) : \pi_{HC}(I) = \{p_1, p_2, \dots, p_n\}\} \subset \mathrm{Hilb}_n(\mathbb{C}^2) \times (\mathbb{C}^2)^n$$

← reduced induced subsch. structure.

is Cohen-Macaulay.

In fact this is equivalent to ①, as follows:

∃ a morphism $Y \xrightarrow{\phi} \mathrm{Hilb}_n(\mathbb{C}^2)$ given by Z_n / S_{n-1} ← $S_{\{1,2,\dots,n-1\}} \subset S_n$

↑ rank n bundle of $\mathbb{C}[x,y]$-algebras

which is iso generically. (exercise)

$(\Rightarrow)$ Suppose X_n is CM. Miracle flatness says:

$$\underset{\text{CM}}{X_n} \xrightarrow[\text{finite}]{\text{projection.}} \underset{\text{smooth}}{Hilb_n(\mathbb{C}^2)}$$

$\uparrow$ deg $n!$

is flat. $\Rightarrow$ X_n induces a morphism $Hilb_n(\mathbb{C}^2) \xrightarrow{\psi} Y$ generically inverse to ϕ.

$\Rightarrow$ inverse everywhere.

$(\Leftarrow)$ If ϕ is iso, its inverse gives an incidence correspondence

$$\begin{tikzcd} & X_n' \arrow[dl, "\text{flat}"'] \arrow[dr] & \\ Hilb_n(\mathbb{C}^2) & & Y \end{tikzcd}$$

with $X_n \simeq X_n'$ generically. X_n reduced $\Rightarrow$ X_n' reduced

$\Rightarrow$ $X_n \simeq X_n'$ everywhere.

$\Rightarrow$ X_n is also flat over $Hilb_n(\mathbb{C}^2)$. $\square$.

So

$$\begin{tikzcd} X_n \arrow[r, "\sim"] \arrow[d, "\rho"'] & Z_n \arrow[d] \\ Hilb_n(\mathbb{C}^2) \arrow[r, "\overset{\phi}{\sim}"', "\psi"] & Hilb_{S_n}((\mathbb{C}^2)^n) \end{tikzcd}$$

and the iso. $K_T(Hilb(\mathbb{C}^2))_{loc} \xrightarrow{\sim} K_{T\times S_n}(*)_{loc}$ is:

$$\mathcal{E} \longmapsto \chi(X_n, \rho^* \mathcal{E})$$

no S_n-action.

$$\overset{\text{projection formula.}}{\|}$$

$$\chi(Hilb_n(\mathbb{C}^2), \underbrace{\rho_* \mathcal{O}_{X_n}}_{\substack{\text{carries an } S_n\text{-action.} \\ \text{(regular rep.)}}} \otimes \mathcal{E})$$

$\mathcal{P} := \rho_* \mathcal{O}_{X_n}$ "Procesi bundle".

is a rank-$n!$ bundle on $Hilb_n(\mathbb{C}^2)$

(tautological bundle of $Hilb_{S_n}((\mathbb{C}^2)^n)$.)

e.g. T-fixed points in $Hilb(\mathbb{C}^2)$:

$$\mathcal{O}_{Z_\lambda} \longmapsto \chi(Hilb_n(\mathbb{C}^2), \mathcal{P} \otimes \mathcal{O}_{Z_\lambda}) = \mathcal{P}|_{Z_\lambda}$$

fiber at Z_λ.

Thm: [Haiman] $\quad F_{q,t}\left(\mathcal{P}|_{z_\lambda}\right) = \tilde{H}_\lambda$

previously known as "Garsia–Haiman $n!$ conjecture" because it required the flatness of $\rho: X_n \to \mathrm{Hilb}_n(\mathbb{C}^2)$.

Pf: check the defining properties of $\tilde{H}_\lambda$.

(normalization) $\quad \langle s_{(n)}, F_{q,t}(\mathcal{P}|_{z_\lambda}) \rangle_{q,t} = 1 \qquad |\lambda| = n.$

Use that $\mathcal{P}|_{z_\lambda}$ is regular rep of $S_n = \bigoplus_{\text{irreps } V} V^{\oplus \dim V}$

In particular contains $\underset{\text{trivial rep.}}{s_{(n)} = F(1_n)}$ exactly once.

(triangularity) $\quad F_{q,t}(\mathcal{P}|_{z_\lambda})[X(1-q)] \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda\}$

Geometric meaning of plethystic substitution:

coordinates $\{x_i, y_i\}_{i=1}^n$ x_i: wt t, y_i: wt q.

Let $L = \{y_1 = y_2 = \cdots = y_n = 0\} \subset (\mathbb{C}^2)^n$

Then $\mathcal{O}_L = \prod_{i=1}^n \dfrac{1}{1-t} \in K_{T \times S_n}(*)$

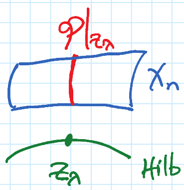
and $\mathrm{tr}_{\mathcal{O}_L} \sigma_\mu = \prod_i \dfrac{1}{\underbrace{1-t^{\mu_i}}}$

permutation conjugacy class μ cancel

$$\Rightarrow \underset{S_n\text{-module}}{F_{q,t}(W \otimes \mathcal{O}_0)} = F_{q,t}(W \otimes \mathcal{O}_L)\Big|_{p_k \mapsto \underbrace{(1-t^k)}\, p_k}$$

$$\overset{\text{by def.}}{=} F_{q,t}(W \otimes \mathcal{O}_L)[X(1-t)]$$

In reverse,



$$\underset{2\text{-variables}}{F_{q,t}(\mathcal{P}|_{z_\lambda})[X(1-q)]} = F_{q,t}\left(\mathcal{P}|_{z_\lambda} \big/ (\vec{y})\right)$$

1-variable !

Better:

$$F_{q,t}(\mathcal{P}|_{z_\lambda}) \cdot \underbrace{F_{q,t}(\mathcal{O}_{\mathrm{Hilb}(\mathbb{C}^2), z_\lambda})}_{\substack{\text{trivial } S_n\text{-module} \\ \text{(some rat. func. in } q,t)}} = F_{q,t}\Big(\underbrace{\mathcal{O}_{X_n, (z_\lambda, 0, 0, \ldots, 0)}}_{\text{denote this } S_\lambda}\Big)$$

and $S_\lambda/(\vec{y})$ is a previously understood object in Springer theory [Garsia-Procesi]

In particular,

$$F_{gt}\left(S_\lambda/(\vec{y})\right) = \text{Hall-Littlewood polys.}$$

$\lim_{q \to 0} \tilde{H}_\lambda$.

were known to have desired triangularity. □.

Note: X_n around $(Z_\lambda, 0, 0, \ldots, 0)$ has explicit presentation as

$$\mathbb{C}[\vec{x}, \vec{y}]/J_\lambda \leftarrow \{s : s(\partial_x, \partial_y)\Delta_\lambda = 0\}$$

which relates it (and $P|_{Z_\lambda}$) to rings of coinvariants.