Equivariant K-theory: Lecture 13

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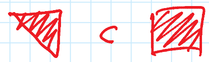
Springer theory + more.

G : complex semisimple Lie group. (connected.)

fix one. → B : Borel subgroup   e.g. 

$\mathcal{B}$: $\{$ all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g} \} \simeq G/B$ (all conjugate under G.)

$\mathcal{N}$: $\{ x \in \mathfrak{g} : \mathrm{ad}_x$ is nilpotent $\}$ "nilpotent cone"

$\tilde{\mathcal{N}}$: $\{ (x, \mathfrak{b}) : x \in \mathfrak{b} \} \subset \mathcal{N} \times \mathcal{B}$

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ ($\mathfrak{h}$: Cartan; $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ nilpotent.)

$$\begin{tikzcd} & \tilde{\mathcal{N}} \cong T^*G/B \arrow[dl, "\text{surjective}"'] \arrow[dr] & \\ \mathcal{N} & & \mathcal{B} = G/B \end{tikzcd}$$

on the locus of regular nilpotent elements. generically, 1-to-1 (exercise.)

fiber at $(x, \mathfrak{b}) = \mathfrak{n} \simeq \mathfrak{n}_-^* = T_{\mathfrak{b}}^* G/B$ (use pairing $\mathfrak{g} \simeq \mathfrak{g}^*$; $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$)

$\Rightarrow$ $\tilde{\mathcal{N}} \simeq T^*G/B$

is a resolution of singularities of $\mathcal{N}$.

"Springer resolution".

! $\begin{matrix} T^*G/B \\ \downarrow \\ \mathcal{N} \end{matrix}$ shares many properties with $\begin{matrix} \mathrm{Hilb}(\mathbb{C}^2) \\ \downarrow \\ \mathrm{Sym}(\mathbb{C}^2) \end{matrix}$: notably, both are equivariant symplectic resolutions. [Kaledin]

A central theme in geometric rep. theory :

some cohom theory. → $H^*\left(\text{interesting correspondence } \begin{matrix} Z \\ \cap \\ X \times X \end{matrix} \right) \circlearrowright H^*(X).$ push-pull.

LHS can be made into an algebra via convolution :   ← assume X is smooth

$$\underbrace{X \times \overbrace{X}}_{Z_{12}} \underbrace{\times X}_{Z_{23}}$$

$$H^*(Z_{12}) \otimes H^*(Z_{23}) \longrightarrow H^*\left(\mathrm{im}\left(\pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(Z_{23}) \xrightarrow{\pi_{13}} X \times X \right)\right)$$

$$(\alpha_{12}, \alpha_{23}) \longmapsto \pi_{13*}(\pi_{12}^* \alpha_{12} \cdot \pi_{23}^* \alpha_{23})$$

projections from $X \times X \times X$

e.g. in K-th., need π_{12}, π_{23} flat for this to make sense.
π_{13} proper

If $\mu : X \to Y$ is a resolution of singularities, set.

Steinberg variety.

$$Z := X \times_Y X = \{(x, x') : \mu(x) = \mu(x')\} \subset X \times X$$

$\Rightarrow$ convolution produces $\star : H^*(Z) \otimes H^*(Z) \to H^*(Z)$
(exercise.) is an associative product on $H^*(Z)$.

Note: many other interesting choices for Z:

$Z_\Delta \underset{\text{diagonal}}{\subset} X \times X \Rightarrow$ convolution = usual intersection product on $H^*(X)$.

e.g. $\otimes$ in $K(X)$.

Returning to Springer theory: let $Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$

(like ordinary homology, but with possibly-infinite cycles.

In Borel-Moore homology:

Thm [Springer, Kazhdan-Lusztig]:

Weyl group of G.

$$H^{BM}_{top}(Z) \simeq \mathbb{Z}[W]$$

$$\circlearrowright$$

$$H^{BM}_{top}(\mathcal{B}_x)$$

stratified by Jordan type of ad_x $\equiv$ partition μ.

$\mu^{-1}(x)$, $x \in \mathcal{N}$

$H^{BM}_{top}(\mathcal{B}_x) \simeq H^{BM}_{top}(\mathcal{B}_y)$ iff x is G-conjugate to y.

(For $G = SL_n$) $\{H^{BM}_{top}(\mathcal{B}_x)\}_x$ is a complete collection of simple W-modules.

What about $K_G(Z)$??

$G \curvearrowright G/B$, T^*G/B, Z

$$Z = \{(x,\mathfrak{b}),(x',\mathfrak{b}') : x = x'\} = \{(x,\mathfrak{b},\mathfrak{b}') : x \in \mathfrak{b}\cap\mathfrak{b}'\}$$

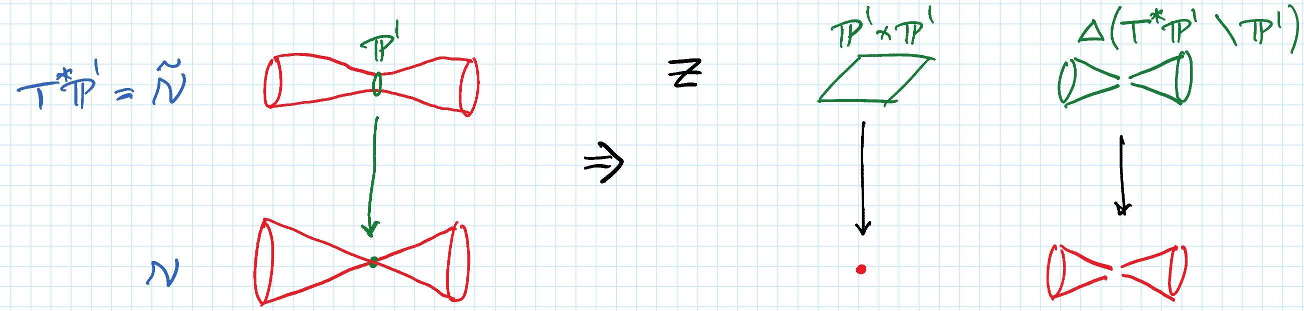
$$\begin{array}{ccc} \cap & & \cap \\ \tilde{N}\times\tilde{N} & \xrightarrow{\sim} & T^*(\mathcal{B}\times\mathcal{B}) \end{array}$$

$$(x_1,\mathfrak{b}_1,x_2,\mathfrak{b}_2) \longmapsto (x_1,\mathfrak{b}_1,-x_2,\mathfrak{b}_2)$$

(to preserve symplectic form.)

eg. $G = SL(2)$. $\quad N = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : -\det = a^2+bc = 0 \right\}$ ← quadric cone in $\mathbb{C}^3$.

$G \curvearrowright \mathbb{P}^1$ with stabilizer $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \Rightarrow \mathbb{P}^1 \simeq G/B$



$$Z = (\mathbb{P}^1\times\mathbb{P}^1) \cup \Delta(T^*\mathbb{P}^1)$$

Even better: $\quad Z = \bigsqcup_{\substack{G\text{-orbits} \\ \text{in} \\ \mathcal{B}\times\mathcal{B}}} \left(\begin{array}{c}\text{conormal bundles} \\ \text{to } G\text{-orbits}\end{array} \right) \subset T^*\mathcal{B}\times T^*\mathcal{B}$

Pf: $T_{(\mathfrak{b}_1,\mathfrak{b}_2)}(G\text{-orbit}) = (u\cdot\mathfrak{b}_1, u\cdot\mathfrak{b}_2) \qquad u \in \mathfrak{g}$

$(x_1,\mathfrak{b}_1,x_2,\mathfrak{b}_2)$ in conormal $\iff \langle u,x_1\rangle + \langle u,x_2\rangle = 0$ ← pairing $T \otimes T$

$\iff x_1 = -x_2.$ ← exactly condition of $Z \subset T^*(\mathcal{B}\times\mathcal{B})$. □.

$$\{G\text{-orbits on } \mathcal{B}\times\mathcal{B}\} \underset{\text{(exercise)}}{\simeq} \{B\text{-orbits on } \mathcal{B}\} \cong W$$

Bialynicki-Birula decomposition.

$$\Rightarrow \quad K_G(Z) \underset{\text{affine bundle thm}}{\cong} \bigoplus_{w\in W} K_G(\mathcal{B})$$

maximal torus.

Compute: $K_G(G/B) \simeq K_B(\{e\}) = R(B) \simeq R(T)$

unipotent groups act trivially on any simple rep.

$\Rightarrow$ as $K_G(pt)$-modules, $\qquad \mathbb{Z}[P]$ where $P = \mathrm{Hom}(T,\mathbb{C}^*)$ is weight lattice

$$K_G(Z) = \bigoplus R(T) = R(T)[W]$$

weight lattice

$$K_G(Z) = \bigoplus_{w \in W} R(T) = R(T)[W]$$

$$= \mathbb{Z}[W \times P]$$

productive in light of the Springer thm relating $H^{BM}_{top}(Z) \simeq \mathbb{Z}[W]$ and Chern iso $K(-) \otimes \mathbb{Q} \simeq H^*(-, \mathbb{Q})$

This is the wrong algebra structure.

Correct alg. structure comes from affine Lie algebras

actually $K_G(Z) \simeq \mathbb{Z}[W \ltimes P]$