Last time:

$$\underbrace{K_G(T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B})}_{\text{Steinberg variety } Z} \circlearrowright K_G(T^*\mathcal{B})$$

G/B flag variety

a convolution algebra

nilpotent cone

push-pull.

We computed $K_G(Z) \simeq \mathbb{Z}[W \ltimes P]$ as modules for $K_G(pt)$.

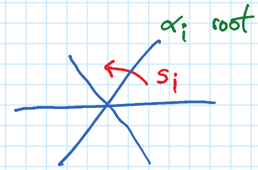
not as algebras.

Weyl grp. weight lattice $\mathrm{Hom}(T, \mathbb{C}^\times)$

maximal torus.

Correct alg. structure on $K_G(Z)$ comes from affine Lie algs:

Def: Ordinary Weyl group: W = group of reflections

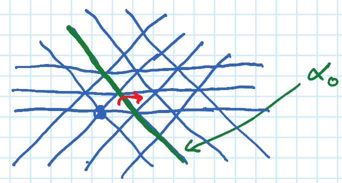


Affine Weyl group W^a = " ______

+ additional affine reflection.

idea: $s_0 s_i$ = translation.

Q = group of translations.



$= W \ltimes Q$

root lattice

$\langle \theta^\vee, - \rangle + 1 = 0$

maximal coroot

Extended affine Weyl group $W^{ae} = W \ltimes P$

$= W^a \rtimes P/Q$

minuscule weights π (with $\langle \pi, \alpha^\vee \rangle = 1$)

Thm: $K_G(Z) = \mathbb{Z}[W^{ae}]$

as algebras.

In fact, can do better:

$$\begin{array}{ccc} T^*G/B = \tilde{\mathcal{N}} & & \mathrm{Hilb}(\mathbb{C}^2) \\ \downarrow & & \downarrow \\ \mathcal{N} & & \mathrm{Sym}(\mathbb{C}^2) \end{array}$$

are conical resolutions, i.e.

$\exists\ \mathbb{C}^\times$ automorphism contracting base to pt.

call weight q.

$\mathbb{C}^\times_q$ = scaling of cotangent direction

$\mathbb{C}^\times_q$ = scaling of $\mathbb{C}^2$ with $q = t_1 t_2$.

in T^*G/B

Thm: $K_{G\times\mathbb{C}^*_q}(Z) = \mathbb{H}^a_q$ is the affine Hecke algebra.

q-deformation of $\mathbb{Z}[W^{ae}]$.

Historically, interest in $\mathbb{H}^a_q$ comes from Langlands program:

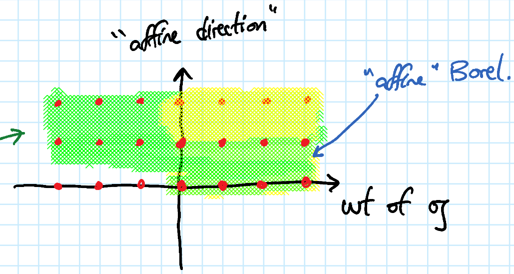
$$\mathbb{C}\left[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)\right] \simeq \mathbb{H}_q$$

finite Hecke alg.
q-deformation of $\mathbb{Z}[W]$

$$\mathbb{C}\left[I\backslash G(\mathbb{Q}_p)/I\right] \simeq \mathbb{H}^a_q$$

Iwahori subgroup

p-adics.



convolution algebra for $\mathrm{Ind}_I^{G(\mathbb{Q}_p)}\mathbb{C}$, which controls an equivalence of categories

$$\left\{\begin{matrix}\text{admissible } G(\mathbb{Q}_p)\text{-modules}\\ \text{generated by } I\text{-fixed vectors}\end{matrix}\right\} \xrightarrow{\sim} \left\{\text{fin. dim. } \mathbb{H}^a_q\text{-modules}\right\}$$

$$V \longmapsto V^I$$

Def: $\mathbb{H}^a_q$ is the $\mathbb{Z}[q^{\pm}]$-alg. with generators

$$e^\lambda T_w \qquad \begin{matrix}\lambda\in P\\ w\in W\end{matrix}$$

and relations:

1. $\{T_w\}_{w\in W}$ generate a finite Hecke algebra

$$\mathbb{Z}[W] \rightsquigarrow \mathbb{H}_q$$
$$T_i^2 = id \qquad (T_i+1)(T_i-q)=0$$

2. $\{e^\lambda\}_{\lambda\in P}$ generate commutative subalg $\simeq \mathbb{Z}[q^{\pm}][P]$

3. $$q\left\{\begin{matrix} T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q e^\lambda & \langle\alpha^\vee,\lambda\rangle = 1 \\ T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha} & \langle\alpha^\vee,\lambda\rangle = 0\end{matrix}\right.$$

3. $\begin{cases} T_{s_\alpha} e^{\lambda} T_{s_\alpha} = q e^{s_\alpha(\lambda)} & \langle \alpha^\vee, \lambda \rangle = 1 \\ T_{s_\alpha} e^{\lambda} = e^{\lambda} T_{s_\alpha} & \langle \alpha^\vee, \lambda \rangle = 0 \end{cases}$

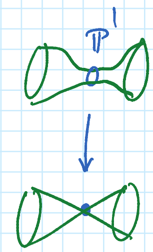
$$T^2 + (1-q)T - q = 0 \quad \Rightarrow \quad T^{-1} = q^{-1} T + (q^{-1} - 1)$$

$$\Rightarrow \quad T_s e^{s(\lambda)} = q e^{\lambda} T_s^{-1} = e^{\lambda} T_s + (1-q) e^{\lambda}$$

<u>Ex</u>: of Thm when $G = SL_2$.

$$\mathbb{H}_q^a = \mathbb{Z}[q^{\pm}] \langle T, X, X^{-1} \rangle \Big/ \left\langle \begin{matrix} (T+1)(T-q) = 0 \\ TX^{-1} - XT = (1-q) X \end{matrix} \right\rangle$$

$$Z = T^*\mathbb{P}^1 \times_{\mathcal{N}} T^*\mathbb{P}^1 = \Delta(T^*\mathbb{P}^1) \bigcup_{\Delta(\mathbb{P}^1)} (\mathbb{P}^1 \times \mathbb{P}^1)$$



Let $\mathcal{O}_n := \left(\Delta(T^*\mathbb{P}^1) \xrightarrow{\pi} \Delta(\mathbb{P}^1) \right)^* \mathcal{O}(n)$

$$\mathcal{Q} := \Omega^1_{\mathbb{P}^1 \times \mathbb{P}^1 / \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega^1_{\mathbb{P}^1}$$

1st factor.

The iso. $K_{G \times \mathbb{C}_q^\times}(Z) \simeq \mathbb{H}_q^a$ is given by

$$-(1+T) \longleftrightarrow q \mathcal{Q}$$
$$X \longleftrightarrow \mathcal{O}_1$$

Check well-defined, i.e. check relations:

of sheaves on $T^*\mathbb{P}^1$

1. $$\mathcal{Q} * \mathcal{Q} = \pi_{13*}\left(\mathcal{O}_{\mathbb{P}^1} \boxtimes (\Omega^1_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}) \boxtimes \Omega^1_{\mathbb{P}^1} \right)$$

$$= \mathcal{Q} \cdot \chi\left(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes (\mathcal{O}_{\mathbb{P}^1} - q^{-1} T_{\mathbb{P}^1}) \right)$$

restriction of Koszul resolution of $\mathcal{O}_{\mathbb{P}^1}$ on $T^*\mathbb{P}^1$ to $\mathbb{P}^1 \subset$

$$= \mathcal{Q}(-1 - q^{-1})$$

$$(T+1)(T-q) = (T+1)^2 - (T+1)(1+q)$$

2. $$Z \xhookrightarrow[\pi]{} \mathbb{P}^1 \times T^*\mathbb{P}^1 \xhookleftarrow[i]{} \mathbb{P}^1 \times \mathbb{P}^1$$

2. $Z \xhookrightarrow[\pi]{} \mathbb{P}^1 \times T^*\mathbb{P}^1 \xhookleftarrow[i]{} \mathbb{P}^1 \times \mathbb{P}^1$

$\left(\begin{array}{c} T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \\ \downarrow \\ \mathbb{P}^1 \times T^*\mathbb{P}^1 \end{array} \right) \Big|_Z$ (id, zero section).

Fact: $K_{G \times \mathbb{C}^\times_q}(Z) \xhookrightarrow[i^* \pi_*]{} K_{G \times \mathbb{C}^\times_q}(\mathbb{P}^1 \times \mathbb{P}^1)$ is injective.

(Remaining computation: exercise.)

Proof idea for general G: $K_{G \times \mathbb{C}^\times_q}(Z) \hookrightarrow \mathrm{End}\, \underbrace{K_{G \times \mathbb{C}^\times}(T^*\mathcal{B})}_{\text{i.e. faithful representation.}}$ is injective.

But $\mathcal{H}^a_q$ is known to have a "polynomial rep.":

$$\mathcal{H}^a_q \curvearrowright R(T)[q^{\pm}] = \mathrm{Ind}_{\mathcal{H}}^{\mathcal{H}^a_q} \mathbb{Z}[q^{\pm}]$$

Demazure–Lusztig operators (starting point of K-th study of $\mathcal{H}^a_q$)

$$T_s \cdot 1 = q^{\ell(s)}$$

$$T_{s_\alpha} \cdot e^\lambda := \frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1} - q\,\frac{e^\lambda - e^{s_\alpha(\lambda)+\alpha}}{e^\alpha - 1}$$

faithful rep. of $\mathcal{H}^a_q$ (since faithful at $q=1$.)

(exercise.) (exercise)

Two faithful reps $\begin{array}{c} K_{G \times \mathbb{C}^\times_q}(Z) \curvearrowright \\ \mathcal{H}^a_q \curvearrowright \end{array} K_{G \times \mathbb{C}^\times_q}(T^*\mathcal{B})$

really are some localization on $\mathbb{P}^1$!

Suffices to check by manual computation that actions of generators agree. e.g.