Equivariant K-theory: Lecture 15

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Last week: a key theme in geometric rep. theory:

$$K_G(Z) \circlearrowright K_G(X)$$

- $K_G(Z)$: an interesting algebra, e.g. affine Hecke alg.
- some $Z \subset X \times X$ forming a convolution alg.
- $K_G(X)$: an interesting rep., e.g. polynomial rep. of affine Hecke alg.

This week: a similar idea geared towards producing quantum groups.
(but somewhat distinct.)

Crash course on quantum groups:

$$\mathcal{U}\mathfrak{g} \rightsquigarrow \mathcal{U}_q\mathfrak{g}$$

- $\mathfrak{g}$: Lie alg. (classical)
- $\mathcal{U}$: universal enveloping (i.e. where $a*b-b*a=[a,b]$.)
- $\rightsquigarrow$: q-deformation
- $\mathcal{U}_q\mathfrak{g}$: Drinfeld-Jimbo-type quantum groups. (not actually groups!...)

e.g. $[h_i, e_j] = \alpha_{ij} e_j \rightsquigarrow q^{h_i} e_j q^{-h_i} = q^{\alpha_{ij}}$ (made "group-like")

Prop: $\mathcal{U}_q\mathfrak{g} \simeq \mathcal{U}\mathfrak{g}$ as algebras over $\mathbb{C}[q^{\pm}]$. (exercise.)

$$\Delta(x) = x\otimes 1 + 1\otimes x \rightsquigarrow$$

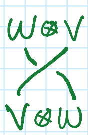
$$\Delta(e_i) = e_i \otimes q^{h_i} + 1\otimes e_i$$
$$\Delta(f_i) = f_i \otimes 1 + q^{-h_i}\otimes f_i$$
$$\Delta(q^{h_i}) = q^{h_i}\otimes q^{h_i}$$

- Δ: coproduct (algebra hom.), $x \in \mathfrak{g}$
- $x\otimes 1 + 1\otimes x$: symmetric in the two factors, i.e. "cocommutative"
- the new Δ: no longer cocommutative

Prop: $\mathcal{U}_q\mathfrak{g} \not\simeq \mathcal{U}\mathfrak{g}$ as Hopf algebras. (Hopf algebra: an alg. with compatible coproduct + "antipode" (a "dualizing" operation))

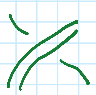
Pf: $\mathcal{U}_q\mathfrak{g}$ not cocomm., $\mathcal{U}\mathfrak{g}$ cocomm. □

Def: A Hopf alg. H is a quantum group ("quasi-triangular") if $\exists\, R \in H\otimes H$ s.t.



$$R\,\Delta(x) = \Delta^{op}(x)\,R \qquad \forall x\in H$$

($\Delta^{op} = (12)\Delta$)



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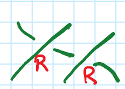
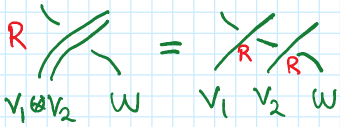


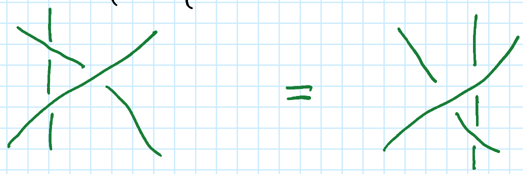
$$(\Delta\otimes 1)R = R^{(13)}R^{(23)}$$



$(\Delta \otimes 1) R = R^{(13)} R^{(23)}$ ← R acting on these factors.

$(1 \otimes \Delta) R = R^{(13)} R^{(12)}$

(Usually only consider $\mathcal{H}$ non-cocomm. ⇒ R nontrivial.)

⇒  Yang-Baxter equation. (YBE.)

$$R^{(12)} R^{(13)} R^{(23)} = R^{(23)} R^{(13)} R^{(12)}$$

Ex: For $\mathcal{U}_q\mathfrak{g}$, R-matrix comes from "Drinfeld double" construction:

If H is a Hopf alg.

$$D(H) := H \otimes (H^\vee)^{op}$$

($H^\vee$: alg. dual; op: opposite coproduct.)

is a quantum group with $R = \sum e_i \otimes e^i$ (e_i: some basis of H; e^i: dual basis.)

Note: $\mathcal{U}_q\mathfrak{g} = D(\mathcal{U}_q\mathfrak{b}) / \langle h_i - \tilde{h}_i \rangle$

($\mathfrak{b}$: any Borel subalg.; h_i, $\tilde{h}_i$: two copies of the Cartan in $D(\cdots)$.)

In fact:

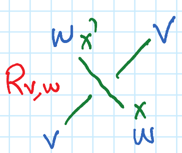
$$\{\text{quantum groups}\} \equiv \{R\text{-matrices}\}$$!

(R-matrices: solutions to YBE.)

"reconstruction" [Faddeev-Reshetikhin-Takhtajan]

Idea: quantum group $A \subset \prod_{V \in \mathfrak{B}} \operatorname{End}(V)$ ← all A-modules, or at least containing some faithful ones.

⇒ A = all matrix elements of "intertwiners" $R_{V,W} : V \otimes W \to W \otimes V$ ← satisfy YBE, ...



$\langle x \mid R_{V,W} \mid x' \rangle_2 \in \operatorname{End}(V)$, $x, x' \in W$ ← matrix elem in only 2^{nd} tensor factor.

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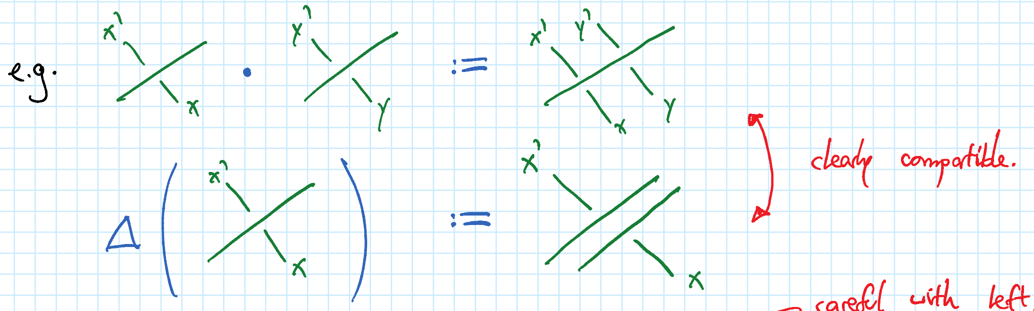

(Antipode is similar: $\mathrm{End}(V) \mapsto \mathrm{End}(V^*)$) ← careful with left/right duals!

Ubiquity of "braiding" operators satisfying YBE (i.e. R-matrices) $\Rightarrow$ ubiquity of quantum groups.

Returning to geometry ...

Thms: [Nakajima]

1. (with Kronheimer) ADHM construction of instantons on $\mathbb{C}^2$ (Hilb($\mathbb{C}^2$)) $\rightsquigarrow$ ADHM construction of instantons on ALE surfaces (Hilb(ADE surfaces))

   || a Nakajima quiver variety for $Q = $ (Jordan quiver: one vertex with a loop) — || a Nakajima quiver variety for Q = affine ADE type

2. For any quiver Q, $\exists\ \mathcal{M}_Q = \mu^{-1}(0) /\!/_\theta\, G$ an alg. symplectic reduction

   $\Rightarrow \mathcal{M}_Q \longrightarrow \mathcal{M}_Q^0$ is a symplectic resolution. (GIT stability is $\theta = 0$)

   $\Rightarrow$ can form a Steinberg variety $Z_Q := \mathcal{M}_Q \times_{\mathcal{M}_Q^0} \mathcal{M}_Q$

(for Q finite ADE) $H^{BM}_{top}(Z_Q) = \mathcal{U}\,\mathfrak{g}_Q$ [Nakajima]

(top-dim Borel–Moore homology.)

3. $K_{\mathbb{C}^\times}(Z_Q) = \mathcal{U}_q\,\hat{\mathfrak{g}}_Q$ ← "quantum affine algebra."

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($\mathbb{C}_q^\times$: scaling of symplectic form.)

Pf. of 3. by explicitly checking generators & relations.
(like what we outlined for Springer theory.) □.