Equivariant K-theory: Lecture 7

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Today: cohomology $H^*(-)$ K-theory $K(-)$ elliptic cohom. $Ell(-)$

topologically, for simplicity
algebraically, see [Levine-Morel]

complex cobordism $U^*(-)$

A generalized multiplicative cohom. theory = a functor $h^* : \underbrace{(\text{top. spaces})}_{\text{(some subcategory of)}} \longrightarrow (\text{rings})$

satisfying Eilenberg-Steenrod axioms except dimension
(homotopy, long exact sequences, etc.)

requiring $h^n(pt) = 0 \quad n > 0$.

A complex orientation for h^* = a choice of iso.

$$h^*(\mathbb{CP}^\infty) \simeq h^*(pt)[\![c_1]\!]$$

Idea: $\mathbb{CP}^\infty = BU(1)$ classifies line bundles.

$$\Rightarrow \quad h^*(BU(n)) \underset{\substack{\text{splitting}\\ \text{principle}}}{=} h^*((\mathbb{CP}^\infty)^n)^{S_n} \simeq h^*(pt)[\![x_1, \dots, x_n]\!]^{S_n}$$

$$\simeq h^*(pt)[\![c_1, c_2, c_3, \dots, c_n]\!]$$

Chern class of vector bundles.

e.g. $H^*(-)$ has ordinary Chern classes.

e.g. $K(-)$ has: $K(\mathbb{CP}^\infty) = \varprojlim_n K(\mathbb{CP}^n) = \varprojlim_n \mathbb{Z}[s^{\pm}] / (1-s)^n$

$$= \mathbb{Z}[\![1-s]\!]$$

$\Rightarrow$ "K-theoretic Chern class" is $c_1^K(\mathcal{L}) := 1 - \mathcal{L} = \Lambda^\bullet_{-1}(\mathcal{L})$

Note: $\mathbb{CP}^\infty$ has a multiplication $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ given by

(classifying map of) $(\mathcal{L}_1, \mathcal{L}_2) \longmapsto \mathcal{L}_1 \otimes \mathcal{L}_2$

$$\Rightarrow \quad \underbrace{h^*(\mathbb{CP}^\infty)}_{h^*(pt)[\![c_1]\!]} \xrightarrow{m^*} \underbrace{h^*(\mathbb{CP}^\infty) \otimes h^*(\mathbb{CP}^\infty)}_{h^*(pt)[\![x, y]\!]}$$

$$c_1 \longmapsto F(x, y)$$

encodes $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$

Def: A formal group law (of rank 1) is (F, A) s.t.

commutative ring

$F(x,y) \in A[[x,y]]$ — A: commutative ring

1. $F(x,0) = F(0,x) = x$
2. $F(x,y) = F(y,x)$
3. $F(F(x,y),z) = F(x,F(y,z))$

$\left.\begin{matrix}1.\\2.\end{matrix}\right\}$ $F(x,y) = x + y + \sum_{i+j>1} a_{ij} x^i y^j$

($\equiv$ formal 1-dimensional groups with a choice of coordinate @ identity.)

e.g. $F_a(x,y) = x+y$ ("additive")

ordinary cohom. $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$

$F_m(x,y) = x+y-xy$ ("multiplicative")

$c_1^K(\mathcal{L}_1 \otimes \mathcal{L}_2) = 1 - \mathcal{L}_1 \otimes \mathcal{L}_2$

$c_1^K(\mathcal{L}_i) = 1 - \mathcal{L}_i$

Homomorphism of formal group laws: $g(u) = u + \cdots$ s.t.

$$g(F_1(x,y)) = F_2(g(x), g(y))$$

Prop: Any F is iso to F_a over $\mathbb{Q}$.

Pf:
$$\frac{\partial}{\partial y}\Big|_{y=0} g(F(x,y)) = \frac{\partial}{\partial y}\Big|_{y=0} (g(x)+g(y))$$
$$\| \qquad\qquad\qquad \|$$
$$g'(x)\, F'(x,0) \qquad\qquad 1$$

$$\Rightarrow \quad g(u) = \int_0^u \frac{dx}{F'(x,0)}$$

involves inverting a series $\Rightarrow$ lives over $\mathbb{Q}$ in general.

called "logarithm" of the formal group law F.

e.g. F_m is iso to F_a by $g(u) = \int_0^u \frac{dx}{1-x} = -\log(1-u)$

(Inverse is $x \mapsto 1 - e^{-x}$.)

$\exists$ a universal formal group law F_U by putting $g(u) = \sum_{n\geq 0} x_n \frac{u^{n+1}}{n+1}$ (x_n: independent formal vars.)

Lives over $\hat{K}$ = ring generated by coefficients of $g(u)$.

(Equivalently, set $F_U(x,y) = \sum A_{i,j} x^i y^j$ and $\hat{K} = \mathbb{Z}[A_{i,j}] / (\text{relations})$.)

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relations from axioms of formal group laws.

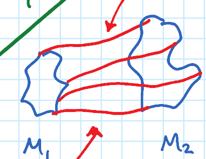
Thm (Lazard, Miščenko, Quillen, ...)

$$(F_U, \hat{K}) = \text{formal group law for complex cobordism } U^*(-).$$

generalized cohom. theory built from complex bordism

$$U_*(X) = \{\text{maps } M \to X\} / \text{"bordism"}$$

stable tangent bundles carry complex structure.



$$\hat{K} = U_*(pt) = \mathbb{Z}[[\mathbb{CP}^1], [\mathbb{CP}^2], \ldots]$$

Any cohom. theory therefore has $h : \hat{K} \to h^*(pt)$.

$$h^*(X) = U^*(X) \otimes_h h^*(pt)$$

Conversely, satisfies all axioms except exactness of long exact sequences.

h is "Landweber exact" if this holds.

$\exists$ some algebraic criterion to check this.

$\Rightarrow$ generalized cohom. theories $\equiv$ Landweber exact formal group laws $\supset$ 1-dim algebraic groups.

Prop: Every 1-dim (connected) alg. group over $\mathbb{C}$ is:

(exercise)

$\mathbb{G}_a$	$\mathbb{G}_m$	E
additive group	multiplicative	elliptic curves.
Spec $\mathbb{C}[t]$	Spec $\mathbb{C}[t^{\pm}]$	projective (& has moduli!)
$\downarrow$	$\downarrow$	$\downarrow$
$H^*(-)$	$K(-)$	$Ell(-)$

In principle, can write elliptic formal group law explicitly :

$$\mathbb{C}/\mathbb{Z}\omega_1+\mathbb{Z}\omega_2 \longrightarrow E = \{ y^2 = \overbrace{1 - 2\delta x^2 + \varepsilon x^4}^{R(x)} \}$$

$$z \longmapsto [\sigma(z) : \sigma'(z) : 1] \subset \mathbb{P}^2$$

related to Weierstrass $\wp$-function.

$$\sigma(z,w) = \frac{\sigma(z)w + z\sigma(w)}{1 - \varepsilon\sigma(z)^2\sigma(w)^2} \quad \Rightarrow \quad F_E(x,y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \varepsilon x^2 y^2}$$

$\sigma(z)$ $\sigma(w)$

Better: equivariant elliptic cohom.