Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) The **gamma function** is defined as
\[
\Gamma(r) := \int_0^\infty e^{-u}u^{r-1} du.
\]
Using a change of variables \( u = st \), show that
\[
\mathcal{L}\{t^{r-1}\} = \frac{\Gamma(r)}{s^r}.
\]
Use integration by parts to show that the gamma function satisfies \( \Gamma(r+1) = r\Gamma(r) \). Finally, show that \( \Gamma(1) = 1 \) to conclude that \( \Gamma(r) = (r-1)! \) when \( r \) is a positive integer.

(2) Using problem 1, solve the IVP
\[
y'' - 3y' + 2y = t^2, \quad y(0) = 0, \quad y'(0) = 1.
\]
(Please feel free to use a computer to find the partial fraction decomposition, but make sure you know in principle how to do it by hand.)

(3) Find the inverse Laplace transforms of
\[
\frac{5e^{-6s} - 11e^{-7s}}{(s-1)(s-2)}, \quad \frac{2s + e^{-2s}}{s^2 - 1}.
\]

(4) Show that if \( \mathcal{L}\{f(t)\} = F(s) \), then
\[
\mathcal{L}\{-tf(t)\} = F'(s).
\]
Explain why this means \( \mathcal{L}\{(-t)^nf(t)\} = F^{(n)}(s) \). In other words, multiplying by \( (-t)^n \) in the time domain is the same thing as taking \( n \) derivatives in the frequency domain.

(5) One way to compute the Laplace transform of \( f(t) \) is to expand it as a series, and take the Laplace transform term by term. Use problem 1 to do this for
\[
\sin t = \sum_{n=0}^{\infty} (-1)^n t^{2n+1} (2n+1)!
\]
to verify that (for \( s > 1 \))
\[
\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}.
\]