HOMEWORK 9 (DUE AUG 08)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

Note: for solutions which are a product of matrices, you do not need to multiply everything out. The point of the problem set is not to test how well you can multiply matrices.

(1) Find all linearly independent eigenvectors for the matrix

\[ P := \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}. \]

(2) Find the Jordan normal form and corresponding basis for the matrix \( P \) in problem 1. (Hint: choose your eigenvectors wisely!) Using this, write the general solution for the homogeneous system \( x' = Px \).

(3) Using problem 2, compute the matrix exponential \( \exp(Pt) \). Check that its columns agree with the general solution you wrote in problem 2.

(4) The \( n \)-th order homogeneous constant-coefficient equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \]

is equivalent to a system of \( n \) first-order equations of the form \( x' = Px \). What is the matrix \( P \)? (It is called the \textit{companion matrix} of the associated characteristic polynomial; there is a normal form for matrices, called the \textit{rational normal form}, whose blocks are companion matrices instead of Jordan blocks.)

(5) Prove that if \( AB = BA \), i.e. \( A \) and \( B \) commute with each other, then

\[ Ae^B = e^B A, \]

i.e. \( A \) and \( e^B \) also commute with each other.