Singular Stochastic PDEs
and Theory of Regularity Structures

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(Based on joint works with Martin Hairer)

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Outline of talk

1. M.Hairer’s theory of regularity structures: what can it do?
   1.1 Well-posedness results for singular SPDEs
   1.2 Universality results for dynamical models
   1.3 Wong-Zakai type results for SPDE.

2. Basic ideas of the theory
Outline of talk

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2. Basic ideas of the theory
Well-posedness of singular stochastic PDEs

- Linear heat equation with space-time white noise $\xi$
  \[
  \partial_t u = \Delta u + \xi
  \]

- Kardar-Parisi-Zhang (d=1) (Hairer 2011)
  \[
  \partial_t h = \Delta h + (\partial_x h)^2 + \xi
  \]

- Dynamical $\Phi^4$ (d=3) (Hairer 2013)
  \[
  \partial_t \phi = \Delta \phi - \phi^3 + \xi
  \]

- Sine-Gordon (d=2) (Hairer and S. 2014)
  \[
  \partial_t u = \frac{1}{2} \Delta u + \sin(\beta u) + \xi
  \]

Difficulty: nonlinearities are meaningless!
If white noise $\xi$ is replaced by smooth noise $\xi_\varepsilon$, such that $\xi_\varepsilon \xrightarrow{\varepsilon \to 0} \xi$, then the (smooth) solutions of

$$\partial_t h_\varepsilon = \Delta h_\varepsilon + (\partial_x h_\varepsilon)^2 + \xi_\varepsilon \quad \text{(KPZ)}$$

$$\partial_t \phi_\varepsilon = \Delta \phi_\varepsilon - \phi_\varepsilon^3 + \xi_\varepsilon \quad \text{(\Phi^4)}$$

$$\partial_t u_\varepsilon = \frac{1}{2} \Delta u_\varepsilon + \sin(\beta u_\varepsilon) + \xi_\varepsilon \quad \text{(Sine-Gordon)}$$

do not converge to any nontrivial limit as $\varepsilon \to 0$. 
Well-posedness of singular stochastic PDEs

One has to renormalize the equations

\[
\begin{align*}
\partial_t h_\varepsilon &= \Delta h_\varepsilon + (\partial_x h_\varepsilon)^2 - A_\varepsilon + \xi_\varepsilon \quad (1D \text{ KPZ}) \\
\partial_t \phi_\varepsilon &= \Delta \phi_\varepsilon - (\phi_\varepsilon^3 - B_\varepsilon \phi_\varepsilon) + \xi_\varepsilon \quad (3D \text{ } \Phi^4) \\
\partial_t u_\varepsilon &= \frac{1}{2} \Delta u_\varepsilon + C_\varepsilon \sin(\beta u_\varepsilon) + \xi_\varepsilon \quad (\text{SG } \beta^2 < 8\pi)
\end{align*}
\]

- The renormalization constants are \textbf{divergent}:

\[
A_\varepsilon \sim \varepsilon^{-1}, \quad B_\varepsilon \sim \varepsilon^{-1} + \log \varepsilon, \quad C_\varepsilon \sim \varepsilon^{-\beta^2/(4\pi)}
\]

- However, the solutions \( h_\varepsilon, \phi_\varepsilon, u_\varepsilon \) \textbf{converge} to nontrivial limits

\[
h_\varepsilon \rightarrow h \quad (\varepsilon \rightarrow 0) \quad \text{etc.}
\]

- These limiting solutions are the intrinsic interpretations of solutions to the singular SPDEs driven by white noise \( \xi \).
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2. Basic ideas of the theory
Dynamical models for interface growth
Dynamical models for interface growth

Stochastic PDE model for interface growth
(proposed by Kardar-Parisi-Zhang 1986)

\[ \partial_t h = \underbrace{\partial_x^2 h}_{\text{smoothing effect}} + \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\xi}_{\text{space-time white noise}} \]

- Kardar-Parisi-Zhang argued (physically) that this equation should describe the large scale behavior of the interface.
- (Weak) Universality: Different microscopic models should all scale to this equation at large scale.
Dynamical models for interface growth

Simplest microscopic model: KPZ equation in weakly asymmetric regime

\[ \partial_t h = \partial_x^2 h + \sqrt{\varepsilon} (\partial_x h)^2 + \xi_1 \]

where \( \xi_1 \) is smooth Gaussian noise with unit correlation length.

Rescale \( h_\varepsilon(x, t) = \varepsilon^{1/2} h(\varepsilon^{-1} x, \varepsilon^{-2} t) \)

\[ \partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \xi_\varepsilon \]

According to well-posedness result by Hairer(2011), \( h_\varepsilon(x, t) - \nu(\varepsilon) t \) converges to nontrivial limit as \( \varepsilon \to 0 \), with vertical speed \( \nu(\varepsilon) = \varepsilon^{-1} C_0 \).
General class of microscopic models

- (Hairer & Quastel, in progress)

\[ \partial_t h = \partial_x^2 h + \sqrt{\varepsilon} P(\partial_x h) + \xi_1 \]

where \( P \) is even polynomial. The rescaled solution \( h_\varepsilon \) has

\[ h_\varepsilon(x, t) - v^{(\varepsilon)}_{\text{ver}} t \to h \]

\( h \) is the solution to KPZ with quadratic nonlinearity \( \lambda (\partial_x h)^2 \), and \( \lambda \) and \( v^{(\varepsilon)}_{\text{ver}} \sim \varepsilon^{-1} \) depend on all coefficients of \( P \) explicitly.

- (Hairer & S. 2015)

\[ \partial_t h = \partial_x^2 h + \sqrt{\varepsilon} (\partial_x h)^2 + \zeta_1 \]

\( \zeta_1 \) is smooth non-Gaussian noise with unit correlation length,

\[ h_\varepsilon(x - v_{\text{hor}} t, t) - v^{(\varepsilon)}_{\text{ver}} t \to h \]

\( h \) is the same solution to KPZ with (Gaussian) white noise; the speeds \( v_{\text{hor}}, v^{(\varepsilon)}_{\text{ver}} \) depend on cumulants of \( \zeta_1 \) explicitly.

\[ v^{(\varepsilon)}_{\text{ver}} = \varepsilon^{-1} C_0 + \varepsilon^{-\frac{1}{2}} C_1 + C_2 \]
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2. Basic ideas of the theory
Wong-Zakai theorem for SPDE

Approximating Itô solution to SDE (Wong & Zakai 1965)

\[ \dot{x}_\varepsilon = h(x_\varepsilon) + g(x_\varepsilon) \dot{B}_\varepsilon \]

then \( x_\varepsilon \) converges to Stratonovich solution. To obtain Itô solution,

\[ \dot{x}_\varepsilon = h(x_\varepsilon) - \frac{1}{2} g'(x_\varepsilon) g(x_\varepsilon) + g(x_\varepsilon) \dot{B}_\varepsilon \]

Approximating Itô solution to SPDE (Hairer & Pardoux 2014)

\[ \partial_t u = \partial_x^2 u + H(u) + G(u)\xi \]

Let \( \xi_\varepsilon \) be smooth Gaussian, \( \xi_\varepsilon \rightarrow \xi \).

\[ \partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) - \varepsilon^{-1} c_0 G'(u_\varepsilon) G(u_\varepsilon) \\
- c_1 G'(u_\varepsilon)^3 G(u_\varepsilon) - c_2 G''(u_\varepsilon) G'(u_\varepsilon) G(u_\varepsilon)^2 + G(u_\varepsilon) \xi_\varepsilon \]
Wong-Zakai theorem for SPDE

Approximating Itô solution to

\[ \partial_t u = \partial_x^2 u + H(u) + G(u)\xi \]

- (Hairer & Pardoux 2014) Let \( \xi_\varepsilon \) be smooth Gaussian, \( \xi_\varepsilon \to \xi \).

\[ \partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) - \varepsilon^{-1} c_0 G'(u_\varepsilon) G(u_\varepsilon) - c_1 G'(u_\varepsilon)^3 G(u_\varepsilon) - c_2 G''(u_\varepsilon) G'(u_\varepsilon) G(u_\varepsilon)^2 + G(u_\varepsilon)\xi_\varepsilon \]

- Let \( \zeta_\varepsilon \) be smooth non-Gaussian, \( \zeta_\varepsilon \to \xi \).

\[ \partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) - H_1(u_\varepsilon) - H_2(u_\varepsilon) + G(u_\varepsilon)\zeta_\varepsilon \]

\[ H_1(u_\varepsilon) = -\varepsilon^{-1} c_0 G'(u_\varepsilon) G(u_\varepsilon) - \varepsilon^{-\frac{1}{2}} c^{(1)} G'(u_\varepsilon)^2 G(u_\varepsilon) - \varepsilon^{-\frac{1}{2}} c^{(2)} G''(u_\varepsilon) G(u_\varepsilon)^2 \]

\[ H_2(u) = -c^{(\alpha)} G'''(u) G(u)^3 - c^{(\beta)} G'(u)^3 G(u) - c^{(\gamma)} G''(u) G'(u) G(u)^2 \]

(Work in progress by Chandra, Hairer & S.)
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Basic ideas of the theory

KPZ equation

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi$$

Difficulty: $h \in C^{\frac{1}{2}-}$

so $(\partial_x h)^2$ can’t be classically defined.

We illustrate the basic ideas using SDE

$$dZ_t = F(Z_t)dB_t \quad F \in C^\infty$$

The same difficulty: $F(Z) \in C^{\frac{1}{2}-}$, $dB \in C^{-\frac{1}{2}-}$,

their product can’t be classically defined.
Basic ideas of the theory

SDE:

\[ dZ_t = F(Z_t)dB_t \quad F \in C^\infty \]

\[ F(Z) \in C^{-1/2}, \ dB \in C^{-1/2}, \text{ product can’t be classically defined.} \]

- Fix \( t \in \mathbb{R}_+ \). There exists a number \( Y_t \) s.t. for \( s \approx t \)

\[ Z_s - Z_t \approx Y_t(B_s - B_t) + \text{(something smoother)} \]

\[ F(Z_s) - F(Z_t) \text{ also behaves like } B_s - B_t. \]

- One can more or less “replace” \( F(Z)dB \) by

\[ B \ dB + \text{(something smoother)}dB \]

The only mission is to define \( B dB \).
Basic ideas of the theory

- Differentiable function \( f \in C^2 \)

\[
f(y) \approx f(x) \cdot 1 + f'(x) \cdot (y - x) + \frac{1}{2} f''(x) \cdot (y - x)^2
\]

"\( f(x) = f(x)1 + f'(x)X + \frac{1}{2} f''(x)X^2 \)"

- Stochastic ODE \( dZ_t = F(Z_t)dB_t \)

\[
Z(s) \approx Z(t) + Y(t)(B_s - B_t)
\]

"\( Z = Z1 + YB \)"

and for the right hand side of equation

"\( F(Z)dB = F(Z)dB + F'(Z)YBdB \)"
Basic ideas of the theory

KPZ equation

\[ \partial_t h = \Delta h + (\partial_x h)^2 + \xi \]

Local behavior of solution to KPZ is described by

\[ H = h \mathbf{1} + \mathcal{I} + \mathcal{Y} + h' X_1 + 2 \mathcal{V} + 2h' \mathcal{E}, \]

Right hand side of equation

\[ (\partial H)^2 + \Xi = \Xi + \mathcal{V} + 2 \mathcal{W} + 2h' \mathcal{I} + \mathcal{Y} + 4 \mathcal{W} + 2h' \mathcal{Y} + 4h' \mathcal{E} + (h')^2 \mathbf{1}. \]

Thank you!