

Testing Composite Hypotheses via Convex Duality ^{*}

Birgit Rudloff [†] and Ioannis Karatzas [‡]

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Abstract

We study the problem of testing composite hypotheses versus composite alternatives, using a convex duality approach. In contrast to classical results obtained by Krafft & Witting [11], where sufficient optimality conditions are obtained via Lagrange duality, we obtain necessary and sufficient optimality conditions via Fenchel duality under some compactness assumptions. This approach also differs from the methodology developed in Cvitanić & Karatzas [6].

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1 Introduction

The problem of Hypothesis Testing is well understood in the classical case of testing a simple hypothesis versus a simple alternative. Suppose one wants to discriminate between two probability measures P (the “null hypothesis”) and Q (the “alternative hypothesis”). In the classical Neyman-Pearson formulation, one seeks a randomized test $\varphi : \Omega \rightarrow [0, 1]$ which is *optimal*, in that it minimizes the overall probability $\mathbb{E}^Q(1 - \varphi)$ of not rejecting P when this hypothesis is false, while keeping below a given significance level $\alpha \in (0, 1)$ the overall probability $\mathbb{E}^P(\varphi)$ of rejecting the hypothesis P when in fact it is true.

In this classical framework an optimal randomized test $\tilde{\varphi}$ always exists and can be calculated explicitly in terms of a reference probability measure R , with respect to which both measures are absolutely continuous (for instance, $R = (P + Q)/2$). It has the randomized 0-1 structure

$$\tilde{\varphi} = 1_{\{L > \delta\}} + \delta \cdot 1_{\{L = \delta\}} \quad (1)$$

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[†]Princeton University, Department of Operations Research and Financial Engineering, Princeton, NJ 08544, USA. E-mail: brudloff@princeton.edu

[‡]Columbia University, Department of Mathematics, MailCode 4438, New York, NY 10027, USA. E-mail: ik@math.columbia.edu

that involves the likelihood ratio $L = (dQ/dR)/(dP/dR)$ of the densities of the null and the alternative hypotheses, the quantile $\mathfrak{z} = \inf\{z \geq 0 : P(L > z) \leq \alpha\}$, and the number $\delta \in [0, 1]$ which enforces the significance-level requirement without slackness, that is, $\mathbb{E}^P(\tilde{\varphi}) = \alpha$.

The problem becomes considerably more involved when the hypotheses are composite, that is, when one has to discriminate between two entire *families of probability measures*; then likelihood ratios of mixed strategies have to be considered. This type of problem also arises in the financial mathematics context of minimizing the expected hedging loss in incomplete or constrained markets; see e.g. Cvitanić [5], Schied [16] and Rudloff [15]. It was shown by Lehmann [12], Krafft & Witting [11], Baumann [4], Huber & Strassen [10], Österreicher [14], Witting [18], Vajda [17] and Cvitanić & Karatzas [6], that duality plays a crucial rôle in solving the testing problem. Most of these papers deal with Lagrange duality; they prove that the typical 0-1 structure of (1) is sufficient for optimality, and that it is both necessary and sufficient if a dual solution exists. An important question then is to decide when a dual solution will exist, and to describe it when it does.

The most recent of these papers, Cvitanić & Karatzas [6], takes a different duality approach. Methods from non-smooth convex analysis are employed, and the set of densities in the null hypothesis is enlarged, in order to obtain the existence of a dual solution – which plays again a crucial rôle.

In the present paper we shall use Fenchel duality. One advantage of this approach is that, as soon as one can prove the validity of strong duality, the existence of a dual solution follows. We shall show that strong duality holds under certain compactness assumptions. This generalizes previous results, insofar as no need to enlarge the set of densities arises, a dual solution is obtained, and thus necessary and sufficient conditions for optimality ensue.

In Section 2 we introduce the problem of testing composite hypotheses. Section 3 gives an overview of the duality results, which are established and explained in detail in Section 4. In Section 5 the imposed assumptions are discussed and possible extensions are given. A comparison of the results and methods of this paper, with those in the extant literature, can be found in the last sections, including Section 6.

2 Testing of Composite Hypotheses

Let (Ω, \mathcal{F}) be a measurable space. A central problem in the theory of Hypothesis Testing is to discriminate between a given family \mathcal{P} of probability measures (composite “null hypothesis”) and another given family \mathcal{Q} of probability measures (composite “alternative hypothesis”).

Suppose that there exists a *reference probability measure* R on (Ω, \mathcal{F}) , that is, a probability measure with respect to which all probability measures $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ are absolutely continuous. We shall use the notation $Z_{\Pi} \equiv d\Pi/dR$ for the Radon-Nikodým derivative of a finite measure Π which is absolutely continuous with respect

to the reference measure, and $\mathbb{E}^\Pi(Y) := \int_\Omega Y d\Pi = \int_\Omega Z_\Pi Y dR$ for the integral with respect to such Π of an \mathcal{F} -measurable function $Y : \Omega \rightarrow [0, \infty)$. Finally, we shall denote the sets of these Radon-Nikodým derivatives for the composite null hypothesis and for the composite alternative hypothesis, respectively, by

$$\mathfrak{Z}_\mathcal{P} := \{Z_P \mid P \in \mathcal{P}\} \quad \text{and} \quad \mathfrak{Z}_\mathcal{Q} := \{Z_Q \mid Q \in \mathcal{Q}\}.$$

Both $\mathfrak{Z}_\mathcal{P}$ and $\mathfrak{Z}_\mathcal{Q}$ are subsets of the non-negative cone \mathbb{L}_+^1 and of the unit ball in the Banach space $\mathbb{L}^1 \equiv \mathbb{L}^1(\Omega, \mathcal{F}, R)$. We shall assume that the mapping $\Omega \times \mathfrak{Z}_\mathcal{P} \ni (\omega, Z) \mapsto Z(\omega) \in [0, \infty)$ is measurable with respect to the product σ -algebra $\mathcal{F} \otimes \mathcal{B}$, where \mathcal{B} is the σ -algebra of Borel subsets of $\mathfrak{Z}_\mathcal{P}$.

We shall denote by Φ the set of all *randomized tests*, i.e., of all Borel-measurable functions $\varphi : \Omega \rightarrow [0, 1]$ on (Ω, \mathcal{F}) . The interpretation is as follows: if the outcome $\omega \in \Omega$ is observed and the randomized test φ is used, then the null hypothesis \mathcal{P} is rejected with probability $\varphi(\omega)$. Thus, $\mathbb{E}^P[\varphi] = \int_\Omega \varphi(\omega) P(d\omega)$ is the overall probability of *type-I-error* (of rejecting the null hypothesis, when in fact it is true) under a scenario $P \in \mathcal{P}$; whereas $\mathbb{E}^Q[1 - \varphi]$ is the overall probability of *type-II-error* (of not rejecting the null hypothesis, when in fact it is false) under the scenario $Q \in \mathcal{Q}$.

We shall adopt the Neyman-Pearson point of view, whereby a type-I-error is viewed as the more severe one and is not allowed to occur with probability that exceeds a given acceptable *significance level* $\alpha \in (0, 1)$, no matter which scenario $P \in \mathcal{P}$ might materialize. Among all randomized tests that observe this constraint

$$\mathfrak{s}(\varphi) := \sup_{P \in \mathcal{P}} \mathbb{E}^P[\varphi] \leq \alpha, \quad (2)$$

we then try to minimize the highest probability $\sup_{Q \in \mathcal{Q}} (1 - \mathbb{E}^Q[\varphi])$ of type-II-error over all scenarios in the alternative hypothesis. We look in other words for a randomized test $\tilde{\varphi}$ that maximizes the smallest *power* with respect to all alternative scenarios

$$\pi(\varphi) := \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi],$$

over all randomized tests φ whose ‘size’ $\mathfrak{s}(\varphi)$, the quantity defined in (2), does not exceed a given significance level α .

Equivalently, we look for a randomized test $\tilde{\varphi} \in \Phi$ that attains the supremum

$$\underline{V} := \sup_{\varphi \in \Phi_\alpha} \pi(\varphi) = \sup_{\varphi \in \Phi_\alpha} \left(\inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \right) \quad (3)$$

of the power $\pi(\varphi)$, over all generalized tests in the class

$$\Phi_\alpha := \left\{ \varphi \in \Phi \mid \sup_{P \in \mathcal{P}} \mathbb{E}^P[\varphi] \leq \alpha \right\}. \quad (4)$$

When such a randomized test $\tilde{\varphi}$ exists, it will be called (max-min) *optimal*.

3 Duality

We shall denote by Λ_+ the set of finite measures on the measurable space $(\mathfrak{Z}_{\mathcal{P}}, \mathcal{B})$. We shall then associate to the maximization problem of (3) the dual minimization problem

$$V^* := \inf_{\substack{Q \in \mathcal{Q} \\ \lambda \in \Lambda_+}} \mathcal{D}(Q, \lambda), \quad (5)$$

where

$$\mathcal{D}(Q, \lambda) := \mathbb{E}^R \left[\left(Z_Q - \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P d\lambda \right)^+ \right] + \alpha \lambda(\mathfrak{Z}_{\mathcal{P}}). \quad (6)$$

Here and in the sequel, we view $\int_{\mathfrak{Z}_{\mathcal{P}}} Z_P(\omega) d\lambda$ as the integral with respect to the measure λ of the continuous functional $\mathfrak{Z}_{\mathcal{P}} \ni Z \mapsto \ell(Z; \omega) := Z(\omega) \in \mathbb{R}$, for fixed $\omega \in \Omega$; see (19) below for an amplification of this point.

The idea behind the setting of (5), (6) is simple: we regard λ as a ‘Bayesian prior’ distribution on the set $\mathfrak{Z}_{\mathcal{P}}$ of densities for the null hypothesis, and its total mass $\lambda(\mathfrak{Z}_{\mathcal{P}}) < \infty$ as a variable whose rôle is to enforce the constraint in (2). More precisely: for any given $Q \in \mathcal{Q}$ and any $\varphi \in \Phi_\alpha$, we have by Tonelli’s theorem the *weak duality*

$$\begin{aligned} \mathbb{E}^Q[\varphi] &= \mathbb{E}^R[\varphi Z_Q] = \mathbb{E}^R \left[\varphi \left(Z_Q - \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P d\lambda \right) \right] + \mathbb{E}^R \left[\varphi \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P d\lambda \right] \\ &\leq \mathbb{E}^R \left[\left(Z_Q - \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P d\lambda \right)^+ \right] + \alpha \lambda(\mathfrak{Z}_{\mathcal{P}}) = \mathcal{D}(Q, \lambda), \end{aligned} \quad (7)$$

for all $\lambda \in \Lambda_+$. Now let us observe that equality holds in (7), if and only if we have both

$$\varphi(\omega) = \begin{cases} 1 & : Z_Q(\omega) > \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P(\omega) d\lambda \\ 0 & : Z_Q(\omega) < \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P(\omega) d\lambda \end{cases}, \quad \text{for } R\text{-a.e. } \omega \in \Omega \quad (8)$$

and

$$\mathbb{E}^R[\varphi Z_P] = \alpha, \quad \text{for } \lambda\text{-a.e. } Z_P \in \mathfrak{Z}_{\mathcal{P}}. \quad (9)$$

It follows from (7) that the inequality $\sup_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \leq \mathcal{D}(Q, \lambda)$ holds for all $\lambda \in \Lambda_+$ and $Q \in \mathcal{Q}$, so

$$\underline{V} \leq \overline{V} := \inf_{Q \in \mathcal{Q}} \left(\sup_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \right) \leq V^* \quad (10)$$

in the notation of (3), (5).

The challenge, then, is to turn this ‘weak’ duality into ‘strong’. That is, to show that equalities $\underline{V} = \overline{V} = V^*$ prevail in (10); that the infimum in (5) is attained by some $(\tilde{Q}, \tilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$; that there exists a $\tilde{\varphi} \in \Phi_\alpha$ for which the triple $(\tilde{\varphi}, \tilde{Q}, \tilde{\lambda})$ satisfies (8), (9); that for this triple equality prevails in (7); and that this same $\tilde{\varphi}$ is optimal for the generalized hypothesis-testing problem, i.e., attains the supremum in (3).

4 Results

In order to carry out the program outlined in the previous section, we shall impose the following assumptions. A discussion of their rôle can be found in Remark 5.1.

Assumption 4.1.

- (i) $\mathfrak{Z}_{\mathcal{Q}}$ is a weakly compact, convex subset of \mathbb{L}^1 .
- (ii) $\mathfrak{Z}_{\mathcal{P}}$ is a compact subset of \mathbb{L}^1 .

Our main result reads as follows.

Theorem 4.2 (Generalized Neyman-Pearson Lemma). *Let \mathcal{P} , \mathcal{Q} be families of probability measures on (Ω, \mathcal{F}) as in Sections 2 and 3, that satisfy Assumption 4.1. For a given constant $\alpha \in (0, 1)$, recall the subclass Φ_{α} of randomized tests in (4).*

There exists then a randomized test $\tilde{\varphi} \in \Phi_{\alpha}$ which is optimal for, that is, attains the supremum in, (3). There exists also a solution to the dual problem of (5), to wit, a pair $(\tilde{Q}, \tilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$ which attains the infimum there.

Furthermore, strong duality is satisfied, in the sense that

- *the optimal test for (3) has the structure of (8), (9), namely*

$$\tilde{\varphi}(\omega) = \left\{ \begin{array}{l} 1 \quad : \quad Z_{\tilde{Q}}(\omega) > \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P(\omega) d\tilde{\lambda} \\ 0 \quad : \quad Z_{\tilde{Q}}(\omega) < \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P(\omega) d\tilde{\lambda} \end{array} \right\}, \quad \text{for } R - a.e. \omega \in \Omega \quad (11)$$

and

$$\mathbb{E}^R[\tilde{\varphi}Z_P] = \alpha, \quad \text{for } \tilde{\lambda} - a.e. Z_P \in \mathfrak{Z}_{\mathcal{P}}; \quad \text{whereas} \quad (12)$$

- $(\tilde{\varphi}, \tilde{Q})$ is a saddle point in $\Phi_{\alpha} \times \mathcal{Q}$ of the functional $(\varphi, Q) \mapsto \mathbb{E}^Q[\varphi]$:

$$\mathbb{E}^{\tilde{Q}}[\varphi] \leq \mathbb{E}^{\tilde{Q}}[\tilde{\varphi}] \leq \mathbb{E}^Q[\tilde{\varphi}], \quad \forall (\varphi, Q) \in \Phi_{\alpha} \times \mathcal{Q}. \quad (13)$$

We shall prove the theorem in several steps, using the following lemmata. The convention of denoting by “max” (resp., “min”) a supremum (resp., infimum) which is attained, will be used freely.

Lemma 4.3. *The supremum in (3) is attained by some randomized test $\tilde{\varphi} \in \Phi_{\alpha}$; and there exists a $\tilde{Q} \in \mathcal{Q}$ such that the saddle-point property (13) holds. In particular, the lower- and upper-values \underline{V} and \overline{V} of (3) and (10), respectively, are the same, i.e.,*

$$\max_{\varphi \in \Phi_{\alpha}} \left(\min_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \right) = \min_{Q \in \mathcal{Q}} \left(\max_{\varphi \in \Phi_{\alpha}} \mathbb{E}^Q[\varphi] \right). \quad (14)$$

Proof. The set Φ of all randomized tests is a weakly* compact subset of the Banach space $\mathbb{L}^\infty \equiv \mathbb{L}^\infty(\Omega, \mathcal{F}, R)$, as it is a weakly* closed subset of the weakly* compact unit ball in \mathbb{L}^∞ (Alaoglu's theorem, e.g. Dunford & Schwartz [7], Theorem V.4.2 and Corollary V.4.3).

To see that Φ is weakly* closed, consider a net $\{\varphi_\alpha\}_{\alpha \in D} \subseteq \Phi$ that converges to φ with respect to the weak* topology in \mathbb{L}^∞ . This means that for all $X \in \mathbb{L}^1$ we have $\mathbb{E}^R[\varphi_\alpha X] \rightarrow \mathbb{E}^R[\varphi X]$. If there existed an event $\Omega_1 \in \mathcal{F}$ with $R(\Omega_1) > 0$ and $\{\varphi > 1\} \subseteq \Omega_1$, then we could choose $\widehat{X}(\omega) = 1_{\Omega_1}(\omega) \in \mathbb{L}^1$ and obtain $\mathbb{E}^R[\varphi \widehat{X}] > R(\Omega_1)$. But this contradicts $\mathbb{E}^R[\varphi \widehat{X}] = \lim_\alpha \mathbb{E}^R[\varphi_\alpha \widehat{X}] \leq R(\Omega_1)$, which follows from $\varphi_\alpha \leq 1$ for all $\alpha \in D$, since $\varphi_\alpha \in \Phi$. Hence, $\varphi \leq 1$ holds R -a.e. It can be shown similarly that $\varphi \geq 0$ also holds R -a.e.

Thus Φ is indeed weakly* closed, hence weakly* compact. Since the mapping $\varphi \mapsto \sup_{P \in \mathcal{P}} \mathbb{E}^P[\varphi]$ is lower-semicontinuous in the weak* topology, the set Φ_α in (4) is weakly* closed, hence weakly* compact. Because of the upper-semicontinuity of the mapping $\varphi \mapsto \pi(\varphi) = \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi]$ in the weak* topology, there exists a $\tilde{\varphi} \in \Phi_\alpha$ that attains the supremum in (3).

The weak* compactness and convexity of Φ_α , and the weak compactness and convexity of \mathfrak{Z}_Q (Assumption 4.1(i)), enable us to apply the von Neumann min-max theorem (see e.g. Aubin [2], Theorem 7, Chapter 7.1, or Aubin [3], section 2.7, pages 39-45); the assertions follow. \square

Let us fix now an arbitrary $Q \in \mathcal{Q}$, and consider as our primal problem the inner maximization in the middle term of (10), namely:

$$\mathfrak{p}(Q) := \sup_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi]. \quad (15)$$

This supremum is always attained, since Φ_α is weakly* compact. We want to show that strong duality holds between (15) and its *Fenchel dual problem* which, we claim, is of the form

$$\mathfrak{d}(Q) = \inf_{\lambda \in \Lambda_+} \mathcal{D}(Q, \lambda) = \inf_{\lambda \in \Lambda_+} \left[\int_{\Omega} \left(Z_Q - \int_{\mathfrak{Z}_P} Z_P d\lambda \right)^+ dR + \alpha \lambda(\mathfrak{Z}_P) \right]. \quad (16)$$

In this case, the typical 0-1 structure of the randomized test $\tilde{\varphi}_Q \in \Phi_\alpha$ that attains the supremum in (15), is necessary and sufficient for optimality.

Lemma 4.4. *Strong duality holds for problems (15) and (16), that is,*

$$\forall Q \in \mathcal{Q} : \quad \mathfrak{d}(Q) = \mathfrak{p}(Q).$$

Moreover, for each $Q \in \mathcal{Q}$, there exists an element $\tilde{\lambda}_Q$ of Λ_+ which attains the infimum in (16); whereas an optimal randomized test $\tilde{\varphi}_Q \in \Phi_\alpha$ that attains the supremum

in (15) has the structure of (8) and (9), namely

$$\tilde{\varphi}_Q(\omega) = \left\{ \begin{array}{l} 1 \quad : \quad Z_Q(\omega) > \int_{\mathfrak{Z}_P} Z_P(\omega) d\tilde{\lambda}_Q \\ 0 \quad : \quad Z_Q(\omega) < \int_{\mathfrak{Z}_P} Z_P(\omega) d\tilde{\lambda}_Q \end{array} \right\}, \quad \text{for } R\text{-a.e. } \omega \in \Omega \quad (17)$$

and

$$\mathbb{E}^R[\tilde{\varphi}_Q Z_P] = \alpha, \quad \text{for } \tilde{\lambda}_Q\text{-a.e. } Z_P \in \mathfrak{Z}_P. \quad (18)$$

Proof. Let \mathcal{L} be the linear space of all continuous functionals $\ell : \mathfrak{Z}_P \rightarrow \mathbb{R}$ on the compact subset \mathfrak{Z}_P of \mathbb{L}^1 (Assumption 4.1(ii)) with pointwise addition, multiplication by real numbers, and pointwise partial order

$$\ell_1 \leq \ell_2 \iff \ell_2 - \ell_1 \in \mathcal{L}_+ := \{\ell \in \mathcal{L} \mid \ell(Z_P) \geq 0, \forall P \in \mathcal{P}\}.$$

We endow \mathcal{L} with the supremum norm $\|\ell\|_{\mathcal{L}} = \sup_{P \in \mathcal{P}} |\ell(Z_P)|$, which ensures that \mathcal{L} is a Banach space (Dunford & Schwartz [7], Section IV.6).

Similarly, we let Λ be the space of finite signed measures $\lambda = \lambda^+ - \lambda^-$ on $(\mathfrak{Z}_P, \mathcal{B})$, with $\lambda^\pm \in \Lambda_+$. We regard this space as the norm-dual of \mathcal{L} , with the bilinear form

$$\langle \ell, \lambda \rangle = \int_{\mathfrak{Z}_P} \ell d\lambda \quad \text{for } \ell \in \mathcal{L}, \quad \lambda \in \Lambda; \quad (19)$$

see Aliprantis & Border [1], Corollary 14.15. (Intuitively speaking, the elements of Λ are generalized Bayesian priors, that may assign negative mass to certain null hypotheses; they are countably additive, however.) We have also the clear identification $\Lambda_+ \equiv \{\lambda \in \Lambda \mid \lambda(B) \geq 0, \forall B \in \mathcal{B}\}$.

Let us define a linear operator $A : (\mathbb{L}^\infty, \|\cdot\|_{\mathbb{L}^\infty}) \rightarrow (\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ by

$$\mathbb{L}^\infty \ni \varphi \longmapsto (A\varphi)(Z_P) := -\mathbb{E}^P[\varphi] = -\mathbb{E}^R[\varphi Z_P] \in \mathbb{R}, \quad (20)$$

for $Z_P \in \mathfrak{Z}_P$; this operator is bounded, thus continuous. We introduce also the constant functionals $\mathbf{1}, \mathbf{0} \in \mathcal{L}$ by

$$\forall Z_P \in \mathfrak{Z}_P : \mathbf{1}(Z_P) = 1 \in \mathbb{R}, \quad \mathbf{0}(Z_P) = 0 \in \mathbb{R}.$$

The constraint of (2) can be rewritten then as

$$\alpha \mathbf{1} + A\varphi \geq \mathbf{0} \iff A\varphi \in \mathcal{L}_+ - \alpha \mathbf{1}.$$

With this notation, for any given $Q \in \mathcal{Q}$ the primal problem (15) can be cast as

$$\begin{aligned} -\mathfrak{p}(-Q) &= \inf_{\varphi \in \mathbb{L}^\infty} (\mathbb{E}^Q[\varphi] + \mathcal{I}_\Phi(\varphi) + \mathcal{I}_{\mathcal{L}_+ - \alpha \mathbf{1}}(A\varphi)) \\ &= \inf_{\varphi \in \mathbb{L}^\infty} (f(\varphi) + g(A\varphi)), \end{aligned} \quad (21)$$

where $-Q$ is interpreted as a finite, signed measure on (Ω, \mathcal{F}) ; cf. the discussion preceding (28). Here and for the remainder of this proof, we use the notation $\mathcal{I}_C(\varphi) := 0$ for $\varphi \in C$, $\mathcal{I}_C(\varphi) := \infty$ for $\varphi \notin C$, as well as

$$f(\varphi) := \mathbb{E}^Q[\varphi] + \mathcal{I}_\Phi(\varphi), \quad g(A\varphi) := \mathcal{I}_{\mathcal{L}_+ - \alpha\mathbf{1}}(A\varphi). \quad (22)$$

• We claim that the *Fenchel dual* of the primal problem in (15) has the form (16). We shall begin the proof of this claim by recalling (from Ekeland & Temam [8], Proposition III.1.1, Theorem III.4.1 and Remark III.4.2) that the Fenchel dual of the problem (21) is given by

$$-\mathfrak{d}(-Q) = \sup_{\lambda \in \Lambda} \left(-f^*(A^*\lambda) - g^*(-\lambda) \right), \quad (23)$$

where $A^* : \Lambda \rightarrow \mathfrak{ba}(\Omega, \mathcal{F}, R)$ the adjoint of the operator A in (20). Here and in the sequel, $\mathfrak{ba}(\Omega, \mathcal{F}, R)$ is the space of bounded, (finitely-)additive set-functions on (Ω, \mathcal{F}) which are absolutely continuous with respect to R (see, for instance, Yosida [19], Chapter IV, section 9, Example 5).

The function $g^*(\cdot)$ is the conjugate of the function $g(\cdot)$, namely

$$\begin{aligned} g^*(\lambda) &= \sup_{\tilde{\ell} \in \mathcal{L}} \left(\langle \tilde{\ell}, \lambda \rangle - \mathcal{I}_{\mathcal{L}_+ - \alpha\mathbf{1}}(\tilde{\ell}) \right) = \sup_{\tilde{\ell} \in \mathcal{L}_+ - \alpha\mathbf{1}} \langle \tilde{\ell}, \lambda \rangle = \sup_{\ell \in \mathcal{L}_+} \langle \ell - \alpha\mathbf{1}, \lambda \rangle \\ &= \sup_{\ell \in \mathcal{L}_+} \langle \ell, \lambda \rangle - \alpha \int_{\mathfrak{Z}_P} d\lambda = \mathcal{I}_{\mathcal{L}_+^*}(\lambda) - \alpha \lambda(\mathfrak{Z}_P), \end{aligned}$$

where $\mathcal{L}_+^* := \{ \lambda \in \Lambda \mid \langle \ell, \lambda \rangle \leq 0, \forall \ell \in \mathcal{L}_+ \}$ is the negative dual cone of \mathcal{L}_+ . The last equality in the above string holds, because \mathcal{L}_+ is a cone containing the origin $\mathbf{0} \in \mathcal{L}$.

To determine the conjugate $f^*(\cdot)$ of the function $f(\cdot)$ at $A^*\lambda$, namely

$$f^*(A^*\lambda) = \sup_{\varphi \in \mathbb{L}^\infty} \left\{ \langle A^*\lambda, \varphi \rangle - \mathbb{E}^Q[\varphi] - \mathcal{I}_\Phi(\varphi) \right\},$$

we have to calculate $\langle A^*\lambda, \varphi \rangle$. By the definition of A^* , the equation $\langle A^*\lambda, \varphi \rangle = \langle \lambda, A\varphi \rangle$ has to be satisfied for all $\varphi \in \mathbb{L}^\infty$, $\lambda \in \Lambda$ (see [1], Chapter 6.8). Thus,

$$\forall \varphi \in \mathbb{L}^\infty, \quad \forall \lambda \in \Lambda : \quad \langle A^*\lambda, \varphi \rangle = - \int_{\mathfrak{Z}_P} \mathbb{E}^R[\varphi Z_P] d\lambda,$$

and the conjugate of the function $f(\cdot)$ at $A^*\lambda$ is evaluated as

$$f^*(A^*\lambda) = \sup_{\varphi \in \Phi} \left(- \int_{\mathfrak{Z}_P} \mathbb{E}^R[\varphi Z_P] d\lambda - \mathbb{E}^Q[\varphi] \right).$$

The dual problem (23) becomes therefore

$$\begin{aligned} -\mathfrak{d}(-Q) &= \sup_{\lambda \in \Lambda} \left[-\sup_{\varphi \in \Phi} \left(-\int_{\mathfrak{Z}_P} \mathbb{E}^R[\varphi Z_P] d\lambda - \mathbb{E}^Q[\varphi] \right) - \mathcal{I}_{-\mathcal{L}_+^*}(\lambda) - \alpha \lambda(\mathfrak{Z}_P) \right] \\ &= \sup_{\lambda \in -\mathcal{L}_+^*} \left[-\sup_{\varphi \in \Phi} \left(-\int_{\mathfrak{Z}_P} \mathbb{E}^R[\varphi Z_P] d\lambda - \mathbb{E}^Q[\varphi] \right) - \alpha \lambda(\mathfrak{Z}_P) \right]. \end{aligned} \quad (24)$$

It is not hard to show that $-\mathcal{L}_+^* = \Lambda_+$, so (24) can be re-cast in the form

$$-\mathfrak{d}(-Q) = \sup_{\lambda \in \Lambda_+} \left[-\sup_{\varphi \in \Phi} \left(-\int_{\mathfrak{Z}_P} \mathbb{E}^R[Z_P \varphi] d\lambda - \mathbb{E}^R[Z_Q \varphi] \right) - \alpha \lambda(\mathfrak{Z}_P) \right]. \quad (25)$$

• Now both (Ω, \mathcal{F}, R) and $(\mathfrak{Z}_P, \mathcal{B}, \lambda)$ for $\lambda \in \Lambda_+$ are positive, finite measure spaces. Furthermore, for every $\varphi \in \Phi$ the mapping $\Omega \times \mathfrak{Z}_P \ni (\omega, Z_P) \mapsto f(\omega, Z_P) := Z_P(\omega)\varphi(\omega) \in \mathbb{R}$ is measurable with respect to the product σ -algebra $\mathcal{F} \otimes \mathcal{B}$, thanks to the measurability assumption imposed in Section 2; whereas for every $\lambda \in \Lambda_+$ and $\varphi \in \Phi$ we have

$$\int_{\mathfrak{Z}_P} \int_{\Omega} |Z_P \varphi| dR d\lambda \leq \left(\sup_{P \in \mathcal{P}} \|Z_P\|_{\mathbb{L}^1} \right) \lambda(\mathfrak{Z}_P) = \lambda(\mathfrak{Z}_P) < \infty,$$

since $\|\varphi\|_{\mathbb{L}^\infty} \leq 1$. Thus, we can apply Tonelli's Theorem (see [7], Corollary III.11.15) and deduce that the order of integration can be interchanged, i.e., for all $\lambda \in \Lambda_+$ and all $\varphi \in \Phi$ we have

$$\int_{\mathfrak{Z}_P} \int_{\Omega} Z_P \varphi dR d\lambda = \int_{\Omega} \int_{\mathfrak{Z}_P} Z_P \varphi d\lambda dR < \infty.$$

In (25) only elements $\lambda \in \Lambda_+$ and $\varphi \in \Phi$ are considered, so we can interchange the order of integration and obtain

$$-\mathfrak{d}(-Q) = \sup_{\lambda \in \Lambda_+} \left(-\sup_{\varphi \in \Phi} \mathbb{E}^R \left[\varphi \left(-Z_Q - \int_{\mathfrak{Z}_P} Z_P d\lambda \right) \right] - \alpha \lambda(\mathfrak{Z}_P) \right). \quad (26)$$

Since $\varphi \in \Phi$ is a randomized test, it follows that the supremum over all $\varphi \in \Phi$ in (26) is attained by some $\bar{\varphi}_{\lambda, -Q} \in \Phi$ of a form similar to (8), namely

$$\bar{\varphi}_{\lambda, -Q}(\omega) = \left\{ \begin{array}{ll} 1 & : -Z_Q(\omega) > \int_{\mathfrak{Z}_P} Z_P(\omega) d\lambda \\ 0 & : -Z_Q(\omega) < \int_{\mathfrak{Z}_P} Z_P(\omega) d\lambda \end{array} \right\}, \quad \text{for } R - \text{a.e. } \omega \in \Omega. \quad (27)$$

Given any finite, signed measure $\Pi = \Pi^+ - \Pi^-$ on (Ω, \mathcal{F}) with $\Pi^\pm \ll R$, let us denote $Z_\Pi = Z_{\Pi^+} - Z_{\Pi^-}$ and

$$\Upsilon_{\lambda, \Pi} := Z_\Pi - \int_{\mathfrak{Z}_P} Z_P d\lambda \in \mathbb{L}^1, \quad (28)$$

and let $\Upsilon_{\lambda, \Pi}^+$ (respectively, $\Upsilon_{\lambda, \Pi}^-$) be the positive (respectively, negative) part of the random variable in (28). With this notation, and recalling (27), the value of the dual problem (26) becomes

$$-\mathfrak{d}(-Q) = \sup_{\lambda \in \Lambda_+} \{ -\mathbb{E}^R[\Upsilon_{\lambda, -Q}^+] - \alpha\lambda(\mathfrak{Z}_P) \},$$

thus

$$\mathfrak{d}(Q) = \inf_{\lambda \in \Lambda_+} \{ \mathbb{E}^R[\Upsilon_{\lambda, Q}^+] + \alpha\lambda(\mathfrak{Z}_P) \}. \quad (29)$$

We deduce from this representation and (28) that the dual $\mathfrak{d}(Q)$, of the primal problem $\mathfrak{p}(Q)$ of (15), is indeed as claimed in equation (16).

- Now strong duality holds if both $f(\cdot)$ and $g(\cdot)$ are convex; if $g(\cdot)$ is continuous at some $A\varphi_0$ with $\varphi_0 \in \text{dom}(f)$; and if $\mathfrak{p}(Q)$ is finite (see Ekeland & Temam [8], Theorem III.4.1 and Remark III.4.2).

Indeed, the existence of a primal solution ensures the finiteness of $\mathfrak{p}(Q)$. In (22) the function $f(\cdot)$ is convex, since Φ is a convex set; and $g(\cdot)$ is convex, since the set $\mathcal{L}_+ - \alpha\mathbf{1}$ is convex. The function $g(\cdot)$ is indeed continuous at some $A\varphi_0$ with $\varphi_0 \in \text{dom}(f)$, provided $A\varphi_0 \in \text{int}(\mathcal{L}_+ - \alpha\mathbf{1})$. If we take $\varphi_0 \equiv 0$, then $\varphi_0 \in \text{dom}(f)$ since $\varphi_0 \in \Phi$, and we see that $A\varphi_0 = \mathbf{0} \in \text{int}(\mathcal{L}_+ - \alpha\mathbf{1})$ since $\text{int}(\mathcal{L}_+) \neq \emptyset$ in the norm topology and $\alpha > 0$. Hence, we have strong duality.

- The existence of a solution to the primal problem $\mathfrak{p}(Q)$ (that is, of a generalized test $\tilde{\varphi}_Q \in \Phi_\alpha$ that attains the supremum in (15)) follows from the weak* compactness of Φ_α . With strong duality established, the existence of a solution to the dual problem, that is, of an element $\tilde{\lambda}_Q \in \Lambda_+$ that attains the infimum in (16), follows (Ekeland & Temam [8], loc. cit.); whereas the values of the primal (respectively, the dual) objective functions at $\tilde{\varphi}_Q$ (respectively, $\tilde{\lambda}_Q$) coincide. To indicate the dependence of these quantities on the selected $Q \in \mathcal{Q}$, we have used the notation $\tilde{\varphi}_Q$ and $\tilde{\lambda}_Q$ for the primal and dual solutions, respectively.

These considerations lead to a necessary and sufficient condition for optimality. Indeed, let us write the expression for $\mathbb{E}^Q[\varphi]$ that appears in the equation of (7), as

$$\mathbb{E}^Q[\varphi] = \mathbb{E}^R[\varphi \Upsilon_{\lambda, Q}^+] - \mathbb{E}^R[\varphi \Upsilon_{\lambda, Q}^-] + \mathbb{E}^R\left[\varphi \int_{\mathfrak{Z}_P} Z_P d\lambda\right]$$

in the notation of (28), and subtract it from the dual objective function $\mathbb{E}^R[\Upsilon_{\lambda, Q}^+] + \alpha\lambda(\mathfrak{Z}_P)$ as in (29). Because of strong duality, this difference has to be zero when evaluated at $(\varphi, \lambda) = (\tilde{\varphi}_Q, \tilde{\lambda}_Q)$, namely:

$$\mathbb{E}^R\left[\Upsilon_{\tilde{\lambda}_Q, Q}^+ (1 - \tilde{\varphi}_Q)\right] + \mathbb{E}^R\left[\Upsilon_{\tilde{\lambda}_Q, Q}^- \tilde{\varphi}_Q\right] + \int_{\mathfrak{Z}_P} (\alpha - \mathbb{E}^R[Z_P \tilde{\varphi}_Q]) d\tilde{\lambda}_Q = 0.$$

Each of these three integrals is nonnegative, so their sum is zero if and only if $\tilde{\varphi}_Q \in \Phi_\alpha$ satisfies the condition (18) of Lemma 4.4 and is of the form (17) or, equivalently, of the form $\tilde{\varphi}_Q \equiv \bar{\varphi}_{\tilde{\lambda}_Q, -Q}$ of (27). \square

Now, we are able to prove our main result.

Proof of Theorem 4.2. With Lemma 4.4 it follows that

$$\min_{Q \in \mathcal{Q}} \left(\max_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \right) = \min_{Q \in \mathcal{Q}} \mathfrak{p}(Q) = \min_{Q \in \mathcal{Q}} \mathfrak{d}(Q) = \min_{(Q, \lambda) \in \mathcal{Q} \times \Lambda_+} \mathcal{D}(Q, \lambda) = V^*$$

in the notation of (5) and (15), (16). From Lemma 4.3 it follows that there exists an element \tilde{Q} of \mathcal{Q} which attains the infimum in (10). For this \tilde{Q} , Lemma 4.4 shows the existence of an element $\tilde{\lambda}_{\tilde{Q}}$ of Λ_+ that attains the infimum in (16). Thus, there exists a pair $(\tilde{Q}, \tilde{\lambda})$ that attains the infimum in (5), and Lemma 4.4 gives the required structural result. \square

Corollary: *It follows that the optimal randomized test has the form*

$$\tilde{\varphi}(\omega) = 1_{\{Z_{\tilde{Q}} > \int_{\mathfrak{F}_P} Z_P d\tilde{\lambda}\}}(\omega) + \delta(\omega) \cdot 1_{\{Z_{\tilde{Q}} = \int_{\mathfrak{F}_P} Z_P d\tilde{\lambda}\}}(\omega) \quad (30)$$

reminiscent of (1), where the random variable $\delta : \Omega \rightarrow [0, 1]$ is chosen so that (12) is satisfied by $\tilde{\varphi}$.

5 Extensions and Ramifications

Remark 5.1. The weak compactness of the set of alternative densities $\mathfrak{Z}_\mathcal{Q}$ in Assumption 4.1(i) seems to be crucial. Without it, we can still get

$$\max_{\varphi \in \Phi_\alpha} \left(\inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \right) = \inf_{Q \in \mathcal{Q}} \left(\max_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \right)$$

by Fenchel duality (endowing \mathbb{L}^∞ with the norm topology). There is no guarantee anymore, however, that the infimum will be attained in \mathcal{Q} . The infimum will be attained at some element $\hat{\mu}$ of the set $\mathfrak{M} \subseteq \{\mu \in \mathfrak{ba}(\Omega, \mathcal{F})_+ \mid \mu(\Omega) = 1\}$, which contains \mathcal{Q} ; but since a Hahn decomposition might not exist for this $\hat{\mu}$, we do not obtain in general the 0-1 structure in (30) of the primal solution with respect to the dual solution.

It seems reasonable to endow \mathbb{L}^∞ with the weak* topology, and to apply Fenchel duality. But then it is tricky to show that a suitable constraint qualification (e.g., that $\rho(\varphi) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi]$ be weakly* continuous at some $\varphi_0 \in \Phi_\alpha$) is satisfied, which is needed to obtain strong duality.

Assuming $\mathfrak{Z}_\mathcal{Q}$ to be weakly compact and convex, as we have done throughout the present work, has enabled us to apply a min-max theorem and to ensure that the infimum in the dual problem is attained within $\mathfrak{Z}_\mathcal{Q} \subseteq \mathbb{L}^1$.

- If we were to drop the compactness Assumption 4.1(ii) on the set of densities $\mathfrak{Z}_{\mathcal{P}}$, then the norm-dual of \mathcal{L} would be $\mathfrak{ba}(\mathfrak{Z}_{\mathcal{P}}, \mathcal{B})$ instead of Λ (recall the definitions of the spaces \mathcal{L} and Λ from the start of the proof of Lemma 4.4). The elements of the space $\mathfrak{ba}(\mathfrak{Z}_{\mathcal{P}}, \mathcal{B})$ are the ultimate generalized Bayesian priors; they are allowed to assign negative weights to sets of possible scenarios, and to be just finitely (as opposed to countably) additive. But in such a setting, Tonelli’s Theorem cannot be applied anymore. It is possible to endow \mathcal{L} with the Mackey topology, to ensure Λ is the topological dual space of \mathcal{L} ; but proving strong duality under this topology is a challenge. Throughout this paper Assumption 4.1(ii) is imposed to ensure that the norm-dual of the space \mathcal{L} is Λ , and that a strong duality result can be obtained.

Remark 5.2. The results in this paper can be extended in several directions. For instance, our proofs have not used the assumption that \mathcal{P} and \mathcal{Q} are families of probability measures. The results still hold if we consider instead two arbitrary subsets of \mathbb{L}^1 , namely \mathcal{G} (in lieu of $\mathfrak{Z}_{\mathcal{P}}$) and \mathcal{H} (in lieu of $\mathfrak{Z}_{\mathcal{Q}}$), that satisfy Assumption 4.1 as well as $\sup_{G \in \mathcal{G}} \|G\|_{\mathbb{L}^1} < \infty$.

Furthermore, instead of a constant $\alpha \in (0, 1)$, we may consider a positive continuous functional $\alpha : \mathcal{G} \rightarrow \mathbb{R}_+$. The corresponding optimization problem is then

$$\sup_{\varphi \in \Phi} \left(\inf_{H \in \mathcal{H}} \mathbb{E}[\varphi H] \right), \quad (31)$$

subject to

$$\mathbb{E}^R[\varphi G] \leq \alpha(G), \quad \forall G \in \mathcal{G}. \quad (32)$$

The problem (31)-(32) is no longer of Hypothesis-Testing form in the classical sense, but its structure is similar to that of testing composite hypotheses. Such so-called “generalized hypothesis-testing problems” arise also in the context of hedging contingent claims in incomplete or constrained markets, for instance when one tries to minimize the expected hedging-loss (see Cvitanić [5], Rudloff [15] or, in a related context, Schied [16]).

This kind of generalized hypothesis testing problem was studied for the case of a simple alternative (i.e., \mathcal{H} being a singleton), and a positive, bounded and measurable function $\alpha(\cdot)$, by Witting [18], Section 2.5.1. For this case it was shown with Lagrange duality that the generalized 0-1 structure (30) of a test is sufficient for optimality. Furthermore, it was shown in [18] that, for a finite set \mathcal{G} , the conditions (17), (18) are necessary and sufficient for optimality. The proof of Lemma 4.4 shows that a generalization of these results is possible even when both the ‘hypothesis’ set \mathcal{G} and the ‘alternative hypothesis’ set \mathcal{H} are infinite, provided they satisfy the above conditions (Assumption 4.1 and $\sup_{G \in \mathcal{G}} \|G\|_{\mathbb{L}^1} < \infty$), and $\alpha(\cdot)$ is a given positive, continuous function.

Related results are obtained in Lehmann [12], in Krafft & Witting [11] which is apparently the first work to introduce ideas of Convex Duality in the theory of Hypothesis Testing, and in Baumann [4], Huber & Strassen [10], Österreicher [14], Vajda

[17], pp. 361-362 and Schied [16]. Lehmann [12] works with a finite set $\mathfrak{Z}_{\mathcal{Q}}$, provides existence results and shows that the composite testing problem can be reduced to one with simple hypotheses (consisting of the optimal mixed strategy). Krafft & Witting [11] and Witting [18] use Lagrange duality and show that, even without any compactness assumptions on the sets $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$, the generalized 0-1 structure (30) of $\tilde{\varphi}$ is sufficient for optimality, as well as necessary and sufficient if a dual solution exists (e.g. when $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$ are finite). In this paper we showed that, under Assumption 4.1, the generalized 0-1 structure of (30) is necessary and sufficient for optimality, due to strong duality with respect to the Fenchel dual problem; then, the existence of a dual solution follows from strong duality. Baumann [4] establishes the existence of a max-min optimal test using duality results from linear programming and weak compactness arguments. The problem is also studied for densities that are contents and not necessarily measures. Huber & Strassen [10] dispense with the assumption that all measures in $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$ be absolutely continuous with respect to a reference measure R , at the expense of assuming that these two sets can be described in terms of “alternating capacities” in the sense of Choquet. Finally, using totally different methods and motivated by optimal investment problems in mathematical finance, Schied [16] studies variational problems of Neyman-Pearson type for convex risk measures and for law-invariant robust utility functionals, and obtains explicit solutions for quantile-based coherent risk measures that satisfy the Huber-Strassen-Choquet alternating capacity conditions.

6 Comparisons and Conclusion

The problem of testing a composite null hypothesis against a simple alternative hypothesis has a long history; it has been considered in a variety of papers as discussed above, and in several books (for instance, Ferguson [9], Witting [18], Lehmann [13] and Vajda [17]). The problem of testing a composite hypothesis against a composite alternative has also been studied; see, for instance, Cvitanić & Karatzas [6] and the references therein, for one of the most recent works on this subject. We want to give here a short overview regarding the similarities and differences between the results in Cvitanić & Karatzas [6] and Theorem 4.2 of the present paper, both in terms of results and of the methods used to obtain them.

Cvitanić & Karatzas [6] introduce the enlargement

$$\mathcal{W} := \{W \in \mathbb{L}_+^1 \mid \mathbb{E}^R[\varphi W] \leq \alpha, \forall \varphi \in \Phi_\alpha\} \supseteq \text{co}(\mathfrak{Z}_{\mathcal{P}}) \quad (33)$$

of the convex hull of the Radon-Nikodým densities of \mathcal{P} . This ‘enlarged’ set \mathcal{W} is convex, bounded in \mathbb{L}^1 , and closed under R -a.e. convergence. Furthermore, it is assumed in Cvitanić & Karatzas [6] that the set of densities of \mathcal{Q} is convex and closed under R -a.e. convergence. The starting point of Cvitanić & Karatzas [6] is the observation

$$\forall Q \in \mathcal{Q}, \forall W \in \mathcal{W}, \forall z > 0, \forall \varphi \in \Phi_\alpha : \quad \mathbb{E}^Q[\varphi] \leq \mathbb{E}^R[(Z_Q - zW)^+] + \alpha z. \quad (34)$$

The existence of a quadruple $(\widehat{Q}, \widehat{W}, \widehat{z}, \widehat{\varphi}) \in \mathcal{Q} \times \mathcal{W} \times (0, \infty) \times \Phi_\alpha$ which satisfies (34) as equality is then shown, and the structure of

$$\widehat{\varphi}(\omega) = 1_{\{\widehat{z}\widehat{W} < Z_{\widehat{Q}}\}}(\omega) + \delta(\omega) \cdot 1_{\{\widehat{z}\widehat{W} = Z_{\widehat{Q}}\}}(\omega) \quad (35)$$

for the optimal randomized test $\widehat{\varphi}$ is deduced. Here the triple $(\widehat{Q}, \widehat{W}, \widehat{z})$ is a solution of the optimization problem

$$\inf_{\substack{z > 0 \\ (Q, W) \in \mathcal{Q} \times \mathcal{W}}} (\alpha z + \mathbb{E}^R[(Z_Q - zW)^+]), \quad (36)$$

and the random variable $\delta : \Omega \rightarrow [0, 1]$ is chosen so that $\mathbb{E}^Q[\widehat{\varphi}] = \alpha$ is satisfied.

The methodology of the present paper obviates the need to introduce the enlargement set \mathcal{W} of (33). Thus, we provide a result about the structure of the solution $\widehat{\varphi}$ in terms of the original families of probability measures \mathcal{P} and \mathcal{Q} ; we do need, however, the set $\mathfrak{Z}_\mathcal{Q}$ to be weakly compact.

• Let us study the relationship between Theorem 4.2 and the results of Cvitanić & Karatzas [6]. With the help of Tonelli's Theorem it is easy to show that we have

$$k \int_{\mathfrak{Z}_\mathcal{P}} Z_P d\lambda \in \mathcal{W}, \quad \forall \lambda \in \Lambda_+,$$

where $k = (\lambda(\mathfrak{Z}_\mathcal{P}))^{-1}$ if $\lambda(\mathfrak{Z}_\mathcal{P}) > 0$ and $k = 0$ if $\lambda(\mathfrak{Z}_\mathcal{P}) = 0$. The case $\lambda(\mathfrak{Z}_\mathcal{P}) = 0$ implies $\lambda(B) = 0$ for all $B \in \mathcal{B}$, and thus $\int_{\mathfrak{Z}_\mathcal{P}} Z_P d\lambda = 0$. If we consider in (34) only elements W of the form $k \int_{\mathfrak{Z}_\mathcal{P}} Z_P d\lambda \in \mathcal{W}$, then the inequality (34) coincides with the weak duality between the primal and dual objective functions $\mathfrak{p}(Q)$ and $\mathfrak{d}(Q)$, and reduces to the inequality in (7); whereas Problem (36) reduces to (5).

To summarize the methodology: Cvitanić & Karatzas [6] proved the existence of a primal and a dual solution that satisfy (34) as an equality; in order to do this, strong closure assumptions had to be imposed. In the methodology of the present paper, the validity of strong duality, hence also the equality in (7), are shown directly via Fenchel duality; then the existence of a dual solution follows.

Both methods lead to a result about the structure of an optimal test. But now it is possible to show the impact of the original family \mathcal{P} on the sets that define the solution $\widehat{\varphi}$ in [6], as in (35):

$$\widehat{z}\widehat{W} = \int_{\mathfrak{Z}_\mathcal{P}} Z_P d\widetilde{\lambda}, \quad (37)$$

where $(\widetilde{Q}, \widetilde{\lambda})$ is the optimal pair, that attains the infimum in (5). This means

$$\widehat{z} = \widetilde{\lambda}(\mathfrak{Z}_\mathcal{P}) \quad \text{and} \quad \widehat{W} = k \int_{\mathfrak{Z}_\mathcal{P}} Z_P d\widetilde{\lambda}, \quad (38)$$

where $k = (\widetilde{\lambda}(\mathfrak{Z}_\mathcal{P}))^{-1}$ if $\widetilde{\lambda}(\mathfrak{Z}_\mathcal{P}) > 0$, and $k = 0$ if $\widetilde{\lambda}(\mathfrak{Z}_\mathcal{P}) = 0$.

Let us highlight the improvements made possible by the methodology of the present paper. With Theorem 4.2 one can provide a result about the structure of the solution $\tilde{\varphi}$ in terms of the original sets \mathcal{P} and \mathcal{Q} ; it is not necessary to embed $\mathfrak{Z}_{\mathcal{P}}$ into the larger set \mathcal{W} of (33). But instead of assuming that $\mathfrak{Z}_{\mathcal{Q}} = \{Z_Q \mid Q \in \mathcal{Q}\}$ is closed under R -a.e. convergence, we need here to assume that this set is weakly compact in \mathbb{L}^1 .

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