# Problems about CW complexes

September 6, 2023

In what follows $X$ is a CW complex and $X^{(k)}$ denotes its $k$-skeleton.

1). Suppose trhat $A \subset X$ is a set that meets the interior of each cell of $X$ in a finite set. Show that $A$ is a closed subset of $X$ and that the topology on $X$ induces the discrete topology on $A$.

2). Show that any compact subset of $X$ is contained in a finite union of open cells, and indeed in a finite sub-complex of $X$.

3). Show that the topology of $X$ is the weak topology induced from the finite sub-complexes of $X$. That is to say $A \subset X$ is closed if and only if its intersection with each finite sub-complex of $X$ is a closed subset of $X$. Show that $X$ has the compactly generated topology.

4). Let $B = A \cup D^n$ the result of attaching an $n$-cell to $A$. Show that $\pi_i(B, A) = 0$ for $i < n$. [Hint: Use either simplicial approximation or Sard's theorem.] Fix $k \geq 1$. Suppose that $X$ is a sub-complex of another CW complex $Y$ where $Y$ is obtained from $X$ by attaching cells of dimension $\geq k + 2$. Show that $\pi_i(Y) = \pi_i(X)$ for every $i \leq k$.

5).(Baby form of obsruction theory) Let $Y$ be a path connected topological space with $\pi_i(Y) = 0$ for all $i \geq k$ and suppose given $f \colon X^{(k)} \to Y$. Show that there is an extension of $f$ to $\hat{f} \colon X \to Y$ and that any two such are homotopic relative to $X^{(k)}$.

6) Show that any map between CW complexes $f \colon X \to Y$ is homotopic to one that sends the $k$-skeleton of $X$ to the $k$-skeleton of $Y$ for every $k \geq 0$.

7). The inclusion $A \subset X$ of a subspace is said to be *cofibration* if it satisfies the *homotopy extension property.* That is to say, if for any space $Z$ given a map $f \colon X \to Z$ and a homotopy $H \colon A \times \to Z$ with $H|_{A\times\{0\}} = f$ then there is a homotopy $F \colon X \times I \to Z$ with $F|_{X\times\{0\}} = f$ and $F|_{A\times I} = H$. It is equivalent to say that $A \times I \cup X \times \{0\} \subset X \times I$ is a retract. Show that $S^{n-1} \subset D^n$ is a cofibration, as is $\coprod_{\alpha \in A} S_\alpha^{n-1} \subset \coprod_{\alpha \in A} D_\alpha^n$. Let $X$ be a CW complex. Show that for any $k \geq 0$, the inclusion $X^{(k)} \subset X^{(k+1)}$ is a cofibration. Show $X^{(k)} \subset X^{(\ell)}$ is a cofibraion for any $k < \ell$. Show that $X^{(k)} \subset X$ is a cofibration. Show that for any subcomplex $X' \subset X$, the inclusion is a cofibration.

8). Show that given any map $f\colon A \to B$ there is a space $B'$, an inclusion $A \subset B'$ that is a cofibrataion, and a homotopy equivalence $F\colon B' \to B$ such that $F|_A = f$. (Hint: Use the mapping cylinder of $f$.)