Condensed Abelian Groups

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The easiest way to see all the structure of condensed abelian groups, it is best to use a different model for condensed sets and groups. The other model is built using the category of extremely disconnected sets, so we begin of study of condensed abelian groups by introducing the alternate category and establishing its equivalence with the category of condensed sets (and groups).

1 Equivalent categories to condensed sets

1.1 Compact Hausdorff Spaces and $\mathbf{Prof}_\kappa$

Theorem 1.1. *Fix κ a strong limit cardinal and consider the following two categories of sheaves:*

- *Sheaves on the Grothendieck site on the category of compact Hausdorff spaces whose underlying sets have cardinality less than κ and whose coverings are finite disjoint unions of maps $\{C_i \to C\}_i$ with a common co-domain whose images cover the common co-domain.*
- *Sheaves on the Grothendieck site induced on subcategory $\mathbf{Prof}_\kappa$.*

The restriction map from the first category to the second is an equivalence of categories.

Proof. This follows easily from the fact that ever compact Hausdorff space cardinality less than κ is the quotient of a profinite set of cardinality less than κ. For example, let $\mathcal{F}$ be a sheaf on compact Hausdorff spaces, X be a compact Hausdorff space, and let S be a profinite set mapping onto X. Then then $\mathcal{F}(X)$ is the equalizer of the two maps $\mathcal{F}(S) \to \mathcal{F}(S \times_X S)$, showing that $\mathcal{F}(X)$ is determined by sheaves on profinite sets and morphisms between them. $\square$

1.2 Sheaves on $\mathbf{Prof}_\kappa$ and the category of Functors preserving (co)-products on the category of κ-Extremely Disconnected sets

Definition 1.2. A condensed set X is a projective if for any surjective map of condensed sets $Y \to X$ there is a section.

Lemma 1.3. *X is projective if and only if for every surjective map $A \to B$ of condensed sets $Hom(X, A) \to Hom(X, B)$ is a surjective map of sets.*

Proof. Given any morphsim $X \to A$, the projection of the fibered product $X \times_B A \to X$ is a surjection and hence has a section σ. The composition of σ followed by the projection of the fibered product to A gives the required morphism $X \to A$. □

Lemma 1.4. *The qc projective elements of the category of κ-condensed sets are the κ-extrremely disconnected sets.*

Proof. Being qc X is a quotient of a representable condensed set $\underline{S} \to X$. Being projective, X has a section back to $\underline{S}$. Thus, X is a retract of S. According to the lemma last time about qc sub-objects of a condensed set, this means that X is representable by a closed subset of $S' \subset S$. But being projective means that for any disconnected set T of cardinality less than κ with a surjective map to S, there is a section $S" \to T$. That is to say S' is an extremely disconnected set. □

1.3 The Equivalence of Categories

We now study the category of contravariant functors $\mathcal{F}$ from the category of extremely disconnected sets of cardinality less than κ to the category of sets, with the property that it sends finite co-products to finite products.

$$\mathcal{F}(\coprod_i E_i) = \prod_i \mathcal{F}(E_i).$$

We denote this category by $\mathbf{EDP}_\kappa$

Theorem 1.5. *The restriction map $Sh(\mathbf{Prof}_\kappa) \to \mathbf{EDP}_\kappa$ is an equivalence of categories.*

The proof of this result takes up the rest of this subsection.

Certainly the restriction map from sheaves on $\mathbf{Prop}_\kappa$ to contravariant functions from the category of extremely disconnected spaces of cardinality less than κ to sets gives a functor that sends finite co-products to finite products. In what follows extremely disconnected sets are denoted E with subscripts and superscripts and general profinite sets are denoted S. All such sets are implicitly required to have cardinality less than κ. We fix $\mathcal{F}$ an object of $\mathbf{EDP}_\kappa$, i.e., a contravariant function from extremely disconnected sets to sets sending finite co-products to finite products. Our goal is to extend (up to isomorphism) $\mathcal{F}$ to a functor on $\mathbf{Prof}_\kappa$ and show that the extension is a sheaf.

Claim 1.6. *Let E and E' be extremely disconnected sets and $\pi\colon E' \to E$ a surjection. Then $\pi^*\colon \mathcal{F}(E) \to \mathcal{F}(E')$ is an injection.*

Proof. There is a section $\tau\colon E \to E'$, meaning that $\pi \circ \tau = \mathrm{Id}_E$. Thus, $\tau^*\pi^*\colon \mathcal{F}(E) \to \mathcal{F}(E)$ is the identity and hence π^* is injective. □

Claim 1.7. *Let E and E' be extremely disconnected sets and $\pi\colon E' \to E$ a surjection. Let E'' be an extremely disconnected set and let $f\colon E'' \to E' \times_E E'$ be a surjection. Then the diagram*

$$\mathcal{F}(E) \xrightarrow{\pi^*} \mathcal{F}(E') \underset{(p_2\circ f)^*}{\overset{(p_1\circ f)^*}{\rightrightarrows}} \mathcal{F}(E'')$$

expresses $\mathcal{F}(E)$ as the equalizer of $(p_1 \circ f)^*$ and $(p_2 \circ f)^*$.

Proof. (of claim) Since E is extremely disconnected there is a map $\tau\colon E \to E'$ with $\pi \circ \tau = \mathrm{Id}_E$. Certainly, any element in the image of π^* is contained in the equalizer of $(p_1 \circ f)^*$ amd $(p_2 \circ f)^*$. Define $\sigma\colon E' \to E' \times_E E'$ by $\sigma(x) = (x, \tau(\pi(x)))$. According to Corollary 3.3 of the lecture notes on the first lecture, there is a lift $\hat{\sigma}\colon E' \to E''$ of σ. Then $(p_1 \circ f) \circ \hat{\sigma} = \mathrm{Id}_{E'}$ and $(p_2 \circ f) \circ \hat{\sigma} = \tau \circ \pi$. For $x \in \mathcal{F}(E')$ if $(p_1 \circ f)^*(x) = (p_2 \circ f)^*(x)$, then $x = \pi^*\tau^*(x)$ and hence $x \in \mathrm{Im}(\pi^*)$. Since we have already established that π^* is an injection, this completes the proof that π^* is an injection with image the equalizer of the two given maps. □

Since extremely disconnected sets do not have a fiber product there is no Grothendieck site on this category analogous to the site on **Prof**. Nevertheless, the objects in $\mathbf{EDP}_\kappa$ do satisfy a version of the sheaf condition as the following corollary points out.

Corollary 1.8. *Suppose we have a covering $\{E_i \to E\}_i$ (in $\mathbf{Prof}_\kappa$) with E and the E_i extremely disconnected and suppose that $f\colon E' \to \coprod_{i,i'} E_i \times_E E_{i'}$*

with E' extremely disconnected and f a surjection. Then the diagram

$$\mathcal{F}(E) \xrightarrow{\pi^*} \prod_i \mathcal{F}(E_i) \underset{(p_2\circ f)^*}{\overset{(p_1\circ f)^*}{\rightrightarrows}} \mathcal{F}(E'')$$

expresses $\mathcal{F}(E)$ as the equalizer of $(p_1 \circ f)^*$ and $(p_2 \circ f)^*$ in $\prod_i \mathcal{F}(E_i)$.

Proof. By the defining axiom $\mathcal{F}(\coprod_i E_I) = \prod_i \mathcal{F}(E_i)$ and similarly for

$$\mathcal{F}((\coprod_i E_i) \times_E (\coprod_{i'} E_{i'})).$$

From this the corollary follows immediately from the previous claim applied to the surjection $\coprod_i E_i \to E$. ☐

Now we begin the definition of the extension of $\mathcal{F}$ (up to isomorphism) to a sheaf on the Grothendieck cite $\mathbf{Prof}_\kappa$. For any profinite set S of cardinality less than κ we construct a diagram

$$S \xleftarrow[\pi]{} E \underset{p_1}{\overset{p_2}{\leftleftarrows}} E \times_S E \xleftarrow[f]{} E'$$

where E and E' are extremely disconnected sets of cardinality less than κ, $\pi\colon E \to S$ and $f\colon E' \to E \times_S E$ are both surjective. We then define $\mathcal{G}_{(E,E',\pi,f)}(S)$ to be the equalizer of $(p_1 \circ f)^*$ and $(p_2 \circ f)^*$ in $\mathcal{F}(E)$

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Claim 1.9. *$\mathcal{G}_{(E,E',\pi,f)}(S) \subset \mathcal{F}(E)$ is independent of the choice of E' and the surjection $f\colon E' \to E \times_S E$.*

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Proof. Given two diagrams as above for S with extremely disconnected sets (E, E') and (E, E'_1) there is a diagram for S with sets (E, E'_2) where E'_2 surjects onto $E'_1 \times_E E'_2$. It follows easily from the injectivity of $\mathcal{F}(E'_1) \to \mathcal{F}(E'_2)$ and of $\mathcal{F}(E') \to \mathcal{F}(E'_2)$ that the two equalizers agree. ☐

Given this, we simplify the notation and denote $\mathcal{G}_{(E,\pi)}(S) \subset \mathcal{F}(E)$ the equalizer for any surjection $f\colon E' \to E \times_S E$.

Claim 1.10. *Suppose that we have a diagram $E_1 \xrightarrow{\rho} E \xrightarrow{\pi} S$ where both maps are surjections. Denote by $\pi_1\colon E_1 \to S$ the composition $\pi \circ \rho$. Then the injection $\rho^*\colon \mathcal{F}(E) \to \mathcal{F}(E_1)$ identifies $\mathcal{G}_{(E,\pi)}$ with $\mathcal{G}_{(E_1,\pi_1)}(S)$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
S \arrow[d, "="] & E_1 \arrow[l, "\pi"'] \arrow[d, "\rho"] & E_1 \times_S E_1 \arrow[l, shift right, "p_2"'] \arrow[l, shift left, "p_1"] \arrow[d, "\rho\times\rho"] & E_1' \arrow[l, "f_1"] \\
S & E \arrow[l, "\pi_1"] & E \times_S E \arrow[l, shift right, "p_2"'] \arrow[l, shift left, "p_1"] & E', \arrow[l, "f"]
\end{tikzcd}$$

with all the vertical arrows being surjections. Using the lifting property for E_1', we extend the commutative diagram by adding a map $\rho_1\colon E_1' \to E'$. Using the fibered product constuction and using the fact that $\rho \times \rho$ is a surjection, we can assume wlog that ρ_1 is surjective. This produces the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G}_{(E,\pi)}(S) \arrow[r, "\pi^*"] & \mathcal{F}(E) \arrow[r, shift left, "(p_1\circ f)^*"] \arrow[r, shift right, "(p_2\circ f)^*"'] \arrow[d, "\rho^*"] & \mathcal{F}(E') \arrow[d, "\rho_1^*"] \\
\mathcal{G}_{(E_1,\pi_1)}(S) \arrow[r, "(\pi_1)^*"] & \mathcal{F}(E_1) \arrow[r, shift left, "(p_1\circ f_1)^*"] \arrow[r, shift right, "(p_2\circ f)^*"'] & \mathcal{F}(E_1'),
\end{tikzcd}$$

where the vertical arrows are injections. Since the diagram commutes, it follows that there is an induced injective map

$$\alpha\colon \mathcal{G}_{(E,\pi)}(S) \to \mathcal{G}_{(E_1',\pi_1)}(S).$$

It remains to show that α is onto. Let $h\colon E_2 \to E_1 \times_E E_1$ be a surjective map from an extremely disconnected set, and denote by q_1 and q_2 the projections of $E_1 \times_E E_1 \to E_1$. We must show that any element in $\mathcal{F}(E_1)$ that is in the equalizer of $(p_1 \circ f_1)^*$ and $(p_2 \circ f_1)^*$ is also in the equalizer of $(q_1 \circ h)^*$ and $(q_2 \circ h)^*$. If this is true then it comes from an element of $\mathcal{F}(E)$ and by the injectivity of ρ_1^* this element is in the equalizer of $(p_1 \circ f)^*$ and $(p_2 \circ f)^*$ meaning that it is an element of $\mathcal{G}_{(E,\pi)}(S)$.

But the natural map $\mu\colon E_1 \times_E E_1 \subset E_1 \times_S E_1$ satisfies $p_i \circ \mu = q_i$ for $i = 1, 2$. Using the lifting property we can define a map $\hat{h}\colon E_2 \to E_1'$ making the following diagram commute:

$$\begin{tikzcd}
 & E_1 & \\
E_1 \times_E E_1 \arrow[ur, shift left, "q_1"] \arrow[ur, shift right, "q_2"'] \arrow[rr, "\mu"] & & E_1 \times_S E_1 \arrow[ul, shift right, "p_2"'] \arrow[ul, shift left, "p_1"] \\
E_2 \arrow[u, "h"] \arrow[rr, "\hat{h}"] & & E_1'. \arrow[u, "f_1"']
\end{tikzcd}$$

The result follows immediately. $\square$

Next we surjections $\pi\colon E \to S$ and $\pi_1\colon E_1 \to S$ where neither dominates the other. In this case choose a surjection $E_2 \to E \times_S E_1$. Let $\pi_2\colon E_2 \to S$ be ther resulting map. This allows us to construct an isomorphism $\mathcal{G}_{(E,\pi)}(S)$ and $\mathcal{G}_{(E_1,\pi_1)}(S)$ by showing that each is identified with $\mathcal{G}_{(E_2,\pi_2)}(S)$, and hence they are identified with each other.

A similar argument with fibered products shows that this identification is independent of the choice of E_2 and dominant map $E_2 \to E \times_S E_1$.

Corollary 1.11. *For surjections $\pi\colon E \to S$ and $\pi_1\colon E_1 \to S$ the resulting sets $\mathcal{G}_{(E,\pi)}(S)$ and $\mathcal{G}_{(E_1,\pi_1)}$ are canonically identified.*

We call the result $\mathcal{G}(S)$.

Notice that $\mathcal{G}$ extends $\mathcal{F}$ in the sense that for every extremely disconnected set E we have $\mathcal{G}(E) = \mathcal{F}(E)$. [To see that take the resolution $E \mapsto E$.] Because of this we rename this function now calling it $\mathcal{F}(S)$ for every profinite set S. It remains to prove two things: (i) $\mathcal{F}$ is a functor on $\mathbf{Prof}_\kappa$ and (ii) it satisfies the two sheaf axioms.

Our next task is to show that $\mathcal{F}$ is a functor. This follows easily from an argument completely analogous to the one given in the proof of Claim 1.10. That is to say: given a map $\varphi\colon S \to S'$ we construct the diagram (though the vertical maps are not necessarily surjective in this case). Applying $\mathcal{F}$ yields a commutaive diagram (though the downward arrows are not necessarily injections):

$$\begin{array}{ccccc}
\mathcal{F}(S) & \xrightarrow{\pi^*} & \mathcal{F}(E) & \overset{(f\circ p_1)^*}{\underset{(f\circ p_2^*}{\rightrightarrows}} & \mathcal{F}(E') \\
 & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho_1^*} \\
\mathcal{F}(S') & \xrightarrow{\pi_1^*} & \mathcal{F}(E_1) & \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} & \mathcal{F}(E_1')
\end{array}$$

It follows immediately that there is an induced map $\varphi^*\colon \mathcal{F}(S) \to \mathcal{F}(S')$. Analogous to the arguments above one shows that using different diagrams of extremely discontinuous sets leads to the 'same' function between these sets. This establishes that we have a functor on the category $\mathbf{Prof}_\kappa$ extending the given functor on extremely discontinuous sets.

Lastly, we must show that $\mathcal{F}$ is a sheaf. The first axiom says that $S' \to S$ is a surjection, then $\mathcal{F}(S) \subset \mathcal{F}(S')$ and is the equalizer of p_1^* and p_2^* mapping

$\mathcal{F}(S') \to \mathcal{F}(S' \times_S S')$. We have a commutative diagram:

$$\begin{tikzcd}
S & S' \arrow[l, "\pi"'] & S' \times_S S' \arrow[l, shift right, "p_2"'] \arrow[l, shift left, "p_1"] \\
E \arrow[u, "\pi\circ\rho"] & E \arrow[l, "="] \arrow[u, "\rho"'] & \\
E \times_S E \arrow[u, shift left, "r_1"] \arrow[u, shift right, "r_2"'] & E \times_{S'} E, \arrow[l, "\mathrm{inc}"] \arrow[u, shift left, "q_1"] \arrow[u, shift right, "q_2"'] &
\end{tikzcd}$$

where the p_i, q_i, r_i are induced by the projection of the fibered product onto its i^{th}-factor. Notice that $\mathrm{inc}^* r_i = q_i$ for $i = i, 2$. Also, the map $p_i \circ (\rho \times \rho)\colon E \times_S E \to S$ is $\rho \circ r_i$. Thus, if $a \in \mathcal{F}(S')$ is in the equalizer of p_1^* and p_2^*, then $\rho^* a$ is in the equalizer of r_1^* and r_2^*. Hence $\rho^* a = (\pi \circ \rho)^*(b)$ for some $b \in \mathcal{F}(S)$. The fact that $(\pi \circ \rho)^* b = \rho^* a$ means that $\pi^* b = a$. This completes the proof that $\mathcal{F}(S)$ is the equalizer of p_1^* and p_2^*.

The second sheaf axiom says that $\mathcal{F}(\coprod_i S_i) = \prod_i \mathcal{F}(S_i)$. For each S_i we choose a diagram

$$\begin{tikzcd}
S_i & E_i \arrow[l, "\pi_i"] & E_i \times_{S_i} E_i \arrow[l, shift right, "p_{i,2}"'] \arrow[l, shift left, "p_{i,1}"] & E_i' \arrow[l, "f_i"]
\end{tikzcd}$$

computing $\mathcal{F}(S_i)$ as the equalizer of $(p_{i,1} \circ f_i)^*$ and $(p_{i,2} \circ f_i)^*$. A diagram computing $\mathcal{F}(S)$ can then be taken to be the disjoint union of the diagrams for the S_i,. Since $\mathcal{F}(\coprod_i E_i) = \prod_i \mathcal{F}(E_i)$ it follows easily that $\mathcal{F}(S)$ is identified with $\prod_i \mathcal{F}(S_i)$. This completes the proof that up to isomorphism every $\mathcal{F} \in EDP_\kappa$ extends to a κ-condensed set and only one up to isomorphism.

We turn now to the morphism sets. Suppose that we have a natural transformation $F\colon \mathcal{F} \to \mathcal{G}$ of elements of $\mathbf{EDP}_\kappa$ and extensions of $\mathcal{F}$ and $\mathcal{G}$, denoted $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$, to κ-condernsed abelian groups. We show that $F\colon \mathcal{F} \to \mathcal{G}$ in EDP_κ extends uniquely to a κ-condensed set morphism $\hat{F}\colon \hat{\mathcal{F}} \to \hat{\mathcal{G}}$.

Given $F\colon \mathcal{F} \to \mathcal{G}$ in EDP_κ and given a κ-condensed set S choose a diagram

$$\begin{tikzcd}
S & E \arrow[l, "\pi"] & E \times_S E \arrow[l, shift right, "p_2"'] \arrow[l, shift left, "p_1"] & E' \arrow[l, "f"]
\end{tikzcd}$$

where E and E' are κ-extremely disconnected sets and π and f are surjective.

Then we have a commutative diagram: .

$$\begin{tikzcd}
\hat{\mathcal{F}}(S) \arrow[r, "\pi^*"] & \mathcal{F}(E) \arrow[r, shift left, "(p_1\circ f)^*"] \arrow[r, shift right, "(p_2\circ f)^*"'] \arrow[d] & \mathcal{F}(E') \arrow[d] \\
\hat{\mathcal{G}}(S) \arrow[r, "(\pi)^*"] & \mathcal{G}(E) \arrow[r, shift left, "(p_1\circ f)^*"] \arrow[r, shift right, "(p_2\circ f)^*"'] & \mathcal{F}(E'),
\end{tikzcd}$$

It follows that there is a unique $\hat{F}(S)\colon \hat{\mathcal{F}}(S) \to \hat{\mathcal{G}}(S)$ making the diagram commute. Arguments analogous to those given in the construction of $\hat{\mathcal{F}}$ show that this map is independent of the choice of diagram above S. It now follows easily that the restriction on morphism sets from the category of κ-condensed sets to the category EDP_κ is a bijection.

This completes the proof of the proposition.

Corollary 1.12. *There is an analogous equivalence of categories between of contravariant functions on* $\mathbf{EDP}_\kappa$ *ti the category of abelian groups that sends finite corpoducts to direct sum (=direct product) and the category of* κ*=condensed abelian groups.*

Proof. The proof goes over sets to abelian groups *mutatis mutandis.* □

2 Categorical Properties

2.1 Exactness and Grothendieck Axioms

The equivalence of the categories $Sh(\mathbf{Prof}_\kappa)$ and $\mathbf{EDP}_\kappa$ extends immediately to an equivalence of condensed abelian groups of cardinality less than κ, denoted $\mathbf{Ab}_\kappa$ and contravariant functors from the category of extremely disconnected sets of cardinality less than κ to abelian groups sending finite co-products to finite products, denoted $\mathbf{EDAb}_\kappa$. This latter description allows us to establish the following result.

Theorem 2.1. $\mathbf{EDAb}_\kappa$ *is an abelian category and satisfies Grothendieck's Axioms AB3, AB4, AB5, ABG, and* AB3**.*

Proof. For objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathbf{EDAb}_\kappa$, the set $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a system of homomorphisms $\mathcal{F}(E) \to \mathcal{G}(E)$ compatible with pullbacks as E ranges over extremely disconnected sets of cardinality lessd than κ. These systems are clearly closed under addition of homomorphisms and replacing each homomorphism by its negative. Hence they from an abelian group. Composition is obviously a bilinear map. This shows that $\mathbf{EDAb}_\kappa$ is an additive category.

Every $F\colon \mathcal{F} \to \mathcal{G}$ has a kernel whose value on E is $\mathrm{Ker}(F(E))\colon \mathcal{F}(E) \to \mathcal{G}(E))$. Since by naturality, any map $F\colon \mathcal{F}(\coprod_i E_i) \to \mathcal{G}(\coprod_i E_i)$ is of the form $\prod_i F_i$, it follows in this case that $\mathrm{Ker}(F) = \prod_i \mathrm{Ker}(F_i)$. This implies that $\mathrm{Ker}(F)$ is an element of $\mathbf{EDAb}_\kappa$. A similar argument works for $\mathbf{Im}(F)$ and for the identification of co-domain of F with the image of F, as well as the identification of $\mathbf{Im}(F)$ with the kernel of the map from the co-domain of F to its cokernel. If the kernel of F is non-trivial then the inclusion of the kernel of F into its domain shows that F is not a monomorphism. A similar argument works to show and epimorphism has trivial cokernel. This establishes that $\mathbf{EDAb}_\kappa$ is an abelian category. One consequence of all this is worth pointing out:

Proposition 2.2. *For any κ-extremely disconnected set E, the map $M \to M(E)$ is an exact functor from $\mathbf{Ab}_\kappa$ to the category of abelian groups. Furthermore this functor commutes with all limits and colimits.*

The Grothendieck axioms AB3, AB4, AB5 say that for every indexed family co-products exist in the category, the co-product of a family of monomorphisms is a monomorphism, and that arbitrary filtered colimits preserve exact sequences. Since all these hold for the category of abelian groups and preserve finite co-products (= finite products) and filtered colimits commute with of finite products, it is clear that these conditions hold for $\mathbf{EDAb}_\kappa$. Since products commute with finite products, Axiom $\mathbf{AB3}^*$ (the existence of products) also holds □

3 Compact Projective Generators for $\mathbf{AB}_\kappa$

Definition 3.1. Let T be a condensed set. The presheaf $\mathbb{Z}[T]_{pre}$ associates to a profinite set S the free abelian group generated by $T(S)$. The sheafification of this functor of this functor to an element of $Sh(\mathbf{Prof}_\kappa)$ is denoted $\underline{\mathbb{Z}[T]}$..

Lemma 3.2. *$T \mapsto \underline{\mathbb{Z}[T]}$ is a left adjoint to the forgetful function from $\mathbf{AB}_\kappa \to \mathbf{Prof}_\kappa$. In particular for any condensed abelian group M we have* $\mathrm{Hom}(\underline{\mathbb{Z}[T]}, M) = \mathrm{Hom}(T, M)$.

Proof. Given a morphism of condensed sets $T \to M$, for each profinite set S of cardinality less than κ, we have a set function $T(S) \to M(S)$. By additivity this gives a map $\mathbb{Z}[T(S)] \to M(S)$. These determine an induced map of presheaves of abelian groups $\mathbb{Z}[T]_{pre} \to M$. By the universal property of the sheafification map this is equivalent to a morphism $\underline{\mathbb{Z}[T]} \to M$ in the category $\mathbf{AB}_\kappa$.

The reverse identification comes by pulling back a map $\underline{\mathbb{Z}[T]} \to M$ via the natural map of presheaves $T \to \mathbb{Z}[T]_{pre} \to \underline{\mathbb{Z}[T]}$ to a morphism $T \to M$. It is clear that these are inverse bijections. $\square$

When $T = \underline{S}$, a representable condensed set, we replace $\underline{S}$ in the notation for a free condensed abelian group and call the free group simpl $\underline{\mathbb{Z}[S]}$.

Corollary 3.3. *For any κ-condensed set S, we have $Hom(\underline{\mathbb{Z}[S]}, M)$ is naturally isomoprhic to $M(S) = Hom(S, M)$. That is to say, $\underline{\mathbb{Z}[S]}$ is the* free *κ-condensed abelian group generated by S.*

From now on E is always an **extremely disconnected** space of cardinality less than κ.

Proposition 3.4. *Let E be fixed. Then $\underline{\mathbb{Z}[E]}$ is a projective element in $\mathbf{AB}_\kappa$.*

Proof. Let $F\colon A \to B$ be a surjective morphism in $\mathbf{AB}_\kappa$. A morphism $\varphi\colon \underline{\mathbb{Z}[E]} \to B$ is identified with an element of $B(E)$. The map $A \to B$ is surjective and evaluating at E is an exact functor, thus $F(E)\colon A(E) \to B(E)$ is also surjective., By the adjoint property we see that φ lifts to a morphism $\underline{\mathbb{Z}[E]} \to A$. $\square$

Corollary 3.5. *The collection $\underline{\mathbb{Z}[E]}$ as E ranges over the extremely disconnected sets of cardinality less than κ is a set of projective generators for $\mathbf{AB}_\kappa$. Furthermore, for each E the functor $M \to Hom(\underline{Z[E]}, M)$ commutes with all kernels and colimits. (The latter is the categorical definition of compactness.)*

Proof. We have just seen that the $\underline{\mathbb{Z}[E]}$ are projective. We must show that given any $M \in \mathbf{AB}_\kappa$ there is a co-product of the $\underline{\mathbb{Z}[E_i]}$ mapping onto M. By a simple application of Zorn's lemma, there is a maximal sub-condensed group $M' \subset M$ which is the image of such a co-product. If $M' \neq M$ then the quotient $M/M' \neq 0$ and hence there is an E such that $(M/M')(E) \neq 0$. [IProof: f M is an object of $\mathbf{AB}_\kappa$ and if for every κ-extremely disconnected set of cardinality less than κ, $M(E) = 0$, then $M = 0$. The reason is that by Theorem 1.5 M is determined by its values on extremely disconnected sets of cardinality less than κ.]

Thus, there is a non-trivial map $E \to M/M'$. By the projectivity of $\underline{\mathbb{Z}[E]}$ this map lifts to a map $E \to M$ since E is extremely disconnected. Adding this to the already existing map from a co-product onto $M' \subset M$ gives us a coproduct with an image larger than M', contracting the maximality of

M'. Hence $M' = M$. This proves that every element in $\mathbf{AB}_\kappa$ is the image of a co-product of $\underline{\mathbb{Z}[E_i]}$ which is a projective element.

For each E, the fact that the functor $M \mapsto Hom(\underline{\mathbb{Z}[E]}, M)$ commutes with colimits and kernels follows immediately from the fact that $\underline{\mathbb{Z}[E]}$ is projective and that the functor $M \mapsto M(E)$ commutes with colimits and kernels. □

3.1 Three functorial properties of $\mathbf{AB}_\kappa$

Symmetric Monodial structure. Given M, N objects of $\mathbf{AB}_\kappa$, the functor $S \mapsto M(S) \otimes N(S)$ sheafifies to give a κ-condensed abelian group $M \otimes N$ representing all bilinear morphisms from $M \times N \to \cdot$. This is a symmetric monoidal structure on $\mathbf{AB}_\kappa$. Furthermore, the functor $T \to \underline{\mathbb{Z}}[T]$ from κ-condensed sets to κ-condensed abelian groups $\underline{\mathbb{Z}[T_1 \times T_2]} \cong \underline{\mathbb{Z}[T_1]} \otimes \underline{\mathbb{Z}[T_2]}$ is symmetric monoidal.

Internal Hom. For κ-condensed abelian groups M, N the group $\mathrm{Hom}(M, N)$ has a natural enrichment to a κ-condensed abelian group, denoted $\underline{\mathrm{Hom}}(M, N)$. There is the usual adjunction

$$\mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, P)) = \mathrm{Hom}(M \otimes N, P).$$

Derived Tensor Product. Since $\mathbf{AB}_\kappa$ has enough projectives so does the derived category. This allows us to form the derived tendor product $\otimes^L$.

4 Description of $\underline{\mathbb{Z}[S]}$

Fix $S \in \mathbf{Prof}_\kappa$. We shall give a description of $\underline{\mathbb{Z}[S]}$.

Definition 4.1. Write S as a limit of finite sets, $S = \lim_i S_i$. Let $\mathbb{Z}[S_i]_{\leq n}$ be subset of elements of norm at most n in the natural ℓ^1 norm on this free abelian group with basis $\mathbb{Z}[S_i]$. Explicitly, for $\sum_j n_i s_j$. with the $n_i \in \mathbb{Z}$ and the s_j distinct elements of S_i the norm is $\sum_i |n_i|$. Given a set function $\psi\colon S_j \to S_i$ between finite sets, the induced homomorphism $\widetilde{\psi}\colon \mathbb{Z}[S_j] \to \mathbb{Z}[S_i]$ between abelian groups preserves these ℓ^1-norms. So for each $n \geq 0$ there is an induced map $\widetilde{\psi}_n\colon \mathbb{Z}[S_j]_{\leq n} \to \mathbb{Z}[S_i]_{\leq n}$. The profinite sets $\mathbb{Z}[S_i]_{\leq n}$ represent κ-condensed sets that we denote $\underline{\mathbb{Z}[S_i]_{\leq n}}$.

Theorem 4.2. *For any $S = \lim_i S_i$, with the S_i being finite sets, there is a functorial isomorphism of κ-condensed abelian groups*

$$\underline{\mathbb{Z}[S]} \xrightarrow{\cong} \cup_n \left(\lim_i \underline{\mathbb{Z}[S_i]_{\leq n}} \right) \subset \lim_i \underline{\mathbb{Z}[S_i]}.$$

In particular, $\underline{\mathbb{Z}[S]}$ is a countable union of profinite sets.

Before beginning the proof of the theorem, let us make a coule of preliminary remarks. Implicit in the statement of the theorem is that the right-hand side of the isomorphism is a κ-condensed abelian group not just a κ-condensed set. This group structure is given as follows: The addition in $\lim_i \mathbb{Z}[S_i]$ gives a map of κ-collapsed condensed sets

$$\lim_i \mathbb{Z}[S_i]_{\leq} n \times \lim_i \mathbb{Z}[S_i[s_i]_{\leq m} \to \lim_i \mathbb{Z}[S_i]_{\leq n+m}.$$

inducing an abelian group structure on the colimit of κ-profinite sets $\cup_n \lim_i \mathbb{Z}[S_i]_{\leq n}$. This yields a κ-condensed group

$$\cup_n \underline{\lim_i \mathbb{Z}[S_i]_{\leq n}}$$

that is the countable union of representable κ-condensed sets.

Now we turn to the proof of the theorem.

Proof. Consider the map of abelian groups $\mathbb{Z}[S] \to \lim_i \mathbb{Z}[S_i]$. It is an injection of groups. The reason is that given a non-zero finite sum $\alpha = \sum_j n_j[s_j] \in \mathbb{Z}[S]$ with the s_j disticnt elements of S, there is an S_i in which the images of the s_j are distinct, so that under the projection of the s_j are disticnt. Hence, under the projection $\mathbb{Z}[S] \to \mathbb{Z}[S_i]$ the element α goes non-trivially.

Claim 4.3. *The map $\underline{\mathbb{Z}[S]} \to \lim_i \underline{\mathbb{Z}[S_i]}$ is an injection.*

Proof. To prove this it suffices to prove that for each profinite set T and element $\alpha \in \mathbb{Z}[\underline{S}(T)]$ that maps to zero in $\lim_i \underline{\mathbb{Z}[S_i]}(T)$, there is a covering $\{T_j \to T\}_j$ such that the restriction $\alpha|_{T_j} = 0 \in \mathbb{Z}[\underline{S}(T_j)]]$. We shall prove this by induction on the number of terms in the expression $\alpha = \sum_i n_i f_i$, where $n_i \in \mathbb{Z}$ and $f_i \colon T \to S$.

If there is only one term, i.e., if $\alpha = nf$, then it is clear that if the image of α in any $\mathbb{Z}[\underline{S_i}(T)]$ is zero if and only if $n = 0$.

Suppose for some $k \geq 2$, for all expressions $\alpha = \sum_{i=1}^r n_i f_i$ for $r < k$, elements that map to zero in $\lim_i \underline{\mathbb{Z}[S_i]}(T)$ there is a covering $\{T_j \to T\}_j$ such that $\alpha|_{T_j} = 0$ for all j. Let $\alpha = \sum_{i=1}^k n_i f_i$ map to zero in $\lim_i \underline{\mathbb{Z}[S_i]}$. For each $1 \leq j < j' \leq n$, let $T_{jj'}$ be the set of $t \in T$ where $f_j(t) = f_{j'}(t)$. Clearly the restriction of α to any $T_{jj'}$ can be written as an expression with at most $k-1$ terms. Thus, by the inductive hypothesis, for each $T_{jj'}$ there is a covering $\{T_{rjj'} \to T_{jj'}\}_r$ with $\alpha|_{T_{rjj'}} = 0$ for all rjj'. It remains only to prove that $\{T_{jj'} \to T\}_{1 \leq j < j' \leq k} \to T$ is a covering. Since the $T_{jj'}$ are closed subsets of T, we need only show $\cup_{jj'} T_{jj'} = T$. But if there is $t \in T$ not it $\cup_{jj'} T_{jj'}$, then the functions $s_j = f_j(t)$ are distinct. There is a projection $S \to S_i$ such that the s_j are all distinct, and hence the image of α in $\mathbb{Z}[\underline{S_i}[T]]$ is non-trivial since its value at $t \in T$ is a non-trivial element of $\mathbb{Z}[S_i]$. This completes the inductive proof. $\square$

Since element of $\mathbb{Z}[\underline{S}(T)]$ can be written as $\sum_i n_i f_i$ for some $n_i \in \mathbb{Z}$ and $f_i \in \underline{S}(T)$. Such an expression maps to an element of $\mathbb{Z}[\underline{S}(T)]_{\leq N}$ where $N = \sum_i |n_i|$. Hence, the image of $\mathbb{Z}[S]_{pre}$ under the map to $\lim_i \underline{\mathbb{Z}[S_i]}$ is contained in

$$\cup_n \lim_i \underline{\mathbb{Z}[S_i]_{\leq n}}.$$

It follows by the universal property of sheafification that the map $\underline{\mathbb{Z}[S]} \to \lim_i \underline{\mathbb{Z}[S_i]}$ factors through a map

$$\underline{\mathbb{Z}[S]} \to \cup_n \lim_i \underline{\mathbb{Z}[S_i]_{\leq n}} \subset \lim_i \underline{\mathbb{Z}[S_i]},$$

with the first map being an injection.

We need to show that the inclusion

$$\underline{\mathbb{Z}[S]} \to \cup_n \lim_i \underline{\mathbb{Z}[S_i]_{\leq n}}$$

is onto.

Let $(S \times \{-1,0,1\})^n$ denote the product of n-copies of $S \times \{-1,0,1\}$. Clearly this is $\lim_i ((S_i \times \{-1,0,1\})^n)$. For each i, the map $(S_i \times \{-1,0,1\})^n \to \mathbb{Z}[S_i]_{\leq n}$ that sends the element $\{s_1, \ldots, s_n, a_1, \ldots, a_n\}$, where the $s_j \in S_i$ and $a_j \in \{-1,0,1\}$, to $\sum_{j=1}^n a_j[s_j]$ is a surjection. Hence,

$$\lim_i ((S_i \times \{-1,0,1\})^n) \to \lim_i (S_i)_{\leq n}$$

is surjective.

We have a commutative diagram of sheaves:

$$\begin{array}{ccc}
\underline{(S \times \{-1,0,1\})^n} & \longrightarrow & \lim_i \underline{(S_i \times \{-1,0,1\})^n} \\
\downarrow & & \downarrow \\
\underline{\mathbb{Z}[S]} & \longrightarrow & \cup_n \left(\lim_i \underline{\mathbb{Z}[S_i]_{\leq n}}\right).
\end{array}$$

The vertical morphism on the right maps to $\lim_i \underline{\mathbb{Z}[S_i]_{\leq n}}$ and is induced by a surjection of profinite sets. Hence, it is a surjective morphism of sheaves onto $\lim_i \mathbb{Z}[S_i]_{\leq n}$. It follows that the image of the bottom map of sheaves contains $\lim_i \underline{\mathbb{Z}[S_i]_{\leq n}}$. Since this is true for all n, the bottom map of sheaves is onto.

This completes the proof of the theorem. $\square$

Let's analyze this a little more closely, considering $\mathbb{Z}[*]$. The presheaf associates to $S \in \mathbf{Prof}_\kappa$ the group $\mathbb{Z}$ and to a function $S \to S'$ the identity homomorphism. There is a map of this presheaf to the sheaf $S \mapsto Cont(S, \mathbb{Z})$, which is a condensed abelian group. The map sends $\mathbb{Z} = \mathbb{Z}[*](S)$ isomorphically onto to the constant functions on S. Since this is an embedding, it follows that the sheaf $\underline{\mathbb{Z}[*]}$ is a subsheaf of $S \mapsto Cont(S, \mathbb{Z})$. On the other hand, given a continuous function $f\colon S \to \mathbb{Z}$ it has only finitely many images, so that we have a finite decomposition $S = \coprod_n S_n$ where S_n is the pre-image of n and n ranges over the finite set in the image of the map. This gives us a covering $\{S_n \to S\}_n$ and f pulls back to a function constant on each S_n, meaning that it is an element in $\prod_n \mathbb{Z}[*](S_n)$ whose image in $Cont(S, \mathbb{Z})$ is f. This proves that the sheaf $\underline{\mathbb{Z}[*]}$ is the sheaf of continuous functions $Cont(S, \mathbb{Z})$. The same type of argument shows that if S_i is a finite set then $\underline{\mathbb{Z}[S_i]}$ is the sheaf that assigns to the profinite set S the group of continuous functions from S to $\mathbb{Z}[S_i]$. The condensed set represented by the finite subset $\mathbb{Z}[S_i]_{\leq n}$ assigns to a profinite set S the continuous functions $S \to \mathbb{Z}[S_i]_{\leq n}$. This is a subsheaf of $\underline{\mathbb{Z}[S_i]}$.