CONDENSED COHOMOLOGY

In the last lectures we introduced the categories of condensed sets and condensed abelian groups and proved that they are better behaved replacements of the categories of T_1-topological spaces and topological abelian groups respectively. The goal of this talk is to give more evidence that this is a good replacement by studying cohomological invariants.

1. Some cohomological preliminaries

We briefly recall the definition of (augmented) simplicial object, a splitting of an augmented simplicial object, and the notion of hypercover.

1.1. **Hypercovers and simplicial objects.**

Definition 1.1.1. Let $\mathscr{C}$ be a category.

(1) Let Δ denote the category whose objects are the sets $[n] = \{0, \ldots, n\}$ for $n \in \mathbb{N}$, and arrows given by order preserving maps. A simplicial object in $\mathscr{C}$ is a functor $F : \Delta^{op} \to \mathscr{C}$. We denote by $(S_n)_{[n]\in\Delta^{op}}$ or $S_\bullet$ a simplicial object in $\mathscr{C}$, with $S_n = F([n])$.

(2) We let $\Delta_+ = \Delta \bigsqcup \{\emptyset\}$ be the category Δ added with an initial object (eq. the empty set). An augmented simplicial object in $\mathscr{C}$ is a functor $F : \Delta_+^{op} \to \mathscr{C}$, we let $S_{-1} = F(\emptyset)$ be the augmented object. We let $(S_n)_{[n]\in\Delta_+^{op}}$ or $S_\bullet \xrightarrow{\varepsilon} S_{-1}$ denote an augmented simplicial set.

(3) We let Δ_* be the category whose objects are sets $\langle n\rangle := \{-1, 0, \ldots, n\}$, and whose morphisms are order preserving maps sending -1 to -1. We have a natural functor

$$\Delta_+ \to \Delta_*$$

by mapping $\emptyset \mapsto \{-1\}$ and $[n] \mapsto \langle n\rangle$. We say that an augmented simplicial object $F : \Delta_+^{op} \to \mathscr{C}$ is split if it admits an extension to a functor $\tilde{F} : \Delta_*^{op} \to \mathscr{C}$. We call $\tilde{F}$ a splitting of F. We let $S_{-1} \xrightarrow{\mu} S_\bullet \xrightarrow{\varepsilon} S_{-1}$ denote the splitting of the augmented simplicial set $S_\bullet \xrightarrow{\varepsilon} S_{-1}$.

1.1.1. *Dold-Kan correspondence.* Let Ab be the category of abelian groups and $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ the category of chain complexes concentrated in homological degrees ≥ 0. Given a simplicial abelian group $A_\bullet$ we can form the Moore complex $MA_\bullet$ with with n-th term A_n and differential $d : A_n \to A_{n-1}$ given by the alternating sum

$$d = \sum_{i=0}^{n} (-1)^i d_i^n$$

where $d_i^n : A_n \to A_{n-1}$ is the i-th face map. The Moore complex functor M has a right adjoint F and the Dold-Kan correspondence gives rise a Quillen equivalence

$$M : \mathrm{Func}(\Delta^{op}, \mathrm{Ab}) \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab}) : F.$$

In particular, given a simplicial abelian group $A_\bullet$, we have natural equivalences $\pi_i(A_\bullet) \cong H^{-i}(MA_\bullet)$. Furthermore, given $A_\bullet \xrightarrow{\varepsilon} A_{-1}$ an augmented simplicial abelian group, any splitting $A_{-1} \xrightarrow{\mu} A_\bullet$ gives rise an homotopy equivalence $MA_\bullet \xrightarrow{\sim} A_{-1}$. Explicitly, the homotopy inverse $h_n A_n \to A_{n+1}$ arises from the map $\langle n+1\rangle \to \langle n\rangle$ sending $i \mapsto i-1$ for $i = 0, \ldots, n$. Furthermore, the Dold-Kan correspondence hold for any abelian category $\mathscr{C}$, so by taking $\mathscr{C} = \mathrm{Ab}^{op}$ it also defines a Quillen equivalence

$$M : \mathrm{Func}(\Delta, \mathrm{Ab}) \to \mathrm{Ch}_{\leq 0}(\mathrm{Ab}) : F$$

between cosimplicial abelian groups and coconective chain complexes.

1.1.2. *Hypercovers.* Let $\mathscr{C}$ be a category with finite limits. Given $n \in \mathbb{N}$, a $(\leq n)$-simplicial object in $\mathscr{C}$ is a functor $F : \Delta^{op}_{\leq n} \to \mathscr{C}$ where $\Delta_{\leq n} \subset \Delta$ is the full subcategory whose objects are the totally order sets $[i]$ for $i = 0, \ldots, n$. Similarly, we can define augmented truncated simplicial objects $\mathrm{Func}(\Delta^{op}_{+,\leq n}, \mathscr{C})$. For $n \geq -1$, the restriction functor $\mathrm{Func}(\Delta^{op}_{+}, \mathscr{C}) \to \mathrm{Func}(\Delta^{op}_{+,\leq n}, \mathscr{C})$ has a fully faithful right adjoint $\mathrm{cosk}_n : \mathrm{Func}(\Delta^{op}_{+,\leq n}, \mathscr{C}) \to \mathrm{Func}(\Delta^{op}_{+}, \mathscr{C})$ called the n-coskeleton functor. An augmented (≤ -1)-simplicial object is the same as an object in $\mathscr{C}$ and the (-1)-coskeleton functor is nothing but the constant augmented simplicial set $X \mapsto (X)_{[n]\in\Delta^{op}_{+}}$. Given $S_0 \to S_{-1}$ an augmented (≤ 0)-simplicial object, the 0-coskeleton functor is the Čech nerve $(S_0^{\times_{S_{-1}} n+1})_{[n]\in\Delta^{op}}$. For $n \geq 1$ an explicit description of the coskeleton functor is not easy to give.

When $\mathscr{C} = \mathrm{Sets}$, and $* \to S_\bullet$ is a pointed augmented simplicial object, the natural map $S_\bullet \to \mathrm{cosk}_n S_\bullet$ gives rise a bijection on connected components $\pi_{-1} S_\bullet = \pi_{-1} \mathrm{cosk}_n S_\bullet$ for $n \geq 0$, and equivalences on homotopy groups $\pi_i S_\bullet \cong \pi_i \mathrm{cosk}_n S_\bullet$ for $0 \geq i < n$. If $X_\bullet$ is an augmented simplicial object in Sets, one has that $X_\bullet = \varprojlim_n \mathrm{cosk}_n X_\bullet$, this is called the Postnikov tower of $X_\bullet$.

Definition 1.1.2. Let $\mathscr{C}$ be a site with finite limits, and let $X \in \mathscr{C}$. An hypercover of X is an augmented simplicial set $X_\bullet \xrightarrow{\varepsilon} X$ such that $X_0 \to X_{-1} := X$ is a cover, and such that for all $n \in \mathbb{N}$ the map $X_{n+1} \to (\mathrm{cosk}_n X_\bullet)_{n+1}$ is a cover.

Theorem 1.1.3 ([Sta22, Tag 01H0]). *Let $\mathscr{C}$ be a site with finite limits, $X \in \mathscr{C}$ and $\mathscr{F}$ an abelian sheaf on $\mathscr{C}$. Then there is a natural equivalence*

$$R\Gamma(X, \mathscr{F}) = \varinjlim_{X_\bullet \to X} R\check{\Gamma}(X_\bullet, \mathscr{F}),$$

where $X_\bullet \to X$ runs over all the hypercovers of X, and $R\check{\Gamma}(X_\bullet, \mathscr{F})$ is the Čech cohomology of the hypercover $X_\bullet$ obtained via the Dold-Kan correspondence of the cosimplicial abelian group $\mathscr{F}(X_\bullet)$.

2. Discrete condensed cohomology

2.1. **Condensed cohomology.** Let CondSet and CondAb be the category of condensed sets and condensed abelian groups respectively. Given $T \in \mathrm{CondSet}$ and $M \in \mathrm{CondAb}$ one can define the *condensed cohomology* of T with values in M:

Definition 2.1.1. We define the condensed cohomology $R\Gamma_{cond}(T, M) = R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[T], M)$. We let

$$H^i_{cond}(T, M) := \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}[T], M)$$

denote the condensed cohomology groups. We can upgrade condensed cohomology from an abelian group to a condensed abelian group by taking internal Hom's:

$$R\underline{\Gamma}_{cond}(T, M) := R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[T], M).$$

Let us now explain how to compute condensed cohomology in practice, for simplicity let us assume that T is a quasi-separated condensed set. By [CS20, Proposition 1.2 (4)] the category of quasi-separated condensed sets is equivalent to the category of ind-compact Hausdorff spaces. Writing $T = \varinjlim_i T_i$ as a filtered colimit of compact Hausdorff spaces we formally find that

$$R\Gamma_{cond}(T, M) = R\lim_i R\Gamma(T_i, M).$$

Now, let us assume that T is a compact Hausdorff space and let $S_\bullet \to T$ be an hypercover by extremally disconnected sets. By construction, we have a weak equivalence of simplicial condensed sets $(S_n)_{[n]\in\Delta^{op}} \simeq (T)_{[n]\in\Delta^{op}}$, taking free condensed abelian groups we get a weak equivalence $(\mathbb{Z}[S_n])_{[n]\in\Delta^{op}} \simeq (\mathbb{Z}[T])_{[n]\in\Delta^{op}}$ which by the Dold-Kan correspondence produces a projective resolution of $\mathbb{Z}[T]$

$$\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[T] \to 0.$$

The objects $\mathbb{Z}[S^n]$ are projective condensed abelian groups, and taking Hom spaces we find an equivalence

$$R\Gamma_{cond}(T, M) \cong [M(S_0) \to M(S_1) \to \cdots].$$

A different procedure to compute condensed cohomology is by taking injective resolutions in a suitable sense: there exists some strong limit cardinal κ such that T and M are κ-small condensed sets. By [Sch19, Proposition 2.9] the mapping space from $\mathbb{Z}[T]$ towards M is the same when considered as κ'-condensed sets for any $\kappa' > \kappa$ strong limit cardinal. Thus, $R\Gamma_{cond}(T, M)$ can be compute as the cohomology in the category of κ-small condensed sets $\mathrm{CondSet}_\kappa$. But $\mathrm{CondSet}_\kappa$ is the category of abelian sheaves of the small Grothendieck site of κ-small profinite sets, thus the category of κ-small condensed abelian groups CondAb_κ has enough injectives ([Sta22, Tag 01DP]) and we can take an injective resolution of M in order to compute $R\Gamma_{cond}(T, M)$. This approach, however, is less explicit than resolving T using extremally disconnected sets.

Remark 2.1.2. Note that using projective resolutions of $\mathbb{Z}[T]$ by extremally disconnected sets only helps to compute the condensed cohomology as abelian groups and not its condensed upgrade. The problem is that the category of extremally disconnected sets is not stable under products (cf. [Sch19, Warning 2.6]), and that computing the S-points of $R\underline{\Gamma}_{cond}(T, M)$ is the same as computing $R\Gamma_{cond}(T \times S, M)$. However, one can still use injective resolutions of M in order to compute the condensed upgrade of condensed cohomology, namely, if T and S are κ-small condensed sets then so is $T \times S$. Later in the talks we will see that after working with solid abelian groups one can actually use projective resolutions by extremally disconnected sets to compute the condensed upgrade of condensed cohomology.

2.2. **Comparison with sheaf cohomology.** Let T be a compact Hausdorff space, a very important topological invariant of T is its sheaf cohomology $R\Gamma_{sheaf}(T, \mathbb{Z})$. When T is a CW complex sheaf cohomology also agrees with singular cohomology, but this breaks for more general spaces: if T is a profinite set then $H^0_{sheaf}(T, \mathbb{Z})$ consists on continuous maps $S \to \mathbb{Z}$ while $H^0_{sing}(T, \mathbb{Z})$ are all maps $T \to \mathbb{Z}$. Our next task is to show that sheaf and condensed cohomology are naturally isomorphic.

Lemma 2.2.1. *Let S be a compact Hausdorff space, let S_{cl} denote the Grothendieck site whose objects are closed subspaces and covers given by finite jointly surjective maps that admit an open refinement. Then there is a natural equivalence*

$$R\Gamma_{sheaf}(S, \mathbb{Z}) \cong R\Gamma(S_{cl}, \mathbb{Z}).$$

Proof. Let S_{open} be the site of open subspaces of S, and let S_{clopen} be the site consisting of locally closed subspaces and covers given by finitely many jointly surjective maps that admit an open refinement. We have natural continuous maps of sites

$$S_{open} \leftarrow S_{clopen} \rightarrow S_{cl}.$$

By definition of the topology on S_{clopen}, any cover of S in S_{clopen} is refined by an open cover, this produces the equivalence $R\Gamma_{sheaf}(S, \mathbb{Z}) \cong R\Gamma(S_{clopen}, \mathbb{Z})$. To prove the equivalence $R\Gamma(S_{clopen}, \mathbb{Z}) \cong R\Gamma(S_{cl}, \mathbb{Z})$ it suffices to see that any open cover of S admits a refinement given by finitely many jointly surjective closed subspaces that admit a refinement by an open cover. This follows from the fact that for any $s \in S$ and any open neighbourhood $s \in U \subset S$, one can construct subspaces $s \in U' \subset Z \subset U$ with Z closed in S and U' open. □

Lemma 2.2.2. *Let $S = \varprojlim_I S_i$ be a cofiltered limit of compact Hausdorff spaces. Then there is a natural equivalence*

$$R\Gamma_{sheaf}(S, \mathbb{Z}) = \varinjlim_{i \in I} R\Gamma_{sheaf}(S_i, \mathbb{Z}).$$

Proof. By Theorem 1.1.3 we can compute sheaf cohomology of S as the colimit of Čech cohomologies of open hypercovers

$$R\Gamma_{sheaf}(S, \mathbb{Z}) = \varinjlim_{U_\bullet \to S} R\Gamma(U_\bullet, S).$$

Moreover, for computing $H^k_{sheaf}(S, \mathbb{Z})$, it suffices to consider the colimit along those open hypercovers of the form $\mathrm{cosk}_{k+1}(U_n)_{[n] \in \Delta^{op}_{\leq k+1}}$. Since S is compact Hausdorff, by Lemma 2.2.1 we can also use closed hypercovers to compute sheaf cohomology, and thus, after refining again by open hypercovers, we can suppose that each U_n is a finite union of open subspaces of S. Then, since $S = \varprojlim_i S_i$, for a fixed k, the $(\leq k+1)$-simplicial hypercover $(U_n)_{[n] \in \Delta^{op}_{\leq k+1}}$ of S arises from the pullback of some $(\leq k+1)$-simplicial hypercover $(U_{n,i})_{[n] \in \Delta^{op}_{\leq k+1}}$ for some $i \in I$. This proves that the $(\leq k+1)$-simplicial open hypercovers of S that arise from some S_i are cofinal, and that we have a natural equivalence of truncated complexes

$$\varinjlim_i \tau^{\leq k} R\Gamma_{sheaf}(S_i, \mathbb{Z}) \cong \tau^{\leq k} R\Gamma_{sheaf}(S, \mathbb{Z}).$$

Since this holds for all $k \in \mathbb{N}$, we have the equivalence

$$\varinjlim_i R\Gamma_{sheaf}(S_i, \mathbb{Z}) \cong R\Gamma_{sheaf}(S, \mathbb{Z})$$

as wanted. $\square$

Theorem 2.2.3 ([Sch19, Theorem 3.2])**.** *Let S be a compact Hausdorff space and let $S_{cond} = \mathbf{Haus}_{/S}$ be the site of compact Hausdorff spaces lying over S, with covers given by jointly surjective maps. The natural morphism of sites $S_{cond} \to S_{cl}$ induces a natural equivalence*

$$R\Gamma_{sheaf}(S, \mathbb{Z}) \cong R\Gamma(S_{cl}, \mathbb{Z}) \xrightarrow{\sim} R\Gamma(S_{cond}, \mathbb{Z}) = R\Gamma_{cond}(S, \mathbb{Z}).$$

Proof. Let us first prove the Theorem when S is extremally disconected. Let us write $S = \varprojlim_i S_i$ as a limit of finite sets. By Lemma 2.2.2 we know that $R\Gamma_{sheaf}(S, \mathbb{Z}) = \varinjlim_i R\Gamma_{sheaf}(S_i, \mathbb{Z}) = C^{lc}(S, \mathbb{Z})$ is the space of locally constant functions of S with values in $\mathbb{Z}$. On the other hand, we have that $R\Gamma_{cond}(S, \mathbb{Z}) = C(S, \mathbb{Z}) = C^{lc}(S, \mathbb{Z})$ is the same as continuous functions from S to $\mathbb{Z}$, or equivalently, locally constant functions. Slightly more generally, if S is profinite set we have that $R\Gamma_{sheaf}(S, \mathbb{Z}) = C^{lc}(S, \mathbb{Z})$, and by taking an hypercover by extremally disconnected sets $S_\bullet \to S$ we find that $R\Gamma_{cond}(S, \mathbb{Z})$ is represented by the complex

$$[C^{lc}(S_0, \mathbb{Z}) \to C^{lc}(S_1, \mathbb{Z}) \to \cdots],$$

we claim that this complex is concentrated in degree 0 and quasi-isomorphic to $C^{lc}(S, \mathbb{Z})$. Indeed, let us write $S = \varprojlim_i S_i$ as a limit of finite sets. We can then write $S_\bullet = \varprojlim_i S_{\bullet,i}$ such that $S_{\bullet,i} \to S_i$ is an hypercover by finite sets. Since S_i is finite, $S_{\bullet,i} \to S_i$ is split and we have an equivalence

$$C(S_i, \mathbb{Z}) \cong [C(S_{0,i}, \mathbb{Z}) \to C(S_{1,i}, \mathbb{Z}) \to \cdots].$$

Taking colimits in i we deduce the claim.

Now we prove the theorem for a general compact Hausdorff space S. Consider the morphism of sites $\alpha : S_{cond} \to S_{cl}$, it suffices to prove that $R\alpha_* \mathbb{Z} = \mathbb{Z}$. This can be proved at the stalks of S, so let $s \in S$, we want to show that $(R\alpha_* \mathbb{Z})_s = \mathbb{Z}$, but we have that

$$(R\alpha_* \mathbb{Z})_s = \varinjlim_{s \in V} R\Gamma_{cond}(V, \mathbb{Z})$$

with V running along all closed neighbourhoods of s in S. Let $S_\bullet \to S$ be an hypercover by profinite sets, then $V_\bullet = S_\bullet \times_S V$ is an hypercover of V by profinite sets, and since condensed $\mathbb{Z}$-cohomology of profinite sets is acyclic, we have that

$$R\Gamma_{cond}(V, \mathbb{Z}) = R\check{\Gamma}(V_\bullet, \mathbb{Z}).$$

By Lemma 2.2.2 we find that

$$\begin{aligned}
(R\alpha_* \mathbb{Z})_s &= \varinjlim_{s \in V} R\Gamma_{cond}(V, \mathbb{Z}) \\
&= \varinjlim_{s \in V} R\check{\Gamma}(V_\bullet, \mathbb{Z}) \\
&= R\check{\Gamma}(S_{\bullet,s}, \mathbb{Z}) \\
&= R\Gamma_{cond}(s, \mathbb{Z}) \\
&= \mathbb{Z}
\end{aligned}$$

as wanted. $\square$

Convention 2.2.4. Let S be a compact Hausdorff space and M a discrete abelian group, from now on we will simply write $R\Gamma(S, M)$ for the condensed/sheaf/Čech cohomology of S with values in M.

Remark 2.2.5. Theorem 2.2.3 holds for any discrete abelian group M. Namely, M has a resolution by free abelian groups, and since a compact Hausdorff space is a qcqs condensed set, $R\Gamma(S, -)$ commutes with direct sums of condensed abelian groups.

We record the following computation of condensed/sheaf cohomology of products of tori for future talks.

Proposition 2.2.6 ([Sch19, Proposition 3.1])**.** *Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then for any set I we have that*

$$H^i(\prod_I \mathbb{T}, \mathbb{Z}) = \bigwedge^i(\bigoplus_I \mathbb{Z}).$$

More precisely, the isomorphism $H^1(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}$ induces an isomorphism $\bigoplus_I \mathbb{Z} = H^1(\prod_I \mathbb{T}, \mathbb{Z})$ by taking pullbacks along each factor, and the cup product induces isomorphisms

$$\bigwedge^i(\bigoplus_I \mathbb{Z}) = H^i(\prod_I \mathbb{T}, \mathbb{Z}).$$

Proof. The result is well known when I is finite. For the general case, write I as a filtered colimit of finite sets and use Lemma 2.2.2. □

3. Condensed cohomology on Banach spaces

We have proven that condensed cohomology of compact Hausdorff spaces with values in discrete abelian groups computes sheaf cohomology. In this section we will study the disjoint situation of condensed cohomology with values in real Banach spaces.

3.1. **Vanishing of cohomology.** We shall prove the following theorem

Theorem 3.1.1 ([Sch19, Theorem 3.3])**.** *Let V be a real Banach space seen as a condensed abelian group. For any compact Hausdorff space S we have that*

$$H^i(S, V) = 0$$

for $i > 0$, while $H^0(S, V) = C(S, V)$ is the space of continuous functions from S to V.

More precisely, if $S_\bullet \to S$ is an hypervover of S by profinite sets S_n, the complex of Banach spaces

$$0 \to C(S, V) \to C(S_0, V) \to C(S_1, V) \to \cdots$$

satisfies the following quantitative version of exactness: if $f \in C(S_n, V)$ satisfies $df = 0$ and $\varepsilon > 0$, then there exists $g \in C(S_{n-1}, V)$ with $dg = f$ and such that $||g|| \leq (1 + \varepsilon)||f||$ (where we endow $C(-, V)$ with the supremum norm).

Remark 3.1.2. Let S be a compact Hausdorff space. Theorem 3.1.1 implies that $R\Gamma_{cond}(S, \mathbb{R}) = C(S, \mathbb{R})$, where we see $\mathbb{R}$ as a condensed abelian group. We have the same vanishing result in sheaf cohomology, namely, let $\mathscr{O}_S$ be the sheaf of real-continuous functions of S, then $R\Gamma_{sheaf}(S, \mathscr{O}_S) = C(S, \mathbb{R})$. Indeed, this follows from the fact that $\mathscr{O}_S$ is a flasque sheaf as it has partitions of unity. The existence of partitions of unity is also a key tool used in the proof of the theorem.

3.1.1. *Tietze's extension theorem.* We shall need the following variation of Tietze's extension theorem[1].

Proposition 3.1.3. *Let X be a compact Hausdorff space, V a real Banach space and $A \subset X$ a closed subspace. Then for any $\varepsilon > 0$, a continuous map $f : A \to V$ can be extended to a continuous map $\widetilde{f} : X \to V$ such that $||\widetilde{f}||_{\sup} \leq (1 + \varepsilon)||f||_{\sup}$.*

Proof. The following was taken from an answer in Mathoverflow of Bill Johnson. If V is finite dimensional we are reduced to the usual Tietze's extension theorem. Suppose that V is infinite dimensional. We can replace V with the closure of the Banach space generated by $f(A)$, in particular, since a compact subspace of a Banach space is separable, we can assume that V is separable. Then, by the Anderson-Kadec theorem [BP75, pg. 189], any infinite dimensional separable Banach space is homemorphic to $\mathbb{R}^{\mathbb{N}}$, so in order to find an extension of f are are reduced to find an extension for each projection onto $\mathbb{R}$ which is the classical Tietze's extension theorem.

Finally, in order to control the norm in terms of the original function f, once we have an extension $\widetilde{f} : X \to V$, we can find a neighbourhood $A \subset U \subset X$ such that $||\widetilde{f}|_U|| \leq (1 + \varepsilon)||f||$, and then use Urysohn's lemma to extend $\widetilde{f}|_U$ to a function $\widetilde{f}' : X \to V$ such that $||\widetilde{f}'|| \leq (1 + \varepsilon)||f||$. □

[1]We shall restrict ourselves to the case of compact Hausdorff spaces

3.1.2. *Proof of Theorem 3.1.1.*

Proof. We first prove the statement when S is a finite set and $S_\bullet \to S =: S_{-1}$ is an hypercover by finite sets. Since S is finite, one can find an splitting $S \to S_\bullet$ which induces a contractible homotopy equivalence of Banach spaces

$$h_n : C(S_n, V) \to C(S_{n+1}, V).$$

Then, if $f \in C(S_n, V)$ is such that $df = 0$, we have that $g = h_n(f)$ is such that $dg = f$ and $||g|| \leq ||f||$ proving what we wanted.

Let us now suppose that $S = \varprojlim_i S_i$ is a profinite set written as a limit of finite sets, we can then write the hypercover $S^\bullet \to S$ as a cofiltered limit of hypercovers of S_i by finite sets.

$$S^\bullet = \varprojlim_i S_i^\bullet.$$

We endow $\varinjlim_i C(S_{n,i}, V)$ with the norm given by the eventually constant colimit of the supremum norms of $C(S_{n,i}, V)$. Then the map $\varinjlim_i C(S_{n,i}, V) \to C(S_n, V)$ is an isometric dense embedding. We want to pass to completions in order to deduce the quantitative vanishing for $C(S_\bullet, V)$

Let $f \in C(S_n, V)$ be such that $df = 0$, we can find $f_i \in C(S_{n,i}, V)$ so that $||f - f_i|| \leq \varepsilon ||f||$. Then $df_i = d(f_i - f)$ has norm $\leq (n+2)\varepsilon||f||$ as d is an alternating sum of $n+2$ restriction maps along the face maps $S_{n+1} \to S_n$. By the case of finite sets, there is $l_i \in C(S_{n,i}, V)$ such that $dl_i = df_i$ and $||l_i|| \leq (n+2)\varepsilon||f||$. Then, after modifying f_i, we can assume that $df_i = 0$ and $||f - f_i|| \leq (n+3)\varepsilon||f||$. Now, by the above step there is $g^{(0)} \in C(S_{n-1,i}, V)$ such that $dg^{(0)} = f_i$ and $||g^{(0)}|| \leq ||f_i|| \leq (1 + (n+3)\varepsilon)||f||$. Up to replacing ε we can assume that $||g^{(0)}|| \leq (1+\varepsilon)||f||$ and $||f - dg^{(0)}|| \leq \varepsilon||f||$. Letting $f^{(1)} = f - dg^{(0)}$ and repeating the process we find a convergence sequence

$$f = d(g^{(0)} + g^{(1)} + \cdots) = dg$$

with $||g^{(m)}|| \leq (1+\varepsilon)||f^{(m)}|| \leq \varepsilon^m(1+\varepsilon)||f||$ and so

$$||g|| \leq \frac{1+\varepsilon}{1-\varepsilon}||f||;$$

up to rescaling ε we get the theorem.

Finally, we prove the theorem for S a compact Hausdorff space. Let $s \in S$ and consider the fiber $S_s^\bullet$. Let $f \in C(S_n, V)$ be such that $df = 0$. Then $df_s = 0$ and by the previous step there is $g_s \in C(S_{n-1,s}, V)$ such that $dg_s = f_s$ and $||g_s|| \leq (1+\varepsilon)||f_s||$. By Tietze's extension theorem (cf. Proposition 3.1.3) there is an open neighbourhood $s \in U_s \subset S$ and an extension $\tilde{g}_s$ of g_s to S_n supported on U_s such that $||\tilde{g}_s|| \leq (1+\varepsilon)||g_s||$. Moreover, we can assume that $||(d\tilde{g}_s - f)|_{U_s}|| \leq \varepsilon||f||$. By compactness, finitely many U_s cover S, say $U_1, \ldots, U_k$ covering S and $g_1, \ldots, g_k \in C(S_{n-1,V})$ with $||g_j|| \leq (1+\varepsilon)^2||f||$ (and after modifying ε with $||g_j|| \leq (1+\varepsilon)||f||$) for $j = 1, \ldots, k$ such that

$$||(dg_j - f)|_{S_n \times_S U_j}|| \leq \varepsilon||f||.$$

Choose a partition of unity $1 = \sum_{j=1}^k \rho_j$ of functions $\rho_j \in C(S_n, \mathbb{R})$ supported in $\{U_j\}_{j=1}^k$, and set $g^{(0)} = \sum_{j=1}^k \rho_j g_j$. Then

$$||g^{(0)}|| \leq (1+\varepsilon)||f||$$

and

$$||dg^{(0)} - f|| = ||\sum_{j=1}^k \rho_j(dg_j - f)|| \leq \varepsilon||f||.$$

Letting $f^{(1)} = f - g^{(0)}$ and repeating the process we can write

$$f = d(g^{(0)} + g^{(1)} + \cdots) = dg$$

with $||g^{(m)}|| \leq (1+\varepsilon)||f^{(m)}|| \leq \varepsilon^m(1+\varepsilon)||f||$. In particular

$$||g|| \leq \frac{1+\varepsilon}{1-\varepsilon}||f||.$$

Up to redefining ε, one obtains the theorem. $\square$

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