3. Kähler manifolds

A Kähler manifold is a complex manifold M with a Kähler form ω which is closed ($d\omega = 0$). A Kähler form is equivalent to a Hermitian metric h. We define these and show how they are related on a single vector space V, then on the tangent bundle of M. (However, on a single vector space, it doesn't make sense to talk about closed forms.)

3.1. **Kähler forms.**

We started with the basic concept of a Kähler form. Suppose that V is a vector space over $\mathbb{C}$: $V \cong \mathbb{C}^n$ and $W_\mathbb{R} = \mathrm{Hom}(V, \mathbb{R})$,

$$W_\mathbb{C} = W_\mathbb{R} \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}.$$

Recall that $W_\mathbb{C} = \mathrm{Hom}_\mathbb{R}(V, \mathbb{C})$ and $W^{1,0} = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C}) \subset W_\mathbb{C}$. Now take $W^{1,1} = W^{1,0} \otimes W^{0,1}$. We want to look at

$$W_\mathbb{R}^{1,1} = W^{1,1} \cap \wedge^2 W_\mathbb{R}.$$

Thus elements of $W_\mathbb{R}^{1,1}$ are **alternating real forms of type** $(1, 1)$**.**

Example 3.1.1. The basic example is $V = \mathbb{C}^n$,

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \frac{i}{2} \sum (dx_j + i dy_j) \wedge (dx_j - i dy_j) = \sum dx_j \wedge dy_j.$$

Since $a \wedge b = a \otimes b - b \otimes a$, for $z = x + iy, z' = x' + iy' \in \mathbb{C}^n$ this form is

$$\omega(z, z') = \sum_j (x_j y_j' - y_j x_j').$$

The first form of ω shows that it lies in $W^{1,1}$. The last form shows it is in $\wedge^2 W_\mathbb{R}$. The scalar $\frac{i}{2}$ is needed to make the form real. All Kähler forms will be equivalent to these.

Lemma 3.1.2. *$\omega \in W_\mathbb{R}^{1,1}$ if and only if $\omega : V \times V \to \mathbb{R}$ is a skew symmetric $\mathbb{R}$-bilinear form so that*

$$\omega(Iu, Iv) = \omega(u, v) \tag{3.1}$$

for all $u, v \in V$.

Proof. First note that the condition $\omega(Iu, Iv) = \omega(u, v)$ is equivalent to the condition

$$\omega(u, Iv) + \omega(Iu, v) = 0 \tag{3.2}$$

since $I^2 = -1$. By definition, $W_\mathbb{R}^{1,1}$ is the set of skew-symmetric forms ω on V so that $\omega_\mathbb{C} = \omega \otimes \mathbb{C}$ lies in $W^{1,1}$. This is equivalent to the condition that $\omega_\mathbb{C}$ vanishes on pairs of vectors from $V^{1,0}$ or from $V^{0,1}$. But $V^{1,0}$ is the set of all vectors of the form

$$\tilde{u} = u - iIu$$

where $u \in V$ and

$$\begin{aligned}\omega(\tilde{u}, \tilde{v}) &= \omega(u - iIu, v - iIv) \\ &= \omega(u, v) - \omega(Iu, Iv) - i(\omega(u, Iv) + \omega(Iu, v))\end{aligned}$$

which is zero if and only if (3.1) and (3.2) hold. □

Note that (3.2) implies that

$$g(u,v) := \omega(u, Iv)$$

is a symmetric bilinear pairing $g : V \times V \to \mathbb{R}$ since

$$g(v,u) = \omega(v, Iu) = -\omega(Iu, v) = \omega(u, Iv) = g(u,v)$$

Definition 3.1.3. A **hermitian form** on a complex vector space V is defined to be a map $h : V \times V \to \mathbb{C}$ so that

(1) $h(u,v)$ is $\mathbb{C}$-linear in u
(2) $h(u,v)$ is $\mathbb{C}$-antilinear in v
(3) $h(v,u) = \overline{h(u,v)}$

Note that (3) implies that $h(v,v) \in \mathbb{R}$. The form h is said to be **positive definite** if $h(v,v) > 0$ for all $v \neq 0$. A positive definite hermitian form on V is also called a **Hermitian metric** on V.

Proposition 3.1.4. *There is a $1-1$ correspondence between hermitian forms h on V and forms $\omega \in W_{\mathbb{R}}^{1,1}$ given by*

$$\omega = -\Im h.$$

Furthermore,

$$h(u,v) = g(u,v) - i\omega(u,v)$$

where $g : V \times V \to \mathbb{R}$ is given by $g(u,v) = \omega(u, Iv)$.

Proof. For the second part, $h(u, Iv) = g(u, Iv) - i\omega(u, Iv)$. Since $h(u,v)$ is conjugate linear in v, $h(u, Iv) = -ih(u,v) = -\omega(u,v) - ig(u,v)$. Comparing complex parts gives $g(u,v) = \omega(u, Iv)$. □

Example 3.1.5. The standard positive definite hermitian form on $\mathbb{C}^n$ is:

$$h(z,z') = \sum z_j \overline{z}'_j = \sum (x_j + iy_j)(x'_j + iy'_j)$$
$$= \underbrace{\sum (x_j x'_j + y_j y'_j)}_{g(z,z')} - i \underbrace{\sum (x_j y'_j - y_j x'_j)}_{\omega(z,z')}$$

Definition 3.1.6. A **Kähler form** on V is a form $\omega \in W_{\mathbb{R}}^{1,1}$ whose corresponding hermitian form h is positive definite. In particular, Kähler forms are nondegenerate.

3.2. **Kähler metrics.** Suppose that (M, I) is an almost complex manifold. Then a **Hermitian metric** h on M is a Hermitian metric h_x on the tangent space $T_{M,x}$ at each point which varies smoothly with $x \in M$. Associated to h we have:

$$\omega = -\Im h$$

which is a 2-form on M which is also in $\Omega_M^{1,1}$ which is equivalent to the equation

$$\omega(Iu, Iv) = \omega(u,v)$$

for any two vectors $u, v \in T_{M,x}$ at any point $x \in M$. By Definition 3.1.6, ω is a Kähler form. However we usually want ω to be closed. We say that the Hermitian metric h is a **Kähler metric** if the corresponding Kähler form ω is closed.

Proposition 3.2.1. *The real part of a Hermitian metric h is a Riemannian metric g on M which is also invariant under I:*

$$g(u,v) = \omega(u, Iv) = g(Iu, Iv)$$

Proof. We know that g is a symmetric real form. If $v \neq 0 \in T_{M,x}$ is a nonzero vector, $h(v,v) = \overline{h(v,v)}$ is a positive real number. So,

$$g(v,v) = h(v,v) > 0$$

So, g is a Riemannian metric on M. □

Note that M is oriented since any complex vector space has a natural real orientation.

Theorem 3.2.2. *Given a Hermitian metric h on the complex manifold M, the volume form on M associated to the Riemannian metric $g = \Re h$ is equal to $\omega^n/n!$.*

To prove this, we need the matrices for general h, ω in local coordinates:

$$z = (z_1, \cdots, z_n) : U \hookrightarrow \mathbb{C}^n, \quad z_j = x_j + iy_j$$

and $\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n)$ centered at $x_0 \in U$. Then $dz_j, d\overline{z}_j$ form bases for $\Omega^{1,0}_{M,x_0}, \Omega^{0,1}_{M,x_0}$. These are $W^{1,0}, W^{0,1}$ for $V = T_{M,x_0}$. So $h \in \Omega^{1,1}_M$ is given by

$$h = \sum \alpha_{ij}\, dz_i \otimes d\overline{z}_j$$

where $\alpha_{ij} : M \to \mathbb{C}$ (notation: $\alpha_{ij} \in \Omega^0_{M,\mathbb{C}}$). These functions are given by:

$$\alpha_{ij} = h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Since $h(u,v) = \overline{h(v,u)}$, $\alpha_{ji} = \overline{\alpha_{ij}}$. Since $-\Im z = \frac{i}{2}(z - \overline{z})$, we have:

$$\omega = -\Im h = \frac{i}{2} \sum (\alpha_{ij}\, dz_i \otimes d\overline{z}_j - \alpha_{ji}\, d\overline{z}_i \otimes dz_j) = \frac{i}{2} \sum \alpha_{ij}\, dz_i \wedge d\overline{z}_j.$$

Proof of Theorem 3.2.2. By a $\mathbb{C}$-linear change of the coordinates $z = (z_1, \cdots, z_n)$, we can arrange for $\frac{\partial}{\partial x_i}$ to be ortho-normal at the point x_0. In other words,

$$\alpha_{ij}(x_0) = h_{x_0}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

Then, at x_0,

$$\omega_{x_0} = \frac{i}{2} \sum dz_j \wedge d\overline{z}_j = \sum dx_j \wedge dy_j$$

by Example 3.1.1. So,

$$\omega^n = \sum_{\pi \in S_n} dx_{\pi(1)} \wedge dy_{\pi(1)} \wedge dx_{\pi(2)} \wedge dy_{\pi(2)} \wedge \cdots \wedge dx_{\pi(n)} \wedge dy_{\pi(n)}$$

which is $n!$ times the volume form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ at the point x_0. Since this hold at every point $x_0 \in M$, $\omega^n/n!$ is the volume form at each point. □

3.3. **Kähler manifolds.**

Definition 3.3.1. A *Kähler manifold* is a complex manifold with a Kähler metric which is, by definition, a Hermitian metric h so that ω is closed ($d\omega = 0$).

Corollary 3.3.2. *On a compact Kähler manifold M, for every $1 \leq k \leq n$, the form ω^k is closed but not exact. I.e., $[\omega^k] \neq 0 \in H^{2k}(M)$.*

Proof. ω^k is clearly closed:

$$d\omega^k = k\omega^{k-1} d\omega = 0$$

If $\omega^k = d\alpha$ then

$$d(\alpha \wedge \omega^{n-k}) = d\alpha \wedge \omega^{n-k} = \omega^n$$

which is impossible since the volume form is nonzero in $H^{2n}(M)$ when M is an oriented compact manifold. □

Corollary 3.3.3. *A compact complex k-submanifold N of a Kähler manifold M cannot be the boundary of a (real) submanifold of M.*

Proof. This follows immediately from Stokes' Theorem. If $N = \partial W$ then:

$$\int_W d\omega^k = 0 = \int_N \omega^k = vol\, N$$

a contradiction, since ω^k is the volume form on N. □

3.4. **Connections.**

Given a real C^∞ k-dimensional vector bundle E on a real manifold X we want to take the derivative of a section of E. This is given by a connection. Recall that a **connection** ∇ on E is a linear map

$$\nabla : A^0(E) \to A^1(E) = \Gamma(T_X^* \otimes_{\mathbb{R}} E)$$

satisfying the Leibnitz equation

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

for all $f : X \to \mathbb{R}$ ($f \in \Omega_X^0$), $\sigma \in \Gamma E = A^0(E) = C^\infty(E)$.

Thus, ∇ takes a section σ of E and gives $\nabla(\sigma) \in A^1(E)$ which is a 1-form on X with coefficients in E:

$$\nabla(\sigma) \in A^1(E) = \Gamma(T_X^* \otimes_{\mathbb{R}} E) = \Gamma\,\mathrm{Hom}_{\mathbb{R}}(T_X, E).$$

We interpret $\nabla(\sigma)$ as the derivative of σ.

Analogy: Let $f : X \to \mathbb{R}$ be a smooth function. Then $D(f) = df : T_X \to \mathbb{R}$ is a 1-form on X:

$$df \in \Gamma T_X^* = \Gamma\,\mathrm{Hom}(T_X, \mathbb{R}).$$

Given a vector field ψ on X, $D_\psi(f) = df(\psi)$ is the *directional derivative* of f in the direction ψ. This is at every $x \in X$. So, $D_\psi(f) \in \Omega_X^0$ is a smooth function on X.

For a connection we write:

$$\nabla_\psi(\sigma) := \nabla(\sigma)\psi \in \Gamma E.$$

This is the ∇-directional derivative of the section σ in the direction of the tangent vector field ψ.

Following our philosophy of concentrating on concepts and definitions (and skipping proofs of theorems), we reformulate the definition of a connection.

Lemma 3.4.1. *If E, E' are vector bundles on X then*

$$\mathrm{Hom}_X(E, E') = \Gamma\,\mathrm{Hom}_{\mathbb{R}}(E, E') = \mathrm{Hom}_{\Omega^0_X}(\Gamma E, \Gamma E').$$

Proof. Locally, a homomorphism of vector bundles $E^k \to E'^\ell$ is given by a map

$$M : X \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^\ell)$$

which is a family of $\ell \times k$ matrices $M(x)$, with entries $m_{ij} : X \to \mathbb{R}$.

Locally, sections of $E = X \times \mathbb{R}^k$ are given by k functions on X:

$$u \in \Gamma E = (u_1, \cdots, u_k) = \sum u_i e_i$$

where e_i are "basic sections" of E and $u_i \in \Omega^0_X$. I.e., ΓE is a free Ω^0_X-module. An Ω^0_X-morphism $M : \Gamma E \to \Gamma E'$ is therefore given by:

$$M(\sigma) = \sum u_i M(e_i) = \sum u_i m_{ij} e'_j = (u_1, \cdots, u_k)M$$

So, M is given by the same data: a matrix with entries $m_{ij} \in \Omega^0_X$. □

For a connection $\nabla : \Gamma E \to A^1(E)$, ΓE, $A^1(E)$ are both Ω^0_X modules. But ∇ is not a homomorphism of Ω^0_X modules:

$$\nabla(f\sigma) = df\,\sigma + f\,\nabla(\sigma).$$

However, if we have another connection ∇',

$$\nabla'(f\sigma) = df\,\sigma + f\,\nabla'(\sigma).$$

So, the difference

$$(\nabla - \nabla')(f\sigma) = f(\nabla - \nabla')\sigma$$

is a homomorphism of Ω_X-modules. Analogously to the proof of Lemma 3.4.1 such morphisms are given by matrices $M = (m_{ij})$ with $m_{ij} \in \Omega^1_X$.

In local coordinates, d is a connection. So, an arbitrary connection is given by $\nabla = d + \varphi$ or:

$$\nabla(f_1, \cdots, f_k) = (df_1, \cdots, df_k) + (f_1, \cdots, f_k)M$$

where M is a $k \times k$ matrix with entries in Ω^1_X.

If X has a Riemannian metric g then recall that the *Levi-Civita connection* $\nabla = \nabla^{LC}$ is the unique connection on $E = T_M$ having the properties:

(1) $dg(\sigma, \tau) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$ (∇ is *compatible* with g)
(2) $\nabla(\tau)\sigma - \nabla(\sigma)\tau = [\sigma, \tau]$ usually written as:

$$\nabla_\sigma(\tau) - \nabla_\tau(\sigma) = [\sigma, \tau]$$

for any two vector fields σ, τ on X.

As discussed in class, Equation (1) says that, for any vector field ψ on X,

$$dg(\sigma, \tau)(\psi) = g(\nabla_\psi \sigma, \tau) + g(\sigma, \nabla_\psi \tau).$$

Explanation: Since $g(\sigma,\tau) \in \Omega^0_X$, all three terms in Equation (1) are 1-forms on X. For example, if $\nabla\sigma = \sum \xi_i\alpha_i$ where ξ_i are vector fields and α_i are 1-forms on X, then

$$g(\nabla\sigma,\tau) = \sum g(\xi_i,\tau)\alpha_i.$$

Applying both sides to the vector field ψ we get:

$$g(\nabla\sigma,\tau)(\psi) = \sum g(\xi_i,\tau)\alpha_i(\psi) = \sum g(\xi_i\alpha_i(\psi),\tau) = g(\nabla\sigma(\psi),\tau) = g(\nabla_\psi\sigma,\tau).$$

Recall that a *connection* on a smooth bundle E over X is a linear map

$$\nabla : \Gamma E = A^0(E) \to A^1(E)$$

satisfying Leibniz rule. When X is a complex manifold and E is a holomorphic bundle, $A^0(E), A^1(E)$ are the same set as before but with <u>more structure</u>:

$$A^1(E) = \Gamma(T^*_X \otimes_{\mathbb{R}} E) = \Gamma(T^*_{X,\mathbb{C}} \otimes_{\mathbb{C}} E) = A^{1,0}(E) \otimes A^{0,1}(E)$$

and $A^0(E) = A^{0,0}(E)$ is still the space of smooth sections of E. Then, any connection

$$\nabla : A^{0,0}(E) \to A^1(E) = A^{1,0}(E) \otimes A^{0,1}(E)$$

has two components $\nabla^{1,0}, \nabla^{0,1}$. Last time we showed that

$$\overline{\partial}_E : A^{0,0}(E) \to A^{0,1}(E)$$

given in local coordinates by $\overline{\partial}_U(f_1,\cdots,f_k) = (\overline{\partial}f_1,\cdots,\overline{\partial}f_k)$ is well defined.

Proposition 3.4.2. *Given a Hermitian metric h on a holomorphic bundle E, there is a unique connection ∇ on E so that*

(1) $dh(\sigma,\tau) = h(\nabla\sigma,\tau) + h(\sigma,\nabla\tau)$ *for all* $\sigma,\tau \in A^{0,0}(E)$ (∇ *is "compatible" with* h)
(2) $\nabla^{0,1} = \overline{\partial}_E$

This unique connection is called the *Chern connection* on E.

Proof. $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where $\nabla^{0,1} = \overline{\partial}_E$ and $\nabla^{1,0}$ is uniquely determined by:

$$dh(\sigma,\tau) = h(\nabla^{1,0}\sigma,\tau) + h(\sigma,\overline{\partial}\tau) \tag{3.3}$$

since $h(\Delta^{0,1}\sigma,\tau) = 0$ and $h(\sigma,\Delta^{1,0}\tau) = 0$. In more detail, let $\nabla^{1,0} = \partial + M$ where, in local coordinates, M is a $k \times k$ matrix with entries in $\Omega^{1,0}_X$. We have:

$$h(u,v) = \sum h_{ij}u_iv_j$$

where

$$h_{ij} = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\right).$$

Then

$$dh(u,v) = h(\partial u,v) + h(u,\overline{\partial}v) + \sum dh_{ij}u_iv_j$$

and

$$h(\nabla^{1,0}u,v) = h(\partial u,v) + h(uM,v).$$

Let M be the solution of the equation

$$h(uM,v) = \sum dh_{ij}u_iv_j$$

which is bilinear in u,v. Then $\nabla^{1,0} = \partial + M$ will satisfy (3.3). □

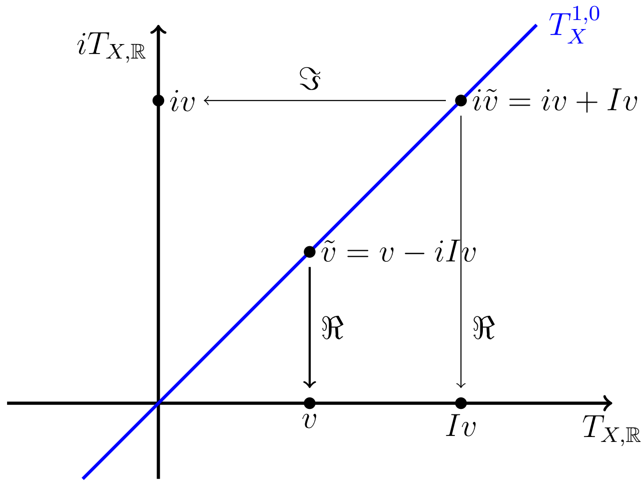

FIGURE 1. (Showing that $\Re(\nabla^{1,0})$ is a connection on $T_{X,\mathbb{R}}$.) Since $i\tilde{v} = iv + Iv = \widetilde{Iv}$, $\Re(i\tilde{v}) = I\Re(\tilde{v})$ making $\Re : T_X^{1,0} \to T_{X,\mathbb{R}}$ an isomorphism of complex vector spaces with i acting as i on $T_X^{1,0}$ and as I on $T_{X,\mathbb{R}}$.

Theorem 3.4.3. *Let (M, I) be a complex manifold with a Hermitian metric $h = g - i\omega$ and let ∇^{LC} be the Levi-Civita connection for g. Then the following are equivalent.*

(1) *h is a Kähler metric ($d\omega = 0$).*
(2) *$\nabla^{LC}(I\sigma) = I\nabla^{LC}(\sigma)$ for every real vector field σ on X.*
(3) *The holomorphic part of the Chern connection is equal to the Levi-Civita connection:*
$$\Re(\nabla^{1,0}) = \nabla^{LC}$$
(See Fig 1.) So, the Chern connection on $T_{X,\mathbb{C}}$ is " $(\nabla^{LC}, \overline{\partial})$ ".

Proof. (3) ⇒ (2). The Chern connection is complex linear and the holomorphic part is the part where $i = I$:
$$\Re(i\sigma) = I\Re(\sigma)$$
for σ a section of $T_X^{1,0}$. So,
$$\nabla^{LC}(I\sigma) = \Re(\nabla^{1,0}(I\sigma)) = \Re(i\nabla^{1,0}(\sigma)) = I\Re(\nabla^{1,0}(\sigma)) = I\nabla^{LC}(\sigma).$$
(2) ⇒ (1). Since $\nabla = \nabla^{LC}$ is compatible with $g = \Re(h)$, we are given that
$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau).$$
Replacing τ with $I\tau$ we get $g(\sigma, I\tau) = \omega(\sigma, \tau)$ and $\nabla I\tau = I\nabla\tau$ by (2). So,
$$d(\omega(\sigma, \tau)) = \omega(\nabla\sigma, \tau) + \omega(\sigma, \nabla\tau).$$
Apply both to vector field ϕ, use rule $df(\phi) = \phi(f)$ and skew-symmetry of ω:
$$\phi(\omega(\sigma, \tau)) = \omega(\nabla_\phi\sigma, \tau) - \omega(\nabla_\phi\tau, \sigma).$$
Cyclically permute the three vector fields:
$$\sigma(\omega(\tau, \phi)) = \omega(\nabla_\sigma\tau, \phi) - \omega(\nabla_\sigma\phi, \tau)$$

$$\tau(\omega(\phi,\sigma)) = \omega(\nabla_\tau\phi,\sigma) - \omega(\nabla_\tau\sigma,\phi).$$

Now use (2.1), the coordinate invariant definition of $d\omega$:

$$d\omega(\phi,\sigma,\tau) = \phi(\omega(\sigma,\tau)) - \sigma(\omega(\phi,\tau)) + \tau(\omega(\phi,\sigma))$$
$$-\omega([\phi,\sigma],\tau) + \omega([\phi,\tau],\sigma) - \omega([\sigma,\tau],\phi)$$

Since $[\phi,\sigma] = \nabla_\phi\sigma - \nabla_\sigma\phi$ and $\omega(\phi,\tau) = \omega(\tau,\phi)$, this is:

$$d\omega(\phi,\sigma,\tau) = \phi(\omega(\sigma,\tau)) + \sigma(\omega(\tau,\phi)) + \tau(\omega(\phi,\sigma))$$
$$+\omega(\nabla_\sigma(\phi) - \nabla_\phi(\sigma),\tau) + \omega(\nabla_\phi(\tau) - \nabla_\tau(\phi),\sigma) + \omega(\nabla_\tau(\sigma) - \nabla_\sigma(\tau),\phi) = 0.$$

(1) $\Rightarrow$ (3). The proof is by reduction to the standard case: When the metric h is constant, $\nabla^{LC} = d$ and $\nabla^{Ch} = (\partial,\overline{\partial})$ and $\Re(\partial) = d$. So, (3) holds. Since these are first order differential equations, it suffices for h to be constant to first order. Thus, assuming (1), we need to show that, at each point, the metric h can be made constant to first order. This follows from the following lemma. □

Lemma 3.4.4. *If h is a Kähler metric on X then, in a nbh of each point, there are holomorphic coordinates z_i so that the matrix of h:*

$$h_{ij} = h\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) = h\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right)$$

is the identity matrix plus $O(|z|^2)$.

Proof. We can choose coordinates which are ortho-normal at the chosen point ($z = 0$). This makes the constant term of h_{ij} the identity matrix. But we also have linear terms:

$$h_{ij} = \delta_{ij} + \epsilon_{ij} + \epsilon'_{ij} + O(|z|^2)$$

where ϵ_{ij} is a linear combination of z_k (ϵ_{ij} are holomorphic)

$$\epsilon_{ij} = \sum \epsilon_{ij}^k z_k$$

and ϵ'_{ij} is a linear combination of $\overline{z}_k$ (ϵ'_{ij} are antiholomorphic):

$$\epsilon'_{ij} = \sum \epsilon'^k_{ij}\overline{z}_k$$

Since h is conjugate symmetric we have:

$$\epsilon'_{ij} = \overline{\epsilon}_{ji}$$

The key property of these numbers is:

<u>Claim:</u> If h is Kähler then

$$\boxed{\epsilon_{ij}^k = \epsilon_{kj}^i}$$

Proof: Since $\partial\delta_{ij} = 0$ and $\partial\overline{z}_k = 0 \Rightarrow \partial\epsilon'_{ij} = 0$, at the point $z = 0$ we have:

$$0 = \partial\omega = \frac{i}{2}\sum_{ij}\partial\epsilon_{ij}\,dz_i\wedge d\overline{z}_j = \frac{i}{2}\sum_{ijk}\epsilon_{ij}^k\,dz_k\wedge dz_i\wedge d\overline{z}_j$$

where we recall that the $\frac{i}{2}$ factor comes from: $-\Im(z) = \frac{i}{2}(z-\overline{z})$ (and $d\omega = 0$ is equivalent to $\partial\omega = 0 = \overline{\partial}\omega$). In order for these terms to cancel, we must have $\epsilon_{ij}^k = \epsilon_{kj}^i$ as claimed.

Now let

$$z_j' = z_j + \frac{1}{2}\sum \epsilon_{ij}^k z_i z_k.$$

Since $\epsilon_{ij}^k z_i z_k$ is symmetric in z_i, z_k, we have

$$dz_j' = dz_j + \sum \epsilon_{ij}^k z_k dz_i = dz_j + \sum \epsilon_{ij} dz_i = dz_j + O(|z|)$$

which implies;

$$dz_j = dz_j' - \sum \epsilon_{ij} dz_i' + O(|z|^2)$$

$$\frac{\partial}{\partial z_i'} = \sum \frac{\partial z_j}{\partial z_i'}\frac{\partial}{\partial z_j} = \sum (\delta_{ij} - \epsilon_{ij})\frac{\partial}{\partial z_j} + O(|z|^2)$$

So, up to terms of second order, we have:

$$h_{ij}' = h\left(\frac{\partial}{\partial z_i'}, \frac{\partial}{\partial \overline{z}_j'}\right) = \sum_{k,\ell} (\delta_{ik} - \epsilon_{ik}) h_{k\ell} (\delta_{j\ell} - \overline{\epsilon}_{j\ell})$$

$$= \sum_{k,\ell} (\delta_{ik} - \epsilon_{ik})(\delta_{k\ell} + \epsilon_{k\ell} + \epsilon_{k\ell}')(\delta_{j\ell} - \epsilon_{\ell j}')$$

since $\overline{\epsilon}_{j\ell} = \epsilon_{\ell j}'$,

$$= \sum \delta_{ik}\delta_{k\ell}\delta_{j\ell} - \epsilon_{ik}\delta_{k\ell}\delta_{j\ell} + \delta_{ik}\epsilon_{k\ell}\delta_{j\ell} + \delta_{ik}\epsilon_{k\ell}'\delta_{j\ell} - \delta_{ik}\delta_{k\ell}\epsilon_{j\ell}'$$

$$= \delta_{ij} - \epsilon_{ij} + \epsilon_{ij} + \epsilon_{ij}' - \epsilon_{ij}' = \delta_{ij}$$

In other words, the matrix (h_{ij}') of h with respect to the new coordinates z_j' is equal to the identity matrix up to second order. This proves the Lemma and completes the proof of Theorem 3.4.3. □

3.5. **Examples of Kähler manifolds.** An easy example is a Riemann surface. This is a complex 1-dimensional and real 2-dimensional manifold. Any Hermitian metric is Kähler since all 2-forms on a real 2-dimensional manifold are closed.

The next example is $\mathbb{C}P^n = \mathbb{P}^n(\mathbb{C})$. We will construct the *Fubini-Study* metric on complex projective space $\mathbb{P}^n(\mathbb{C})$ and showed that it is a Kähler metric. This will imply that all smooth projective varieties over $\mathbb{C}$ are Kähler manifolds.

The outline of the construction is:

$$L \mapsto (L,h) \mapsto \omega_L \leftrightarrow h_\omega$$

Given a holomorphic line bundle L on a complex manifold X, chose a hermitian form h on L. Then, there is an associated 2-form ω_L on X (called the *Chern form* of (L,h)). This 2-form ω_L is associated to a Hermitian metric h_ω on X ($h_\omega \neq h$) which, if we are lucky, will be positive definite and therefore a Kähler metric. We will apply this to the canonical line bundle S^* over $X = \mathbb{P}^n(\mathbb{C})$ to obtain the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$.

A line bundle L over X is the union over open sets U_i of $U_i \times \mathbb{C}$. For each U_i, take the unit section $\sigma_i(v) = (v,1)$. These in general don't match. So, there are functions $g_{ij} : U_i \cap U_j \to \mathbb{C}^\times$ so that

$$\sigma_i(v) = g_{ij}(v)\sigma_j(v)$$

for all $v \in U_i \cap U_j$. Since $\sigma_j = g_{jk}\sigma_k$ we have $\sigma_i = g_{ij}\sigma_j = g_{ij}g_{jk}\sigma_k$. So,

$$\boxed{g_{ik} = g_{ij}g_{jk}}$$

on $U_i \cap U_j \cap U_k$. Conversely, any collection of maps $g_{ij} : U_i \cap U_j \to \mathbb{C}^\times$ satisfying the equations in the box will uniquely determine a line bundle. If the g_{ij} are holomorphic functions, the line bundle will be a holomorphic bundle.

For example, $g^*_{ij} := \frac{1}{g_{ij}}$ is another collection of functions satisfying the same (boxed) equations. So, $\{g^*_{ij}\}$ gives another holomorphic line bundle L^* which one can show is the dual bundle to L.

Let h be a Hermitian metric on L. Let $h_i : U_i \to \mathbb{R}^+$ be the positive function given by $h_i(v) = h(\sigma_i(v), \sigma_i(v))$. Since $\sigma_i = g_{ij}\sigma_j$ we get:

$$h_i = h(\sigma_i, \sigma_i) = g_{ij}\overline{g_{ij}}h(\sigma_j, \sigma_j) = g_{ij}\overline{g_{ij}}h_j$$

Lemma 3.5.1. *Conversely, any family of functions $h_i : U_i \to \mathbb{R}^+$ satisfying the equations $h_i = g_{ij}\overline{g_{ij}}h_j$ gives a Hermitian metric on L.*

Proof. Let h'_i be another collections of functions so that $h'_i = g_{ij}\overline{g_{ij}}h'_j$. On each U_i let $f_i = h'_i/h_i$. Then $f_j = h'_j/h_j = g_{ji}\overline{g_{ji}}h'_i/g_{ji}\overline{g_{ji}}h_i = h'_i/h_i = f_i$. So, $f = f_i = f_j$ is a globally defined function on X and $h' = fh$ is another metric on L. □

For example, $h^*_i = \frac{1}{h_i}$ satisfies

$$h^*_i = g^*_{ij}\overline{g_{ij}}^* h^*_j$$

Therefore, h^*_i gives a metric on L^*.

Let

$$\omega_i = \frac{1}{2\pi i}\partial\overline{\partial}\log h_i.$$

Note that

$$\log h_i = \log g_{ij} + \log \overline{g_{ij}} + \log h_j.$$

Since g_{ij} is holomorphic, $\overline{\partial} \log g_{ij} = 0$. Since $\overline{g_{ij}}$ is antiholomorphic, $\partial \log \overline{g_{ij}} = 0$. So,

$$\omega_i = \frac{1}{2\pi i} \partial\overline{\partial} \log h_i = \frac{1}{2\pi i} \partial\overline{\partial} \log h_j = \omega_j.$$

So, $\omega = \omega_i$ is a well-defined 2-form on all of X. Also $d\omega = \partial\omega + \overline{\partial}\omega = 0$ since $\partial^2 = 0 = \overline{\partial}^2$.

Theorem 3.5.2. *Given a holomorphic line bundle L on a complex manifold X and a Hermitian metric h on L, there is a closed form ω on X of type $(1,1)$ given locally by*

$$\omega = \frac{1}{2\pi i} \partial\overline{\partial} \log h.$$

We call ω the Chern form of (L, h).

Now let $X = \mathbb{P}^n(\mathbb{C})$. Recall that this is the quotient space of $\mathbb{C}^{n+1} \backslash 0$ modulo the relation

$$(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)$$

for any $\lambda \neq 0 \in \mathbb{C}$. The equivalence class is denoted $[z_0, \cdots, z_n]$. Another interpretation is that $\mathbb{P}^n(\mathbb{C})$ is the set of one dimensional subspaces Δ of $\mathbb{C}^{n+1}$. Each such Δ is uniquely determined by any nonzero vector $(z_0, \cdots, z_n) \in \Delta$ and we make the identification $\Delta = [z_0, \cdots, z_n]$.

Let S be the *tautological line bundle* over $\mathbb{P}^n(\mathbb{C})$ given by

$$S = \{(\Delta, v) \mid \Delta \in \mathbb{P}^n(\mathbb{C}) \text{ and } v \in \Delta\} \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}.$$

This is "tautological" since the fiber over the point $\Delta \in \mathbb{P}^n(\mathbb{C})$ is the space $\Delta \subset \mathbb{C}^{n+1}$.

Let U_i be the open subset of $\mathbb{P}^n$ given by

$$U_i = \{[z] \mid z_i \neq 0\}.$$

Let σ_i be the section of S over U_i given by

$$\sigma_i(\Delta) = \sigma_i([z_0, \cdots, z_n]) = \left(\frac{z_0}{z_i}, \cdots, \frac{z_i}{z_i} = 1, \cdots, \frac{z_n}{z_i}\right).$$

This is well-defined since, e.g., the jth coordinate is

$$\frac{z_j}{z_i} = \frac{\lambda z_j}{\lambda z_i}$$

$\sigma_i(\Delta)$ is the unique element of Δ with ith coordinate equal to 1. Comparing this with

$$\sigma_j([z]) = \left(\frac{z_0}{z_j}, \cdots, \frac{z_i}{z_j}, \cdots, \frac{z_n}{z_j}\right)$$

we see that

$$\sigma_i = \frac{z_j}{z_i} \sigma_j.$$

So, the transition functions for S are $g_{ij} = z_j / z_i$ with dual $g^*_{ij} = z_i / z_j$.

Since the line bundle S is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ it gets a metric by restricting the standard metric on $\mathbb{C}^{n+1}$ given by $h(z, z') = \sum z_j \overline{z}'_j$. So $h(z) = h(z, z) = \sum |z_j|^2$. Since the ith coordinate of σ_i is 1 we get:

$$h(\sigma_i) = 1 + \sum_{j \neq i} |z_j|^2.$$

On the dual bundle S^* (called the *canonical bundle* over $\mathbb{P}^n$) we have

$$h^*(\sigma_i^*) = \frac{1}{1 + \sum |z_j|^2}.$$

Using $|z_j|^2 = z_j \overline{z_j}$, the Chern form of S^* on U_i is

$$\omega_i = \frac{1}{2\pi i} \partial \overline{\partial} \log \left(\frac{1}{1 + \sum z_j \overline{z}_j} \right).$$

We calculated this step by step using first the equation

$$\overline{\partial} \log \frac{1}{f} = -\frac{\overline{\partial} f}{f} = -\frac{1}{f} \sum \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

to get:

$$\overline{\partial} \log \left(\frac{1}{1 + \sum z_j \overline{z}_j} \right) = \frac{-\sum z_j d\overline{z}_j}{1 + \sum |z_j|^2}.$$

Apply ∂ using the quotient rule to get:

$$\partial \overline{\partial} \log \left(\frac{1}{1 + \sum z_j \overline{z}_j} \right) = -\left(\frac{(1 + \sum |z_j|^2) \sum dz_j \wedge d\overline{z}_j - \sum z_i \overline{z}_j dz_i \wedge d\overline{z}_j}{(1 + \sum |z_j|^2)^2} \right)$$

where we used the formula $\partial(f d\overline{z}_j) = \sum \frac{\partial f}{\partial z_i} dz_i \wedge d\overline{z}_j$. At the origin $z = 0$ we get

$$\omega = \frac{i}{2\pi} \sum dz_j \wedge d\overline{z}_j$$

which is the standard form corresponding to (a positive scalar multiple of) the standard metric with matrix equal to the identity matrix (divided by π). So, the corresponding metric h_ω is positive definite at the point $z = 0$. However, the space $\mathbb{P}^n(\mathbb{C})$ is homogeneous (the same at every point). This is easier to see if we use a vector space without a basis: Let V be any $n + 1$ dimensional vector space over $\mathbb{C}$ and let $\mathbb{P}(V)$ be the space of 1-dimensional subspaces of V. Then it is clear that every point is the same as every other point. The tautological bundle S and its dual are also defined without choice of coordinates. So, we can choose coordinates to make any point the center point $z = 0$. So, the canonically defined metric h_ω is positive definite at every point.

Theorem 3.5.3. *The hermitian form h_ω corresponding to the canonical Chern form ω on the dual S^* of the tautological line bundle over $\mathbb{P}^n(\mathbb{C})$ is positive definite and therefore a Kähler metric.*

This form h_ω is called the *Fubini-Study metric* on $\mathbb{P}^n(\mathbb{C})$.

Corollary 3.5.4. *Every smooth projective variety over $\mathbb{C}$ is a Kähler manifold.*