The Kähler Identities

Let M be a complex manifold with a Hermitian metric. We define three second-order differential operators:

$$\begin{aligned}\Delta_d &= dd^* + d^*d,\\ \Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial},\\ \Delta_\partial &= \partial\partial^* + \partial^*\partial.\end{aligned}$$

The purpose of this note is to prove that when the metric is Kähler, these three operators agree up to constant factors (Theorem 7 below).

We begin by working on $\mathbb{C}^n$ with the Euclidean metric $\overline{g}$. We wish to derive a simple expression for $\overline{\partial}^*$ on Euclidean space. Recall that for a 1-form ξ and a q-form α, $\xi \vee \alpha$ is the unique $(q-1)$-form such that

$$\langle \xi \vee \alpha, \beta\rangle = \left\langle \alpha, \overline{\xi} \wedge \beta\right\rangle$$

for all $(q-1)$-forms β. It is characterized by $\xi \vee \alpha = \xi^{\#} \lrcorner\, \alpha$, where $\xi^{\#}$ is the vector dual to ξ via the metric, from which it follows easily that the map $\alpha \mapsto \xi \vee \alpha$ is an antiderivation and

$$\begin{aligned}dz^j \vee dz^k &= \left\langle dz^j, dz^{\overline{k}}\right\rangle = \overline{g}_{\mathbb{C}}\left(dz^j, dz^k\right) = 0,\\ dz^j \vee dz^{\overline{k}} &= \left\langle dz^j, dz^k\right\rangle = \overline{g}_{\mathbb{C}}\big(dz^j, dz^{\overline{k}}\big) = \overline{g}^{j\overline{k}}.\end{aligned}$$

If α is a q-form on $\mathbb{C}^n$, we define $\partial_j\alpha$ to be the q-form obtained by applying $\partial/\partial z^j$ to the coefficients of α in standard coordinates, and $\partial_{\overline{j}}\alpha$ is defined similarly.

Lemma 1. *For any smooth q-forms α and β on $\mathbb{C}^n$, we have*

$$\begin{aligned}\overline{\partial}\alpha &= dz^{\overline{j}} \wedge \partial_{\overline{j}}\alpha,\\ \partial\alpha &= dz^j \wedge \partial_j\alpha,\\ \partial_j \langle \alpha, \beta\rangle &= \langle \partial_j\alpha, \beta\rangle + \langle \alpha, \partial_{\overline{j}}\beta\rangle,\\ \partial_j(dz^k \vee \alpha) &= dz^k \vee (\partial_j\alpha).\end{aligned}$$

Proof. These are all obvious once everything is expanded out out in standard coordinates, using the fact that the coefficients of the metric are constants. ☐

Lemma 2. *With respect to the Euclidean metric $\overline{g}$ on $\mathbb{C}^n$, the formal adjoint of $\overline{\partial}$ is given by*

$$\overline{\partial}^*\alpha = -dz^j \vee \partial_j\alpha.$$

Proof. For any smooth, compactly supported function f on $\mathbb{C}^n$,

$$\int_{\mathbb{C}^n} \partial_j f \, dV_{\overline{g}} = 0,$$

as can be seen easily by applying the fundamental theorem of calculus to the real and imaginary parts. Therefore if α is a smooth q-form and β is a smooth $(q-1)$-form, both compactly supported,

$$\begin{aligned}
0 &= \int_{\mathbb{C}^n} \partial_j \left\langle dz^j \vee \alpha, \beta \right\rangle dV_{\overline{g}} \\
&= \int_{\mathbb{C}^n} \left\langle dz^j \vee \partial_j \alpha, \beta \right\rangle dV_{\overline{g}} + \int_{\mathbb{C}^n} \left\langle dz^j \vee \alpha, \partial_{\overline{j}} \beta \right\rangle dV_{\overline{g}} \\
&= \int_{\mathbb{C}^n} \left\langle dz^j \vee \partial_j \alpha, \beta \right\rangle dV_{\overline{g}} + \int_{\mathbb{C}^n} \left\langle \alpha, dz^{\overline{j}} \wedge \partial_{\overline{j}} \beta \right\rangle dV_{\overline{g}} \\
&= (dz^j \vee \partial_j \alpha, \beta) + (\alpha, \overline{\partial} \beta). \qquad \square
\end{aligned}$$

Now suppose M is a complex manifold with a Kähler metric g, and let ω denote its Kähler form. We define a smooth bundle homomorphism $L \colon \Lambda^{p,q} M \to \Lambda^{p+1,q+1} M$ by

$$L\alpha = \omega \wedge \alpha,$$

and let $L^* \colon \Lambda^{p+1,q+1} M \to \Lambda^{p,q} M$ be its pointwise adjoint, so that $\langle L^* \alpha, \beta \rangle = \langle \alpha, L\beta \rangle$ for all α, β.

Lemma 3. *On any Kähler manifold, the following identities hold:*

(a) $[\overline{\partial}^*, L] = i\partial$.

(b) $[\partial^*, L] = -i\overline{\partial}$.

(c) $[L^*, \overline{\partial}] = -i\partial^*$.

(d) $[L^*, \partial] = i\overline{\partial}^*$.

Proof. We will prove (a) first on Euclidean space. Let $\overline{g}_{k\overline{l}} = \frac{1}{2}\delta_{kl}$ denote the components of the Euclidean metric in standard coordinates. Using the fact that the $\vee$ product is an antiderivation and the metric coefficients are constants, we compute

$$\begin{aligned}
[\overline{\partial}^*, L]\alpha &= \overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge (\overline{\partial}^* \alpha) \\
&= -dz^j \vee \partial_j \left(i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \alpha \right) + i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \left(dz^j \vee \partial_j \alpha \right) \\
&= -dz^j \vee \left(i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \partial_j \alpha \right) + i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \left(dz^j \vee \partial_j \alpha \right) \\
&= -0 + i\overline{g}_{k\overline{l}} g^{j\overline{l}} \, dz^k \wedge \partial_j \alpha - i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \left(dz^j \vee \partial_j \alpha \right) \\
&\qquad + i\overline{g}_{k\overline{l}} \, dz^k \wedge dz^{\overline{l}} \wedge \left(dz^j \vee \partial_j \alpha \right) \\
&= i dz^j \wedge \partial_j \alpha \\
&= i\partial\alpha.
\end{aligned}$$

Now let (M, g) be an arbitrary Kähler manifold. Since both operators $\overline{\partial}^*$ and L are coordinate-independent, we may compute $[\overline{\partial}^*, L]$ at a point $p \in M$ in any coordinates we choose. Because g is Kähler, we can choose holomorphic coordinates such that the metric coefficients and their first derivatives at p match those of the Euclidean metric. Then the above computation shows that $[\overline{\partial}^*, L] = 0$ at p, because it involves no more than one derivative of the metric coefficients. This proves (a). Then (b) follows from (a) by conjugation, and (c) and (d) follow from (a) and (b) by taking adjoints, noting that $[P, Q]^* = [Q^*, P^*]$. ☐

Lemma 4. *On any Kähler manifold, the following identities hold:*

$$\overline{\partial}^* \partial + \partial \overline{\partial}^* = 0,$$
$$\partial^* \overline{\partial} + \overline{\partial} \partial^* = 0.$$

Proof. Lemma 3 shows that $\overline{\partial}^* = -i[L^*, \partial]$. Using this and the fact that $\partial^2 = 0$, we compute

$$\begin{aligned}
\overline{\partial}^* \partial + \partial \overline{\partial}^* &= -i[L^*, \partial]\partial - i\partial[L^*, \partial] \\
&= -iL^* \partial\partial + i\partial L^* \partial - i\partial L^* \partial + i\partial\partial L^* = 0.
\end{aligned}$$

The second identity follows from the first by conjugation. ☐

Lemma 5. *On any Kähler manifold,* $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$.

Proof. Lemma 4 yields

$$\begin{aligned}
\Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= \partial\partial^* + \partial\overline{\partial}^* + \overline{\partial}\partial^* + \overline{\partial}\overline{\partial}^* + \partial^*\partial + \partial^*\overline{\partial} + \overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} \\
&= (\partial\partial^* + \partial^*\partial) + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + (\overline{\partial}\partial^* + \partial^*\overline{\partial}) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) \\
&= \Delta_\partial + 0 + 0 + \Delta_{\overline{\partial}}.
\end{aligned}$$

☐

Lemma 6. *On any Kähler manifold,* $\Delta_\partial = \Delta_{\overline{\partial}}$.

Proof. From Lemma 3 again we get $\partial^* = i[L^*, \overline{\partial}]$. Thus

$$\begin{aligned}
\Delta_\partial &= \partial\partial^* + \partial^*\partial \\
&= i\partial[L^*, \overline{\partial}] + i[L^*, \overline{\partial}]\partial \\
&= i\partial L^* \overline{\partial} - i\partial\overline{\partial}L^* + iL^*\overline{\partial}\partial - i\overline{\partial}L^*\partial.
\end{aligned}$$

On the other hand, since $\overline{\partial}^* = -i[L^*, \partial]$,

$$\begin{aligned}
\Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \\
&= -i\overline{\partial}[L^*, \partial] - i[L^*, \partial]\overline{\partial} \\
&= -i\overline{\partial}L^*\partial + i\overline{\partial}\partial L^* - iL^*\partial\overline{\partial} + i\partial L^*\overline{\partial}.
\end{aligned}$$

Because $\partial\overline{\partial} = -\overline{\partial}\partial$, these two expressions are equal. ☐

Theorem 7. *On any Kähler manifold,* $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$.

Proof. Just combine Lemmas 5 and 6. ☐