TOWARDS THE SOLID ANALYTIC RING: LOCALLY COMPACT ABELIAN GROUPS

1. The ring of solid integers

In the previous talk we introduced the notion of analytic ring and proved that it has a good category of complete modules. Our next task is to construct non-trivial examples of such objects. The first analytic ring constructed, and the one mostly used in algebraic and non-archimidean geometry, is the ring of solid integers. With no more additional words let us give the main definition:

Definition 1.1. The analytic ring of solid integers $\mathbb{Z}_{\blacksquare}$ is the analytic ring with underline condensed ring $\mathbb{Z}$ and whose measures at $S \in \text{Prof}$ are given by

$$\mathbb{Z}_{\blacksquare}[S] = \varprojlim_i \mathbb{Z}[S_i],$$

where we write $S = \varprojlim_i S_i$ as a limit of finite sets.

Let us discuss very briefly discuss the definition of the solid integers. In Lecture 3 we saw that the free condensed abelian group generated by a profinite set $S = \varprojlim_i S_i$ has the following shape:

$$\mathbb{Z}[S] = \bigcup_n \mathbb{Z}[S]_{\ell^1 \leq n},$$

where $\mathbb{Z}[S]_{\ell^1 \leq n}$ is the profinite set written as

$$\mathbb{Z}[S]_{\ell^1 \leq n} = \varprojlim_i \mathbb{Z}[S_i]_{\ell^1 \leq n},$$

where $\mathbb{Z}[S_i]_{\ell^1 \leq n}$ is the finite set of sums $\sum_{s \in S_i} a_s s$ with $\sum_s |a_s| \leq n$. Then, we can think of $\mathbb{Z}[S]$ as some kind of ℓ^1-$\mathbb{Z}$-valued measures on the profinite set S. On the other hand, let $C(S, \mathbb{Z})$ be the space of continuous functions from S to $\mathbb{Z}$. Since $\mathbb{Z}$ is discrete we can write

$$C(S, \mathbb{Z}) = \varinjlim_i C(S_i, \mathbb{Z}),$$

taking $\mathbb{Z}$-duals we see that

$$\mathbb{Z}_{\blacksquare}[S] = C(S, \mathbb{Z})^{\vee}.$$

Thus, $\mathbb{Z}_{\blacksquare}[S]$ is the space of $\mathbb{Z}$*-valued Radon measures* of $C(S, \mathbb{Z})$ (in analogy to the space of real or complex valued Radon measures that we will recall in a later talk).

Example 1.2. Let $S = \mathbb{N} \sqcup \{\infty\}$ be the one point compactification of the integers. We can write

$$\mathbb{N} \sqcup \{\infty\} = \varprojlim_{n \in \mathbb{N}} \{1, \ldots, n\} \cup \{\infty\}.$$

We have

$$\mathbb{Z}_{\blacksquare}[\mathbb{N} \sqcup \{\infty\}] = \prod_{n \in \mathbb{N} \cup \{\infty\}} \mathbb{Z} T^n.$$

In particular

$$\mathbb{Z}_{\blacksquare}[\mathbb{N} \sqcup \{\infty\}]/(\infty) = \mathbb{Z}[[T]]$$

is isomorphic to a power series ring over $\mathbb{Z}$ in one variable.

Our main goal is to prove the following theorem.

Theorem 1.3 ([Sch19, Theorem 5.8]). *The object $\mathbb{Z}_\blacksquare$ is an analytic ring. More concretely, for any profinite set S and any connective complex $C_\bullet$ with terms C_i isomorphic to an arbitrary direct sum of $\mathbb{Z}_\blacksquare[S']$ for varying profinite sets S', the natural map*

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}_\blacksquare[S], C_\bullet) \to R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], C_\bullet)$$

is an equivalence. Furthermore, there is a full subcategory Solid $\subset$ CondAb *of solid abelian groups with compact projective objects given by $\mathbb{Z}_\blacksquare[S]$ for S-profinite, and such that the animation of* Solid *is naturally equivalent to the category $\mathscr{D}_{\geq 0}(\mathbb{Z}_\blacksquare)$ of completed connective $\mathbb{Z}_\blacksquare$-modules.*

In order to prove Theorem 1.3 we need to study the objects $\mathbb{Z}_\blacksquare[S]$ and condensed Ext functors between them. In Example 1.2 we saw that the solid abelian group generated by $\mathbb{N}\sqcup\{\infty\}$ is isomorphic to $\prod_{\mathbb{N}\sqcup\{\infty\}}\mathbb{Z}$, we will show in the next lectures that any solid abelian group generated by a profinite set has this shape, namely, it is a direct product of copies of $\mathbb{Z}$. Thus, we need to study Ext's functors of direct sums of objects $\prod_I \mathbb{Z}$. To accomplish this goal we shall use the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

and translate the problem to the study of Ext functors of products of tori and copies of $\mathbb{R}$. This naturally leads to the study of locally compact abelian group inside the category of condensed abelian groups.

2. Locally compact abelian groups and condensed mathematics

Recall that a Hausdorff topological space X is *locally compact* if any point $x \in X$ has a basis consisting con compact neighbourhoods of x. A Hausdorff topological abelian group is said *locally compact* if it is locally compact as a topological space. By definition, a locally compact Hausdorff space X has closed points and it is compactly generated, in particular its sheaf $\underline{X}$ on Prof is a condensed set and the underline topological space $\underline{X}(*)_{top}$ is naturally homeomorphic to X. Thus, we do not lose any topological information by working with $\underline{X}$ instead.

We recall the classification theorem of locally compact abelian groups:

Theorem 2.1. (1) *Let A be a locally compact abelian group, then there is an integer $n \in \mathbb{N}$ and an isomorphism $A \cong \mathbb{R}^n \times A'$ where A' admits an open compact subgroup.*
(2) *The Pontrjagin dual functor $A \mapsto \mathbb{D}(A) = \mathrm{Hom}^{co}(A, \mathbb{T})$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the torus and the Hom^{co} space has the compact open topology) takes values in locally compact abelian groups, and induces a contravariant equivalence in the category of locally compact abelian groups. The biduality map $A \to \mathbb{D}(\mathbb{D}(A))$ is an isomorphism.*
(3) *The Pontrjagin duality functor $A \mapsto \mathbb{D}(A)$ restricts to a contravariant duality between compact abelian groups and discrete abelian groups.*

The first point to check is that the Pontrjagin duality functor is naturally given by the $\underline{\mathrm{Hom}}$ functor as condensed abelian groups:

Proposition 2.2 ([Sch19, Proposition 4.2]). *Let X and Y be Hausdorff topological spaces with X compactly generated, let $\mathrm{Map}^{co}(X,Y)$ denote the space of continuous maps $X \to Y$ endowed with the compact open topology. Then there is a natural equivalence*

$$\underline{\mathrm{Map}^{co}(X,Y)} \cong \underline{\mathrm{Map}}(\underline{X}, \underline{Y}),$$

where $\underline{\mathrm{Map}}(\underline{X}, \underline{Y})$ is the condensed set defined by

$$\underline{\mathrm{Map}}(\underline{X}, \underline{Y})(S) = \mathrm{Map}(X \times S, Y).$$

In particular, if A and B are Hausdorff abelian groups with A compactly generated, we have an equivalence of condensed abelian groups

$$\underline{\mathrm{Hom}^{co}(A,B)} \cong \underline{\mathrm{Hom}}(A, B).$$

Proof. It suffices to show the statement about topological spaces. Indeed, for A and B Hausdorff abelian groups with A compactly generated, $\mathrm{Hom}^{co}(A,B)$ is the equalizer of some finite diagrams involving $\mathrm{Map}^{co}(A,B)$

(same for $\underline{\mathrm{Hom}}(A, B)$). More precisely, Hom^{co} is the equalizer of the diagram

$$\begin{array}{ccc} \mathrm{Map}^{co}(A,B) & \xrightarrow{f\circ(-+_A-)} & \mathrm{Map}^{co}(A\times A,B) \\ & \searrow^{f\times f} & \uparrow{\scriptstyle(-+_B-)\circ f} \\ & & \mathrm{Map}^{co}(A\times A, B\times B), \end{array}$$

resp. for $\underline{\mathrm{Hom}}$.

By definition, $\mathrm{Map}^{co}(X, Y)$ has the topology generated by the subspaces $\mathcal{U}(K, V)$ of functions $f : X \to Y$ mapping a compact $K \subset X$ into an open $V \subset Y$. Since X is Hausdorff and locally compact, the evaluation map

$$X \times \mathrm{Map}^{co}(X, Y) \to Y$$

is continuous. Thus, for a profinite set S and a continuous map $S \to \mathrm{Map}^{co}(X, Y)$ the associated map $X \times S \to Y$ is continuous. This induces a natural map

$$\underline{\mathrm{Map}^{co}(X, Y)} \to \underline{\mathrm{Map}}(\underline{X}, \underline{Y})$$

that we claim is an isomorphism. This boils down to show that for any profinite set S the map

$$\mathrm{Map}(S, \mathrm{Map}^{co}(X, Y)) \to \mathrm{Map}(X \times S, Y)$$

is a bijection. The map is clearly a injection, to see that it is surjective let $F : X \times S \to Y$ be a continuous map. For $s \in S$ let $f_s : X \to Y$ be the fiber of s. We want to show that for $K \subset X$ compact and $V \subset Y$ open, the space of $s \in S$ such that $f_s(K) \subset V$ is open. Let $s \in S$ be such that $f_s(K) \subset V$. Since F is continuous, $F^{-1}(V)$ is an open subspace of $X \times S$ containing $K \times s$, we can then find open subspaces $K \subset W \subset X$ and $s \in U \subset S$ such that $W \times U \subset F^{-1}(V)$, then $K \times U \subset F^{-1}(V)$ and for all $s' \in U$ we have $f_{s'}(K) \subset V$, proving what we wanted. □

Our next task it to describe the $R\underline{\mathrm{Hom}}$ spaces between locally compact abelian groups, this is encoded in the following key computation.

Theorem 2.3 ([Sch19, Theorem 4.3])**.** *Let $A = \prod_I \mathbb{T}$ be the compact condensed abelian group, where I is any index set.*

(1) *For any discrete abelian group M we have*

$$R\underline{\mathrm{Hom}}(A, M) = \bigoplus_I M[-1],$$

where the map $\bigoplus_I M[-1] \to R\underline{\mathrm{Hom}}(A, M)$ is induced by the maps

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \to R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, M) \xrightarrow{p_i^*} R\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, M) = R\underline{\mathrm{Hom}}(A, M),$$

using the pullback under the projection $p_i : \prod_I \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to the i-th factor, for $i \in I$.

(2)
$$R\underline{\mathrm{Hom}}(A, \mathbb{R}) = 0.$$

Remark 2.4 ([Sch19, Remark 4.4]). An outstanding consequence of Theorem 2.3 is that

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R}) = \mathbb{R}.$$

Indeed, by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$, this is equivalent to part (2) for $A = \mathbb{T}$.

In Lecture 4 we showed that condensed cohomology of (locally) compact Hausdorff spaces with discrete coefficients can be computed as sheaf cohomology. We also showed that condensed cohomology of compact Hausdorff spaces with values in $\mathbb{R}$ is trivial and equal to the continuous real valued functions. In the statement of Theorem 2.3 we have extensions between compact abelian groups, discrete abelian groups and $\mathbb{R}$, a way one could try to compute them is by comparing somehow the Ext spaces with the condensed cohomology of the underlying spaces. More concretely, we would like to compare the abelian group A with the free objects $\mathbb{Z}[A]$ generated by A. This is achieved thanks to an unpublished theorem of Deligne:

Theorem 2.5. *(Eilenberg-MacLane, Breen, Deligne) There is a functorial resolution for an abelian group A*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0, \tag{2.1}$$

where the first maps

$$\mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

send $[(a,b)] \mapsto [a+b]-[a]-[b]$, $[a] \mapsto a$,and all n_i and $r_{i,j}$ are non-negative integers.

We will not cover the proof of this important theorem on this lecture (hopefully in a future one), its proof involves studying the stable homotopy theory (actually stable homology) of Eilenberg-MacLane spaces, see [Sch19, Appendix to Lecture IV].

An immediate consequence of the theorem is the existence of a spectral sequence from condensed cohomology to Ext functors:

Corollary 2.6. *For condensed abelian groups A,M and S an extremally disconnected set, there is a natural spectral sequence*

$$E_1^{i_1,i_2} = \prod_{j=1}^{n_{i_1}} H^{i_2}(A^{r_{i_1,j}} \times S, M) \Rightarrow \underline{\mathrm{Ext}}^{i_1+i_2}(A,M)(S).$$

Proof. Just use the resolution (2.1), use that $R\underline{\mathrm{Hom}}(A,M)(S) = R\mathrm{Hom}(A \otimes \mathbb{Z}[S], M)$, and that

$$R\mathrm{Hom}(\mathbb{Z}[A] \otimes \mathbb{Z}[S], M) = R\mathrm{Hom}(\mathbb{Z}[A \times S], M) = R\Gamma(A \times S, M).$$

□

Proof of Theorem 2.3. We start by proving part (i). First, let us assume that I is finite, then we are reduce to proving that

$$R\underline{\mathrm{Hom}}(\mathbb{T}, M) = M[-1].$$

By taking the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$ this boils down to proving that $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$. By Corollary 2.6 we have a spectral sequence

$$E_1^{i_1,i_2} = \prod_{j=1}^{n_{i_1}} H^{i_2}(\mathbb{R}^{r_{i_1,j}} \times S, M) \Rightarrow \underline{\mathrm{Ext}}^{i_1+i_2}(\mathbb{R},M)(S). \tag{2.2}$$

The spaces $\mathbb{R}^n \times S$ are locally compact, eg. by writing $\mathbb{R} = \bigcup_k [-k,k]$. We can then compute

$$R\Gamma(\mathbb{R}^n \times S, M) = R\varprojlim_n R\Gamma([-k,k]^n \times S, M).$$

But $R\Gamma([-k,k] \times S, M)$ is the same as sheaf cohomology since M is discrete, this one is homotopic invariant, so the zero section $S \to [-k,k]^n \times S$ induces an equivalence $R\Gamma([-k,k]^n \times S, M) = R\Gamma(S,M) = C(S,M)$. Therefore, the zero section $S \to \mathbb{R}^n \times S$ induces an equivalence

$$R\Gamma(\mathbb{R}^n \times S, M) = C(S,M).$$

This shows that, thanks to the naturality of the resolution (2.1), the zero section $0 \to \mathbb{R}$ induces an equivalence for the spectral sequence (2.2), and so an equivalence

$$R\underline{\mathrm{Hom}}(\mathbb{R}, M) = R\underline{\mathrm{Hom}}(0, M) = 0.$$

For a general index I, by the spectral sequence (2.2), we have that

$$R\underline{\mathrm{Hom}}(\prod_I \mathbb{T}, M) = \varinjlim_{J \subset I} R\underline{\mathrm{Hom}}(\prod_J \mathbb{T}, M) = \varinjlim_I \bigoplus_J M[-1] = \bigoplus_I M[-1],$$

where J runs over finite subsets of I.

Now we prove (ii). By the aciclicity of real condensed cohomology on compact Hausdorff spaces, the resolution (2.1) shows that $R\underline{\mathrm{Hom}}(A, \mathbb{R})(S)$ is computed by a Banach complex with terms given by the Banach spaces $\bigoplus_{j=1}^{n_i} C(A^{r_{i,j}} \times S, \mathbb{R})$. Now, consider the multiplication by 2 in $R\underline{\mathrm{Hom}}(A, \mathbb{R})(S)$. It is represented by multiplication by 2 in $\mathbb{R}$, and also by multiplication by [2] in A. In other words, there is an homotopy h_n between $2\times -$ and $\circ[2]$ in the complex

$$0 \to C(A\times S,\mathbb{R}) \to C(A^2\times S,\mathbb{R}) \to \cdots \to \bigoplus_{j=1}^{n_i} C(A^{r_{i,j}}\times S,\mathbb{R}) \to \cdots .$$

Assume that $f\in \bigoplus_{j=1}^{n_i} C(A^{r_{i,j}}\times S,\mathbb{R})$ is such that $df=0$. Then $2f-[2]^*f = dh^*_{i-1}(f)$ and we can write

$$f = \frac{1}{2}([2]^*f + dh^*_{i-1}(f)),$$

iterating this formula we find that

$$f = d(\frac{1}{2}h^*_{i-1}(f) + \frac{1}{4}h^*_{i-1}([2]^*f)+\cdots),$$

but h^*_{i-1} is a map of Banach spaces, so bounded, and $[2]^*$ is bounded by 1. This shows that the previous series converges and that f is also a boundary. We deduce that

$$R\underline{\mathrm{Hom}}(A,\mathbb{R}) = 0$$

which finishes the proof. $\square$

References

[Sch19] Peter Scholze. Lectures on Condensed Mathematics. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf, 2019.