Modern Geometry: Fall, 2021: Part 1 – The Basics

October 3, 2021

Introduction

These notes represent the first part of the lecture notes for the course. They are divided into four sections. The first starts with a brief review of the history of geometry and then introduces the basic objects that will concern us for the entire course: smooth and complex manifolds, their structure sheaves of functions and Lie Groups. The second section introduces vector bundles with the prime example of the tangent bundle and other tensor budnles of a smooth manifold. We introduce vector fields with their Lie bracket and prove the existence and uniqueness of integral curves for a vector feild..

1 Basic Definitions of Sheaves, Manifolds, and Lie groups

1.1 Classical Geometry: Euclidean geometries

Geometry began with Euclid's axioms for plane geometry. This is a synthetic geometry. The basic notions are *points* and *lines*: with the usual Euclidean axioms: through any two points there is a unique line, two lines meet in at most one point, and if they do not meet then they are call *parallel*; given a line and a point not on the line there is a unique parallel to the line through the point, etc.

The standard realization of this geometry as the usual Cartesian plane with coordinates (x, y) and distance function

$$d((x, y), (x', y')) = \sqrt{(x-x')^2 + (y-y')^2}.$$

Points are the usual points of the space and lines are straight lines infinite in both directions. These are *geodesics*, in the sense that they are length minimizing curves. This model satisfies all the Euclidean axioms

1.1.1 Early Generalizations

Dropping the parallel axiom there is *spherical geometry*: the model space is the unit sphere in Cartesian 3-space; the geodesics are great circles. All lines meet: on the sphere, hence there are no parallel lines. Unfortunately lines meet in two points rather than the one required by the Euclidean axiom, but this defect is remedied by passing to the projective plan where antipodal points are identified. Again the great circles are geodesics, i.e., locally length minimizing curves.

Much later in the 19^{th} came Lobachevskian geometry where through a point not on a line there are infinitely many lines parallel to the given line. A model for this geometry is the open unit disk in the plane with lines being circular arcs orthogonal to the unit circle. These lines are length minimizing for the Poincaré metric on the open unit disk, or equivalently the (open) upper-half space model where lines are vertical lines and circular arcs perpendicular to the x-axis.

All of these geometries are *homogeneous* in the sense that given any two points there is an isomorphism of the geometry taking the first point to the second, and given two lines through a point there is an isomorphism of the geometry fixing the point and sending the first line to the second.

1.1.2 What Is Modern Geometry?

The word *geometry* comes from the roots *geo* meaning earth and *metric* meaning measure. It was originally conceived of as the study of measurement on the earth (distances, areas, etc). In modern geometry the objets consist of a space (almost alway a topological space) together with some extra structure that allows for some type of measurement on the space. The most direct generalization of Euclidean geometry is *Riemannian geometry* where the spaces are smooth manifolds and notion of measurement is a local (infinitessimal) one of angels and lengths. The study also goes under the name of differential geometry, especially when considering curves, surfaces and other submanifolds of a given ambient manifold and their curvatures inside the ambient manifold. More exotic is Spin geometry which arises from the fact that the orthogonal group has a non-trivial double covering, giving rise to spinor fields . Related to Riemannian geometry is *conformal geometry* where the local measurement is angles only.

There is *symplectic geometry* where the measurement is of areas of surfaces and higher dimensional volumes of even dimensional submanifolds. There is the closely related study of contact geometry, which concerns odd dimensional manifolds ('boundaries' of symplectic manifolds).

In the context of spaces endowed with complex structure there is an analogue of Riemannian geometry called *hermitian geometry*, and an important subcase of Kähler geometry. More special than this are Calabi-Yau geometry (meaning Kähler and complex sympletic) and Hyperkähler geometry based on the quaterions.

Other modern uses of the term geometry include complex geometry where the spaces are endowed with a complex structure, and algebraic geometry where the spaces are defined by polynomial functions (and those polynomial functions are part of the structure of the space). In these uses of the word geometry, there is extra structure but it is not directly a measurement as we usual conceive of it, rather geometry in this context refers to the fact that there are interesting pictures of the spaces in question associated with the extra structure.

Most of the spaces that we will encounter will be smooth manifolds, but singularities (and singularity development) are often important aspects of geometry. This is especially true in algebraic geometry. We will only touch on singularities in this course.

1.2 Topological Manifolds

A topological manifolds is a space (without singularities) on which various types of geometry exist as extra structure.

In every dimension $n \geq 0$ there is a Euclidean space, denoted $\mathbb{R}^n$. Its points are ordered n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ of real numbers. There is the Euclidean distance function

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

This is metric in the usual topological sense, e.g., it is a continuous function of two variables, symmetric in the two variables; it satisfies the triangle inequality; and if $d(\mathbf{x}, \mathbf{y}) = 0$, then $\mathbf{x} = \mathbf{y}$. In the usual way, this metric defines a topology. The resulting topological space endowed with this distance function is *Euclidean n-space.*

Definition 1.1. An *n-dimensional manifold* is a paracompact, Hausdorff space with the property that every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. An *n-dimensional manifold with boundary* is a paracompact Hausdorff space with the property that every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or of the half-space $x^1 \geq 0$ in $\mathbb{R}^n$.

Invariance of domain implies that each point of a compact manifold with boundary with the second type of neighborhood in which it maps to a point of the boundary $\{x_1 = 0\}$ does not have a neighborhood of the first type. The subset of points satisfying this condition form a closed subset and inherit the structure on an $(n-1)$-dimension manifold without boundary.

Alternatively, we could define a manifold by giving a Hausdorff topological space M and a countable atlas of open subsets $\{U_i \subset M\}$ in M (*atlas* in the sense that $\cup_i U_i = M$), together with homeomorphisms $\varphi_i : U_i \to V_i$ where the V_i are open subsets in $\mathbb{R}^n$ or in $\{x^1 \geq 0\}$.

Or we could start with open subsets $V_i \subset \mathbb{R}^n$ or in $\{x^1 \geq 0\} \subset \mathbb{R}^n$ and for each ordered pair $\{i,j\}$ an open subset $W_{i,j} \subset V_i$ and homeomorphisms $h_{i,j} : W_{i,j} \to W_{j,i}$ (called *transition functions* or *overlap functions*) that satisfy $h_{j,k} \circ h_{i,j} = h_{i,k}$ on $h_{i,j}^{-1}(W_{j,k}) \cap W_{i,k}$ for all triples $\{i,j,k\}$. Also, we assume that for all i we have $W_{i,i} = V_i$ and $h_{i,i} = \mathrm{Id}_{V_i}$ and for all $\{i,j\}$ we have $h_{j,i} = h_{i,j}^{-1}$. In this presentation we have to assume that the quotient space

$$\cup_i V_i / \{x \in W_{i,j} \equiv h_{i,j}(x) \in W_{j,i}\}$$

is a Hausdorff space.

Using the first definition, for each open subset $U_i \subset M$, the homeomorphism $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$ or $\subset \{x^1 \geq 0\}$ produces local coordinates $\{x^1, \ldots, x^n\}$ on U_i. These are the pull back via φ_i of the usual Cartesian coordinates restricted to the open V_i in $\mathbb{R}^n$. These local coordinates determine and are determined by the hopmeomorphism of U to an open subset V in $\mathbb{R}^n$ or $\{x^1 \geq 0\}$.

Of course, these are only local coordinates. We define $W_{i,j} \subset V_i$ as $\varphi_i(U_i \cap U_j)$ and the overlap function $h_{i,j} : W_{i,j} \to W_{j,i}$ to be $\varphi_j \circ \varphi_i^{-1}$. These functions give the transition formulae for one set of local coordinates in terms of the other on the overlap.

1.3 Sheaves

We begin with a very important general notion: that of a *sheaf* on a topological space X. Let $Op(X)$ be the category with objects the open subsets of X and the morphisms the inclusions of open sets. A *presheaf* (of real vector spaces) is a contravariant functor from $Op(X)$ to the category of vector spaces. In down-to-earth terms this means an assignment for each open subset $U \subset X$ of a vector space $\mathcal{F}(U)$ and if $V \subset U$ a 'restriction map' $r_{U,V}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ compatible with compositions in the sense that $r_{V,W} \circ r_{U,V} = r_{U,W}$.

A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the following *gluding axiom*: For every open cover $U = \cup_i U_i$ of an open set $U \subset X$ the following sequence is exact:

$$0 \to \mathcal{F}(U) \xrightarrow{\prod_i r_i} \prod_{\mathcal{F}}(U_i) \overset{\prod_{i\neq j}(r_{i,j}-r_{j,i})}{\longrightarrow} \prod_{i\neq j} \mathcal{F}(U_i \cap U_j),$$

where $r_i = r_{U,U_i}$ and $r_{i,j} = r_{U_i,U_i \cap U_j}$. Notice that by the composition axiom the composition of the two mappings is zero. Hence, the sheaf condition is that any element in the kernel of $\prod_{i,j}(r_{i,j} - r_{j,i})$ is in the image under $\prod_i r_i$ of a unique element of $\mathcal{F}(U)$. This means that given an element $\{s_i\} \in \prod_i \mathcal{F}(U_i)$ there is an element $s \in \mathcal{U}$ whose restriction to U_i is equal to s_i for each i if and only if the $\{s_i\}$ satisfy the compatibility condition that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, and furthermore such an s is unique.

1.4 Smooth manifolds

Definition 1.2. We say that a real-valued function $\psi\colon V \to \mathbb{R}$ on an open subset of a Euclidean space is *smooth* if it has continuous partial derivatives of all orders. Similarly, a function on an open subset of $\{x^1 \geq 0\}$ is smooth if it has a smooth extension to some open subset of $\mathbb{R}^n$. More generally, a map between open subsets of Euclidean space or Euclidean half-space $\psi\colon V \to W$ is *smooth* iffy the composition of ψ with all the Euclidean coordinate functions restricted to W are smooth.

Notice that the composition of smooth functions is smooth and that a function is smooth if and only if it is smooth in some neighborhood of every point of the domain.

Definition 1.3. Let M be an n-dimensional manifold. A *smooth atlas* for an n-dimensional manifold M consists of an atlas $\{U_i \subset M\}$ and of homeomorphisms $\varphi\colon U_i \to V_i$ to open subsets of $V_i \subset \mathbb{R}^n$ or $V \subset \{x^1 \geq 0\}$ with the property that the overlap functions $h_{i,j}\colon W_{i,j} \to W_{j,i}$ are smooth. This is equivalent to requiring that the differentials of the $h_{i,j}$ are C^∞ at every point of their domains. It is equivalent to say that the local coordinates for one coordinate are smooth functions of the coordinates from any other coordinate patch on their overlap.

Given a smooth atlas we say that a function $f\colon U \to \mathbb{R}$ from an open subset of M to $\mathbb{R}$ is *smooth* (with respect to the given smooth atlas) if for each $p \in U$ and any U_i in the atlas containing p the function $f \circ \varphi_i^{-1}$ is smooth function on $\varphi_i(U_i \cap U) \subset V_i \subset \mathbb{R}^n$. Notice this condition is independent of the choice of U_i in the smooth atlas containing p. Let $C^\infty(U)$ denote the vector space of smooth functions on U. Clearly if $U' \subset U$ then restriction of functions defines a linear map $C^\infty(U) \to C^\infty(U')$ and if $U'' \subset U'$ the restriction from U to U'' is the composition of the restriction from U to U' followed by the restriction from U' to U''. Hence, this defines a presheaf on M. Since a function is determined by its values and a function is C^∞ if and only if it is C^∞ in a neighborhood over every point of its domain, this presheaf satisfies the gluing axiom and hence is a sheaf. It is the *sheaf of smooth functions* defined by the smooth atlas. Two smooth atlases define the *same smooth structure* on M if their sheaves of smooth functions are identical. Hence, a smooth structure on M is an equivalence class of smooth atlases.

It is easy to see that two smooth atlases define the same smooth structure iff the coordinate functions of all the charts in one atlas are smooth functions with respect to the other atlas and vice-versa.

Definition 1.4. An *n-dimensional smooth manifold* is a topological n-dimensional manifold endowed with a smooth structure. A *smooth manifold* is a disjoint union of n-dimensional smooth manifolds for various n.

It is an elementary exercise to show a sheaf on an n-dimensional manifold M is the sheaf of smooth functions with respect to some smooth atlas on M if and only if it is locally isomorphic to the sheaf of smooth functions on $\mathbb{R}^n$.

Definition 1.5. If M and N are smooth manifolds, then a continuous map $\varphi\colon M \to N$ is *smooth* iff for every open set $U \subset N$ and ever $f \in C^\infty(U)$ we have $\varphi^* f \in C^\infty(\varphi^{-1}(U))$.

Smooth manifolds and smooth maps between them form a category, the *smooth category*; an isomorphism in this category is called a *diffeomorphism*.

This category has (countable) sums given by disjoint union and finite products given by Cartesian product.

A *smooth submanifold* of a manifold M is a subset $X \subset M$ with the property that for every $x \in X$ there is a coordinate chart $U \subset M$ with local coordinates $(x_1, \ldots, x_n)$ with the property $X \cap U$ is given by the vanishing a a subset of these coordinates. Clearly, a submanifold inherits the structure of a smooth manifold such that the inclusion is a smooth map.

The usual product of topological spaces produces a product on the category of smooth manifolds: the smooth coordinate charts for $M \times N$ are the product of the charts of M with those for N. This extends to an associative product of finitely many smooth manifolds.

1.5 Complex manifolds

A *complex atlas* on a manifold M is an atlas of coordinate charts $\varphi_i\colon U_i \to V_i$ where the V_i are open subsets of $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$ in the usual way) with the property that the overlap functions $h_{i,j}\colon W_{i,j} \to W_{j,i}$ are *holomoprphic*, meaning that the differential of the $h_{i,j}$ at every point of its domain is a complex linear map from $\mathbb{C}^n$ to itself. (Complex manifolds do not have boundaries in the sense that smooth ones do, since the boundary is real codimension 1 whereas all complex manifolds have even real dimension.)

Completely analogously to the smooth case, a complex atlas defines a sheaf of complex-valued holomorphic functions on open subset of M. Here, the vector spaces assigned to open subsets are complex vector spaces, instead of real vector spaces. A *complex structure* is an equivalence class of *complex atlases*, where two complex atlases are equivalent if and only if they define the same sheaf of holomorphic functions. Complex manifolds form a category with the morphisms being continuous maps that pull local holomorphic functions on the range back to local holomorphic functions on the range.

There is a natural product (finite products) in the category of complex manifolds which on the level of topological spaces is the usual product.

Notice that underlying a complex manifold of (complex) dimension n is a smooth manifold of (real) dimension $2n$.

1.6 Lie Groups

A *Lie group* is a smooth manifold G that has a group structure given by a multiplication map $\mu\colon G \times G \to G$ and inverse map $\iota\colon G \to G$ with the property that both these structure maps are smooth.

Examples of Lie Groups include countable discrete groups, and in parrticular finite groups, the circle, the general linear group, denoted $GL(n, \mathbb{R})$ of all linear isomorphisms of $\mathbb{R}^n$, the orthogonal group, the group of translations of a real vector space. The 3-sphere has a group structure because it is identified with the group of quaterions of unit norm and these are closed under quarerionic multipliciation.

A morphism between Lie groups is a c smooth map that is also a group homomorphism. A Lie subgroup of a Lie group G is a smooth submanifold that is also a subgroup. The orthogponal group $O(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$.

The group $GL(n, \mathbb{C})$ of complex linear isomorphisms of $\mathbb{C}^n$ is also a Lie group. The unitary group $U(n)$ consisting of all complex linear transformations that preserve the standard hermitian inner product on $\mathbb{C}^n$ is a Lie subgroup of $GL(n, \mathbb{C})$.

A finite product of Lie groups is a Lie group.

Definition 1.6. Let G be a Lie group and let M be a smooth manifold. Then a *smooth action* of G on M is an action

$$\mu \colon G \times M \to M$$

that is a smooth map.

Exercise: Show that if G is a finite group acting freely on a smooth maniold M there is a unique smooth structure on the quotient M/G with the property that the smooth functions on M/G are exactly those whose pull back to M are smooth functions. The smooth functions on M/G are identified in this way with the smooth functions on M invariant under G.

The antipodal action of $\{\pm 1\}$ on $\mathbb{R}^n$ is defined by $\mu(\pm, (x_1, \ldots, x_n)) = (\pm x_1, \ldots, \pm x_n)$. It is a smooth action. whose restriction to the unit sphere S^{n-1} is a free smooth action.

The quotient of S^n by the antipodal action is denoted $\mathbb{R}P^n$ and is called the *real projective n-space*. It is naturally identified with the space of real lines (1-dimensional linear suspaces) of $\mathbb{R}^{n+1}$. It follows that $|rP^n$ is a smooth manifold and its smooth functions are identified with the smooth functions on Sn invariant under the antipodal action.

Definition 1.7. A *complex Lie group* G is a complex manifold with a group structure with the property that the multiplication map $\mu \colon G \times G \to G$ and the inverse map $\iota \colon G \to G$ are holomorphic mappings (i.e., morphisms of complex manifolds)..

Any closed complex submanifold of $G(n, \mathbb{C})$ that is closed under matrix multiplication is a complex Lie group. These are called *linear algebraic groups* (defined over the complex numbers). The matrix coordinates are holomorphic functions on any linear algebraic group. It follows easily from the fact holomorphic functions on compact complex manifolds are locally constant that any positive dimensional linear algebraic group (over $\mathbb{C}$) is non-compact.

An example of a compact complex Lie group is given by $\mathbb{C}/\Lambda$ where Λ is a *lattice* in $\mathbb{C}$, i.e., a subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ generated by two elements of $\mathbb{C}$ that are linearly independent over $\mathbb{R}$. By a complex-linear change of coordinate on $\mathbb{C}$ we can assume that the subgroup is generated by 1 and an element of the form $\tau = a + bi$ with $b > 0$. The resulting element τ in the upper half-plane is well-defined up to fraction linear transformations

$$\tau \mapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an element of $SL(2, \mathbb{Z})$, the group of integral 2×2 matrices with determinant 1.

1.7 Examples

Suppose that $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function and at each $x \in F^{-1}(0)$ the differential $df_x\colon \mathbb{R}^{\to}\mathbb{R}$ is non-zero (meaning that for each $x \in F^{-1}(0)$ there is at least one partial derivative of F at x that is non-zero. Then by the implicit function theorem, for each $x \in F^{-1}(0)$ here is a neighborhood $U(x) \subset F^{-1}(0)$ and one of the coordinates, say for simplicity of notation x_{n+1} such that U_x is the graph of a function $x_{n+1} = \varphi_x(x_1, \ldots, x_n)$ defined on an open subset V_x of the coordinate hyperplane defined by $x_{n+1} = 0$. This produces a local coordinate chart of the neighborhood U_x. The union of all such charts is easily seen to define a smooth atlas. The sheaf of C^∞-functions determined by this atlas is exactly the restriction to $F^{-1}(0)$ of the sheaf of C^∞-functions on $\mathbb{R}^{n+1}$.

Indeed the smooth function F does not need to be defined on all of $\mathbb{R}^{n+1}$, just on an open subset of $\mathbb{R}^{n+1}$.

Taking

$$f(x_1, \ldots, x_{n+1}) = 1 - \sum_i x_i^2$$

we see that the unit sphere $S^n \subset \mathbb{R}^{n+11}$ is a smooth n-dimensional manifold.

More generally, if $U \subset \mathbb{R}^{n+k}$ is an open subset and $F\colon U \to \mathbb{R}^k$ is a smooth function with the property that at each point $x \in F^{-1}(0)$ the differential $DF_x\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ has rank k (ie., is a surjective linear map), the $F^{-1}(0)$ inherits the structure of a smooth n-manifold.

Let $O(n)$ be the group of real orthogonal matrices. Then $O(n)$ is a smooth manifold of dimension $n(n-1)/2$. The reason is that $O(n) \subset M(n)$, where $M(n)$ is the space of $n \times n$ matrices which is identified with $\mathbb{R}^{n^2}$. The orthogonal group is defined by the $n(n+1)/2$ conditions that say the columns of the matrices have inner product $+1$ with themselves and the inner product between two different columns is zero. The gradients of these $n(n+1)/2$ defining equations are linearly independent at any element of $O(n) \subset M(n \times n)$. This establishes that $O(n)$ is a smooth submanifold of $M(n \times n)$ and hence of $GL(n, \mathbb{R})$. Hence, it is a Lie group.

We have embeddings $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ and these lead to embddings $O(n-1) \subset O(n)$ making $O(n-1)$ a sub-Lie group of $O(n)$; it is the subgroup of matrices that fix $e_n = (0, 0, \ldots, 0, 1) \in \mathbb{R}^n$. Thus the map $O(n) \to S^{n-1}$ that sends $A \in O(n)$ to Ae_n is a smooth action and it identifies $O(n)/O(n-1)$ with S^{n-1}. Similarly. $O(n)/O(n-k)$ is identified with the space of orthonormal k-frames $\{f_1, \cdots, f_k\}$ in $\mathbb{R}^n$. That is to say $\langle f_i, f_j \rangle = \delta_{i,j}$. These spaces inherit manifold structures from the smooth structure on $O(n)$. As function on the quotient space is smooth if and only if its pullback to $O(n)$ is smooth.

There is a similar story for the unitary groups $U(n) \subset GL(n, \mathbb{C})$ of matrices that preserve the standard hermitian form on $\mathbb{C}^n$. The quotient of $U(n)/U(n-1)$ is identified with the unit sphere in $\mathbb{C}^n$; i.e., S^{2n-1}.

Notice that even though the unitary group is defined via complex structures it is not a complex Lie group. Indeed the odd unitary groups are of odd dimension.

Real projective n-space. Consider the space $\mathbb{R}P^n$ of linear subspaces of dimension 1 (lines) in $\mathbb{R}^{n+1}$. Each such line meets unit sphere S^n in a pair of antipodal points. Thus, the space is identified with the quotient of S^n by the antipodal action of the group of two elements. The smooth structure is the induced one. A point $x \in \mathbb{R}P^n$ is determined any any point $(x_0, \ldots, x_n)$, different from 0, in the line. Any two such points differ by coordinate-wise multiplication by a non-zero real number. We denote the "projective" coordinates by $[x_0, \ldots, x_n]$, not all zero, where implicitly we can replace the coordinates by any non-zero multiple. This is a smooth n-dimensional manifold.

Complex projective n-space. $\mathbb{C}P^n$ is the space of 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$. The complex projective coordinates are $[z_0, \ldots, z_n]$,not all zero; two-such are equivalent if they differ by coordinate-wise multiplication b a non-zero complex numbers. A complex atlas is given by $U_i, 0 \leq i \leq n$ consists of all points whose projective coordinates have non-zero entry in position i.

The homeomorphism $\varphi_i \colon U_i \to \mathbb{C}^n$ defined by

$$\varphi_i([z_0, \ldots, z_n]) = (z_0/z_i, \ldots, z_{i-1}/z_i.z_{i+1}/z_i, \ldots, z_n/z_i).$$

One sees easily that the overlap functions are holomorphic.

Suppose that $f(z_0, \ldots, z_n)$ is a homogeneous polynomial of degree d with the property that ∇f is non-zero at every point $(z_0, \ldots, z_n) \neq (0, \ldots, 0)$ satisfying $f(z_0, \ldots, z_n) = 0$, then the solutions of $f = 0$ is a union of complex lines and hence defines a subset of $\mathbb{C}P^n$. An easy computation shows that the locus in $\mathbb{C}P^n$ is a comnplex codimension-1 submanifold of $\mathbb{C}P^n$, called a *smooth hypersurface of degree d.*. A hypersurface of degree 1 is isomorphic to $\mathbb{C}P^{n-1}$.

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2 Vector Bundles, the tangent bundle and other tensor bundles of smooth manifolds, vector fields

2.1 Vector Bundles

2.2 The basic definition

In this lecture all real vector spaces are implicitly assumed to be finite dimensional. There are analogues for infinite dimensional spaces but one has to specify topologies or other structures, e.g., Banach spaces, Hilbert spaces, or Frechet spaces.

Definition 2.1. Let X be a topological space. A *family of real vector spaces over* X is a continuous map $\pi\colon \mathcal{E} \to X$ and two maps, a sum map

$$\mu\colon \mathcal{E} \times_X \mathcal{E} \to \mathcal{E}$$

and a scalar multiplication map

$$\mathbb{R} \times \mathcal{E} \to \mathcal{E}$$

both commuting with the natural projections to X making each fiber $\pi^{-1}(x)$ a real vactor space. A real vector space is the same thing as a family of real vector spaces over a point.

Families of real vector spaces over topological spaces are the objects of a category. The morphisms are commutative diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}' \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi'}\big\downarrow \\ X & \longrightarrow & X' \end{array}$$

commuting with the structure maps. There is also a pull back mapping: if $\pi\colon \mathcal{E} \to X$ is a family of real vector spaces and $f\colon Y \to X$ is a continuous map, then there is a family $f^*\mathcal{E} \to Y$ and a morphism

$$\begin{array}{ccc} f^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

defined by $f^*\mathcal{E} = \mathcal{E} \times_X Y$ with the naturally induced projection and structure maps. In particular, given a family of real vector spaces over X and a subspace $Y \subset X$, there is the *restriction* of the family to Y.

We say that a family of real vector spaces is *locally trivial* if for every $x \in X$ the restriction of the family to a neighborhood U of x is isomorphic to the trivial family $V \times U$ where V is a real vector space and the structure maps are pulled back from the usual structure maps on V. Said another way $\mathcal{E}|_U$ is isomorphic to the pullback of a family over a point.

Definition 2.2. A*real vector bundle* is a locally trivial family of real vector spaces.

There is an analogue of coordinate charts for real vector bundles. If $\pi\colon \mathcal{E} \to X$ is a real vector bundle, then there is an open covering $\{U_\alpha\}_\alpha$ of X, vector spaces V_α, and isomorphisms $\varphi_\alpha\colon \mathcal{E}|_{U_\alpha} \to V_\alpha \times U_\alpha$ of vector bundles.

The overlap functions are then isomorphisms $(U_\alpha \cap U_\beta) \times V_\alpha \to (U_\alpha \cap U_\beta) \times V_\beta$ covering the identity of $(U_\alpha \cap U_\beta)$. Such an isomorphism is given by a continuous map $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to Iso(V_\alpha, V_\beta)$. If X is connected, then we can choose identifications of all the V_α with a fixed vector space V. Then the overlap functions are given by continuous maps $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(V)$. These satisfy the cocycle conditions:

- $h_{\alpha\alpha} = \mathrm{Id}_{U_\alpha}$
- $h_{\beta\alpha} = h_{\alpha\beta}^{-1}$
- $h_{\beta\gamma} \circ h_{\alpha\beta} = h_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

There is full subcategory of real vector bundles in families of vector spaces.

A subbundle of a vector bundle $\mathcal{E} \to X$ is a family of subspaces $\mathcal{F}_x \subset \mathcal{E}_x$ for all $x \in X$, such that there are local trivializations of $\mathcal{E}$ that also give local trivializations of $\mathcal{F}$.

2.3 Examples

Associated to any real vector space V for every topological space X there is the trivial vector bundle $V \times X$ pulled back from the bundle $V \to \{*\}$ by the unique map $X \to \{*\}$. As another example in trivial vector bundle $S^1 \times \mathbb{R}^2 \to S^1$ consider the sublocus of all (θ, x) where x lies on the line $\mathbb{R} \cdot (cos(\theta/2), sin(\theta/2))$. This is a subvector bundle whose total space is isomorphic to the open Möbius band.

2.3.1 Sheaves of Sections

Definition 2.3. A *sheaf of rings* over a topological space X is a contravariant functor from the category of open subsets of X to rings satisfying he gluing axiom.

For example, the sheaf of real valued functions on X, denoted $\mathcal{O}(X)$ is a sheaf of rings over X.

Definition 2.4. Given a sheaf of rings $\mathcal{R}$ over X a *sheaf of $\mathcal{R}$-modules* is a sheaf of abelian groups on X such that for each open set $U \subset X$ the group associated to U has the structure of an $\mathcal{R}(U)$-module and the restriction maps are compatible with the ring and module structures.

Definition 2.5. A*section* of a vector bundle $\pi\colon \mathcal{E} \to X$ over an open set U is a map $s\colon U \to \mathcal{E}$ satisfying $\pi \circ s = \mathrm{Id}_U$. These from an abelian group under the structure maps of the vector bundle (called fiberwise addition and scalar multiplication).

Proposition 2.6. *Let $\mathcal{E} \to X$ be a vector bundle. The functor that assigns to each open set $U \subset X$ the abelian group of sections of this bundle over U is a sheaf of abelian groups. Fiberwise scalar multiplication makes this sheaf into a sheaf of $\mathcal{O}(X)$-modules. [Recall that $\mathcal{O}(X)$ is the sheaf of rings of local real-valued functions on X.]*

Remark 2.7. A vector bundle is isomorphic to a trivial bundle $X \times V$ if and only if its sheaf of sections is a free module of finite rank.

2.4 Smooth Vector Bundles

Let M be a smooth manifold. A *smooth* vector bundle over M is a vector bundle $\mathcal{E} \to M$ together with a smooth structure on $\mathcal{E}$ such that the projection to M is a smooth submersion (surjective differential) with smooth local trivializations such that the structure maps for addition and scalar multiplication are smooth maps. In this case by $\mathcal{O}(M)$ we mean the sheaf of smooth functions on M. It is a sheaf of rings and a subsheaf of the sheaf of continuous functions. A *smooth section* of a smooth vector bvundle $\pi\colon \mathcal{E} \to M$ over an open set U is a smooth function $s\colon U \to \mathcal{E}$ with $\pi \circ s = \mathrm{Id}_U$. These form a sheaf of abelian groups under fiberwise addition and scalar multipliciation and indeed form a module over the sheaf of smooth functions. A smooth vector bundle is trivial as a smooth vector bundle if and only if this module is a a free module of finite rank.

The overlap functions for smooth local trivializations of a smooth vector bundle are smooth maps from $h_{ij}\colon U_i \cap U_j \to GL(n, \mathbb{R})$.

2.5 Complex Vector Bundles

We can replace real vector spaces by complex vector spaces and produce the categories of a family of complex vector spaces over a topological space and of complex vector bundles over topological spaces. There is a forgetful functor that associates to a family of complex vector spaces the underlying family of real vector spaces and associates to a complex vector bundle the underlying real vector bundle. In the case of complex vector bundles the overlap functions for local trivializations are maps from the intersections of the open sets in the base to $GL(n, \mathbb{C})$.

As above when we work over a smooth manifold we have the category of smooth complex vector bundles. The sheaf of smooth sections is a sheaf of modules over the sheaf of smooth complex-valued functions on the base.

When we work over a complex manifold we have the category of so-called *holomorphic* vector bundles where the total space is a complex manifold, the projection mapping is a holomorphic surjection, the local trivializations are holomorphic isomorphisms and the structure maps are holomorphic. In this case the overlap functions for holomorphic trivializations are holomorphic maps $h_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$. The sheaf of local holomorphic sections is a sheaf of modules over the sheaf of holomorphic functions on the base and the vector bundle is holomorphically trivial. if and only if this sheaf of modules is free of finite rank.

2.6 Tangent Bundle

Let M be a smooth n-manifold. Denote by $I_x(M)$ the ideal of the ring of smooth functions on M that vanish at x.

Lemma 2.8. *$I_x(M)/(I_x(M))^2$ is an n-dimensional real vector space.*

Proof. For any open subset $U \subset M$ containing x, the restriction map on smooth functions induces a ring homomorphism sending $I_x(M)$ to $I_x(U)$. The kernel consists of functions on M vanishing near x. The map is not onto, but any C^∞-function on U is the sum of the restriction of a C^∞-function on M and a function on U vanishing near x. [Use a bump function to damp a given function on U to 0 outside a compact set and extend by 0 to all the rest of M.]

Functions on M and U vanishing a neighborhood of x are contained in $(I_x(M))^2$ and $(I_x(U))^2$, respectively. [Multiply by a function that vanishes near x and is 1 off the support of the function.] It follows that the induced linear map

$$I_x(M)/(I_x(M))^2 \to I_x(U)/(I_x(U))^2$$

induced by restriction is an isomorphism.

Thus, it suffices to prove the result for any neighborhood of x. We choose a coordinate chart containing x; i.e., a smooth isomorphism $\varphi\colon U \to V$ where V is an open subset of $\mathbb{R}^n$ and $x \in U$. We can assume that $\varphi(x) = 0$. Cleary, φ identifies $I_x(U)/(I_x(U))^2$ with $I_0(V)/(I_0(V))^2$ and the latter is identified with $I_0(\mathbb{R}^n)/(I_0(\mathbb{R}^n))^2$.

Let F be a smooth function on $\mathbb{R}^n$ vanishing at the origin. Then by the chain rule

$$F(\mathbf{x}) = \int_0^1 \nabla F(t\mathbf{x}) \cdot \mathbf{x} dt = \sum_i x_i \int_0^1 \frac{\partial F(t\mathbf{x})}{\partial x^i} dt.$$

We set $h_i(\mathbf{x}) = \int_0^1 \frac{\partial F(t\mathbf{x})}{\partial x^i} dt.$, so that $F(\mathbf{x}) = \sum_i x_i h_i(x)$ and $h_i(0) = \frac{\partial F(0)}{\partial x^i}$. Thus, $F \in (I_0(\mathbb{R}^n))^2$ if and only if $h_i(0) = 0$ for all $1 \le i \le n$. Hence, the map $F \to (h_1(0), \ldots, h_n(0))$ determines an isomorphism from $I_0(\mathbb{R}^n)/(I_0(\mathbb{R}^n))^2$ to $\mathbb{R}^n$.

□

Notice that a basis for $T_x(\mathbb{R}^n)$ are the linear maps $F \mapsto \frac{\partial F}{\partial x^i}|_x$ for $1 \le i \le n$.

Definition 2.9. The tangent space to M at $x \in M$, denoted T_xM is the vector space of linear maps from $I_x(M)(/I_xM)^2$ to $\mathbb{R}$. Notice that a basis for $T_x(\mathbb{R}^n)$ is $(\frac{\partial}{\partial x^i})_x$ for $1 \le i \le n$.

Consider TM the union over all $x \in M$ of the tangent spaces $T_x(M)$. We give this family of vector spaces a topology so that it forms a smooth vector bundle. First, if M is an open subset of $\mathbb{R}^n$ we define a global trivialization of TM by using the basis $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ at each point of M. This determines a function $\cup_{x \in M} T_xM \to M \times \mathbb{R}^n$ that is a linear isomorphism on each tangent space, and hence endows TM with the structure of a smooth vector bundle.

More generally, for any manifold M this construction produces a smooth vector bundle structure on $\cup_{x \in U} T_x(M)$ for any coordinate chart $U \subset M$. To see that this determines a global smooth bundle structure we must show that the smooth trivializations on the tangent bundle over the intersection of two charts determined by each of the charts differ by a bundle isomorphism given by a smooth map from the intersection to $GL(n, \mathbb{R})$. This follows immediately from the fact that the overlap function is $U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ is given by $(x, v) \mapsto (h_{\alpha,\beta}(x), Dh_{\alpha,\beta}(x)(v)$, which is a smooth isomorphism.

We denote this smooth vector bundle TM, and call it the *tangent bundle* of M.

The tangent bundle construction is a functor from the category of smooth manifolds and smooth maps to the category of smooth vector bundles and smooth vector bundle maps. It assigns to a smooth manifold M its tangent bundle TM and to a smooth map $f\colon M \to N$ the map that sends $v \in T_xM$ to $Df_x(v)$. As before the chain rule gives a formula in local coordinates on M near x and local coordinates on N near y for Df in a neighborhood of x. This formula shows that Df is a smooth bundle map.

Definition 2.10. Let M be a smooth manifold. A *vector field* on M is a smooth section of TM. In local coordinates $(x^1,\dots,x^n)$ on a coordinate chart for M any vector field is given as $\sum_i u_i(x^1,\dots,x^n)\frac{\partial}{\partial x^i}$ for some smooth functions u_i.

Remark 2.11. Often, to simplify notation, we denote $\partial/\partial x^i$ by ∂_i.

Notice that if $f\colon M \to N$ is a diffeomorphism then the smooth bundle map $Df\colon TM \to TN$ sends smooth local sections of TM to smooth local sections of TN. Thus, there is a map induced by Df from the vector fields on M to those on N. In particular, a self-diffeomorhism of M acts on the vector fields on M.

Claim 2.12. *Given a diffeomorphsims $f\colon M \to N$ with $f(\mathbf{x}) = \mathbf{y}$, the pull-back map f^* on smooth functions defines a map $I_{\mathbf{y}}(N) \to I_{\mathbf{x}}(M)$. Using local coordinates x^i for M near $\mathbf{x}$ and y^j for N near $\mathbf{y}$ and the resulting basis $\{\partial/\partial x^i\}(\mathbf{x})$ and $\{\partial/\partial y^j\}(\mathbf{y})$ for the dual spaces $T_{\mathbf{x}}M$ and $T_{\mathbf{y}}N$ the map dual to*

$$f^*\colon I_{\mathbf{y}}(N)/(I_{\mathbf{y}}(N))^2 \to I_{\mathbf{x}}(M)/)I_{\mathbf{x}}(M))^2$$

is given by matrix multiplication by the matrix of partial derivatives of f at $\mathbf{x}$:

$$Df_{\mathbf{x}} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial y^1}{\partial x^n}(\mathbf{x}) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{\partial y^n}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial y^n}{\partial x^n}(\mathbf{x}) \end{pmatrix}$$

Proof. Let $\varphi \in I_{\mathbf{y}}(N)$. We write $\varphi = \sum_j y^j\varphi_j$ where $\varphi_j(\mathbf{y}) = (\partial\varphi/\partial y^j)(\mathbf{y})$. Then

$$f^*\varphi = \sum_j y^j(x^1,\dots,z^n)\varphi(y(x^1,\dots,x^n)).$$

Module $(I_x(M))^2$ we have

$$y^j(x^1,\dots,x^n) = \sum_i (\partial y^j/\partial x^i)(\mathbf{x})$$

so that in $I_{\mathbf{X}}(M)/(I_{\mathbf{x}}(M))^2$ we see that

$$f^*\varphi = \sum_j \sum_i x^i \left[(\partial y^j/\partial x^i)(\mathbf{x})(\partial \varphi/\partial y^j)(\mathbf{y})\right].$$

It follows that under the dual map

$$(\partial/\partial x^i)(\mathbf{x}) \mapsto \sum_j (\partial y^j)/\partial x^i)(\mathbf{x})(\partial\varphi/\partial y^j)(\mathbf{y}).$$

Hence, the matrix for the dual map in the given bases is the matrix of partial derivatives for $Df(\mathbf{x})$. $\square$

Proposition 2.13. *Let $f\colon M \to N$ be a diffeomorphism; let χ be a vector field on M and φ a smooth function on N. Then*

$$Df(\chi)(f^*\varphi)(\mathbf{f}(\mathbf{x})) = \chi(f^*\varphi)(\mathbf{x}).$$

Said another way

$$Df(\chi) = (f^{-1})^*(\chi \circ f^*).$$

Proof. We have just established that the map $T_xM \to T_yN$ is given by the usual differential $Df(\mathbf{x})$. Of course, $\chi(f^*\varphi)(\mathbf{x}) = \chi(\varphi \circ f)(\mathbf{x})$ which by the chain rule is $Df(\mathbf{x})(\chi(\varphi))(f(\mathbf{x}))$. This equation for all $\mathbf{x} \in M$ is eqivalent to the equation

$$f^*\left(Df(\mathbf{x})(\chi(\varphi))\right) = \chi(f^*\varphi).$$

Taking $(f^{-1})^*$ of this equation gives the second equation in the propositioin. $\square$

This is the naturaity equation for the action of vector fields on smooth functions under diffeomorphisms.

There is a generalization of this. Suppose that $f\colon M \to N$ is a smooth. This means for each $x \in M$, there is a local coordinate system on an open set $U$of N centered at $f(x)$ such that $f\colon f^{-1}(f(N)\cap U) \to U$ is an embefding with image the intersection of U with a coordinate hyperplane, In this case there is an induced map $Df\colon TN \to TM$ covering embeds each T_xN as the coordinate linear subspace of $T_{f(x)}M$ using the trivialization of $TM|_U$ coming from th given coordinates on U.

2.6.1 Lie Bracket of Vector Fields

A basic structure of this action is that vector fields form a Lie algebra defined by

$$[\chi_1, \chi_2](f) = \chi_1(\chi_2(f)) - \chi_2(\chi_1(f)).$$

Lemma 2.14. *As given by the above formula* $[\chi_1, \chi_2]$ *is a vector field.*

Proof. It suffices to compute in local coordinates where $\chi_1 = \sum a_i \partial_i$ and $\chi_2 = \sum_j b_j \partial_j$. Then

$$\begin{aligned}
[\chi_1, \chi_2](f) &= \sum_{i,j} [a_i \partial_i (b_j \partial_j(f)) - b_j \partial_j (a_i \partial_i(f))] \\
&= \sum_{i,j} [a_i(\partial_i(b_j))\partial_j(f) + a_i b_j \partial_i(\partial_j(f)) - b_j(\partial_j(a_i))\partial_i(f)) - b_j a_i \partial_i(\partial_j(f))].
\end{aligned}$$

Since cross partials are equal, the second derivative terms cancel out and we are left with

$$\begin{aligned}
[\chi_1, \chi_2](f) &= \sum_{i,j} a_i(\partial_i(b_j))\partial_j(f) - b_j(\partial_j(a_i))\partial_i(f) \\
&= \sum_{i,j} \{a_i \partial_i(b_j)\partial_j - b_j \partial_j(a_i)\partial_i\}(f).
\end{aligned}$$

This proves that the space of vector fields is closed under this bracket operation. Since composition of vector fields as operators on smooth functions is associative, it follows that the bracket operation makes the space of vector fields into a *Lie algebra*; i.e., a vector space with a bilinear operation satisfying

$$\begin{aligned}
[\chi_1, \chi_2] + [\chi_2, \chi_1] &= 0 \\
[[\chi_1, \chi_2], \chi_3] + [[\chi_3, \chi_1], \chi_2] + [[\chi_2, \chi_3], \chi_1] &= 0.
\end{aligned}$$

□

Definition 2.15. The bracket of vector fields is usually called the *Lie bracket* of vector fields. The second equation is called the *Jacobi identity.*

Since the action of vector fields on smooth functions is natural under diffeomorphisms, meaning

$$\begin{aligned}
Df(\chi_1)(Df(\chi_2))(\varphi) &= (f^{-1})^* \chi_1(f^* D\chi_2)(\varphi) \\
&= (f^{-1})^* \chi_1(f^*(f^{-1})^* \chi_2(f^* \varphi)) = (f^{-1})^*(\chi_1(\chi_2(f^* \varphi))).
\end{aligned}$$

It follows that the bracket operation is natural under diffeomorphisms. If $f\colon M \to N$ is a diffeomorphism, then the differential of f, $Df\colon TM \to TN$ is an isomorphism of smooth vector bundles. As such it sends vector fields on M to vector fields on N preserving the bracket operation.

2.6.2 Computing Lie brackets along submanifolds

Suppose that $f\colon N \to M$ is a smooth embedding and let A, B be vector fields on N. Pushing forward via f we get partial vector fields $Df(A), Df(B)$ defined along $f(N) \subset M$. That is to say that at each point of $x \in f(N)$ we have tangent vectors $Df(A), Df(B) \in TxM$. Indeed these tangent vectors are tangent to the submanifold $f(N)$.

Lemma 2.16. *Let $\widetilde{A}, \widetilde{B}$ be any vectors fields on M extending the partial vector fields $Df(A)$ and $Df(B)$ respetively. Then at any point $x \in f(N)$ we have*

$$Df([A,B])(f^{-1}(x)) = [\widetilde{A}, \widetilde{B}]_x.$$

Proof. If suffices to work locally. We choose coordinate charts U for N near x and $U \times U'$ for M near $f(x)$ such that f is the natural identification $U \to U \times \{0\} \subset U \times U'$. In these coordinates the lemma is immediate. $\square$

2.7 The Lie Algebra of a Lie Group

Let G be a Lie group. Denote by $\mathfrak{g} = T_eG$ its tangent space at the identity. A vector field χ on G is said to be *left-invariant* if for every $g \in G$ the differential of left multiplication by g leaves χ invariant.

Theorem 2.17. *The left-invariant vector fields on G form a (real) vector space, invariant under the bracket of vector fields, and hence a sub-Lie algebra of the Lie algebra of all vector fields on G. A left-invariant vector field is determined by its value at $e \in G$. Assigning to each left-invariant vector field its value at the identity of G determines a (real) linear isomorphism between the vector space of left-invariant vector fields and $\mathfrak{g}$. Transporting the Lie bracket structure via this isomorphism gives us a Lie algebra structure on $\mathfrak{g}$.*

Proof. The map $g \mapsto D(g\cdot)$ defines a smooth action of G as bundle isomorphism of TG covering the let multiplication of G on itself. Thus, fixing $\chi_e \in \mathfrak{g}$, defining $\chi_g = D(g\cdot)(\chi_e)$ determines a smooth vector field on G (the orbit of χ_e under the given smooth action of G.) Clearly, it is left-invariant and the only left-invariant vector field with agrees with $\chi_e \in T_eG$. This establishes the isomorphism between left-invariant vector fields and $\mathfrak{g}$.

Since the given action of G on TG is by smooth vector bundle isomorphisms, it preserves the Lie bracket. If follows that the Lie bracket of two left-invariant vector fields is itself left-invariant. □

Definition 2.18. $\mathfrak{g}$ with the bracket given by the above theorem is called *the* Lie algebra of G. Typically the symbol $\mathfrak{g}$ us used to refer to this Lie algebra.

Examples. 1. Since $GL(n,\mathbb{R})$ is an open subset of the vector space $M(n\times n)$ of $n \times n$ matrices, the tangent space to $GL(n.\mathbb{R})$ at any point is identified with$M(n\times n)$. Let $A \in M(n\times n)$. For any $g \in GL(n,\mathbb{R})$ we have the matrix gA. This is a smoothly varying family of matrices and hence a vector field over $GL(n,\mathbb{R})$. It is the left invariant vector field whose value at $T_eGL(n,\mathbb{R})$ is A.

2. Let $G \subset GL(n,\mathbb{R})$ be a sub-Lie group. Then for any $g \in G$ the linear space T_gG is identified with a subspace of $M(n \times N)$. For each $g \in G$ and for any $A \in T_eG \subset M(n \times n)$ the matrix gA is contained in T_eG and this family defines the left invariant vector field on G whose value at T_eG is A.

3. Let $A \in M(n \times n)$ be a non-zeo matrix. For any $t \in \mathbb{R}$ the exponential series $e^{tA} = \exp(tA)$ is well-defined and converges. For every $t \in \mathbb{R}$, the determinant of e^{tA} equals $e^{t\det(A)}$ and hence is non-zero. The map $\mathbb{R} \to GL(n,\mathbb{R})$ defined by by $t \mapsto e^{tA}$ is a homomorphism from $\mathbb{R} \to GL(n,\mathbb{R})$ whose differential at the identity sends $(\partial/\partial t)|_0$ to $A \in T_eGL(n,\mathbb{R})$. This is an integral curve for the left-invariant vector field on $GL(n,\mathbb{R})$ whose value at the origin is A. It follows that if $A \in T_eG$ for some Lie subgroup $G \subset GL(n,\mathbb{R})$, then this homomorphism takes values in G.

2.7.1 The Lie Algebra of $GL(n,\mathbb{R})$ and its subgroups

The tangent space to $GL(n,\mathbb{R})$ at the identity (and indeed at any point) is $M(n \times n)$. We have seen that the left invariant vector fields are of the form $\chi(g) = g \cdot A$ for g varying over $GL(n,\mathbb{R})$ and $A \in M(n \times n)$ fixed. We wish to compute the Lie bracket on $M(n \times n)$ coming from the Lie bracket of left-invariant vector fields. To do this we must compute $[gH, gK](e)$ for arbitrary $H, K \in M(n\times n)$. Recall that if in local coordinate $(x^1, \ldots, x^n)$ we have vector fields $H(\mathbf{x}) = \sum_i h^i\partial_i$ and $K(\mathbf{x}) = \sum_i k^i\partial_i$ then for any point p

in the coordinate patch we have

$$[H,K](p) = \sum_j H(k^j)(p)\partial_j - \sum_i K(h^i)(p)\partial_i.$$

In our case $GL(n,\mathbb{R})$ is an open subset of $M(n\times n)$ which is a vector space with coordinates $x^{i,j}$.

Let $\epsilon^{i,j}\colon M(n\times n)\to\mathbb{R}$ be function that associates to a matrix its $(i,j)^{th}$ coordinate For $H\in M(n\times n)$ viewed as a tangent vector at the identity element we have two basic formula:

1. The derivation H is equal to $\sum_{i,j}\epsilon^{i,j}(H)\partial_{i,j}|_e$.
2. $H(f(g)) = \frac{d}{dt}|_{t=0}(f(e^{tH}))$.

The general formula for the bracket of $gH = \sum_{i,j}(gH)^{i,j}\partial_{i,j}$ and $gK = \sum_{i,j}\epsilon^{i,j}(gK)\partial_{i,j}$ gives

$$\begin{aligned}
[gH,gK](e) &= \sum_{i,j} H(\epsilon^{i,j}(gK))\partial_{i,j}|_e - \sum_{i,j} K(\epsilon^{i,j}(gH))\partial_{i,j}|_e \\
&= \sum_{i,j}\frac{d}{dt}|_{t=0}\epsilon^{i,j}(e^{tH}K)\partial_{i,j}|_e - \sum_{i,j}\frac{d}{dt}|_{t=0}\epsilon^{i,j}(e^{tK}H)\partial_{i,j}|_e \\
&= \sum_{i,j}\epsilon^{i,j}(HK)\partial_{i,j}|e - \sum_{i,j}\epsilon^{i,j}(KH)\partial_{i,j}|_e = \sum_{i,j}\epsilon^{i,j}(HK-KH)\partial_{i,j}|_e.
\end{aligned}$$

The last term is the derivation at $e\in G$ given by $(HK-KH)\in M(n\times n)$. This proves that the restriction of bracket of left-invariant vector fields on $GL(n,\mathbb{R})$ induces the Lie algebra structure on $M(n\times n)$ given by

$$[H,K] = HK - KH.$$

Corollary 2.19. *If $G\subset GL(n,\mathbb{R})$ is a sub Lie group, then $T_eG\subset M(n\times n)$ is a sub Lie algebra under the bracket $A\otimes B\mapsto [A,B]$.*

2.8 Tensors and Differential Forms

Associated to a (finite dimensional) real vector space V is the dual space, denote V^* consisting of all linear functions on V. Given a finite dimensional vector space V there is the graded algebra, the tensor algebra on V, denoted $T(V)$. It is $\oplus_{n\geq 0}T^n(V)$ where

$$T^n(V) = \underbrace{V\otimes\cdots\otimes V}_{n\text{ times}}.$$

Juxtaposition of tensors defines an associative algebra structure. There is also the *exterior algebra*, denoted $\Lambda^*(V)$, which is the quotient of the tensor algebra by the two-sided ideal generated by

$$v_1 \otimes v_2 + v_2 \otimes v_1 = 0.$$

If V has dimension n, then $\Lambda^*(V)$ is of dimension 2^n. If $\{e_1, \ldots, e_n\}$ is a basis for V then $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ with $k \geq 0$ form a vasis for $\Lambda^*(V)$.

All these operations extend from vector spaces to vector bundles, smooth vector bundles, and holomorphic vector bundles. Dual to the tangent bundle TM of a smooth manifold is the *cotangent bundle*, denoted T^*M. Sections of this bundle are called *differential* 1*-forms* on M. In local coordinates $(x^1, \ldots, x^n)$ on an open subset $U \subset M$ a differential 1-form is written is $\sum_i f_i(x^1, \ldots, x^n)dx^i$ where $\{dx^i\}_{1\leq i\leq n}$ is the dual basis to $\partial/\partial x^i$. In a different set of coordinates $(y^1, \ldots, y^n)$ the transformation law for the formula for a differential 1-form is determined by

$$dy^j = \sum_i \frac{\partial y^j}{\partial x^i} dx^i.$$

More generally, a differential k-form is a section of $\Lambda^k T^*M$. The sum over all k of the differential k-forms make a graded algebra with over the ring a smooth functions multiplication given by wedge product. This graded algebra is denoted $\Omega^*(M)$. Indeed the differential forms form a sheaf of graded algebras over the sheaf of rings of smooth functions. An element of $\Omega^k(M)$ is a differential form of *degree* k. (In particular $\Omega^0(M)$ is the ring of smooth functions.) A differential k-form is determined by its values on k-tuples of vector fields. Furthermore, given a function on k-tuples of vector fields it is evaluation of a differential k-form if and only if the function is skew-symmetric under interchange of any pair of vector fields and the function is linear over multiplication by smooth functions in each variable.

There is an important extra piece of structure that comes from dualizing the Lie bracket on vector fields. The general principal is that the dual to the.Lie bracket is a differential on the exterior algebra of the dual. In this context we define $d\colon \Omega^0(M) \to \Omega^1(M)$ by

$$df(\chi) = \chi(f).$$

In local coordinates $df = \sum_i \partial_i(f)dx^i$. Then we define $d\colon \Omega^1(M) \to \Omega^2(M)$ by

$$d\omega(\chi_1, \chi_2) = \chi_1(\omega(\chi_2)) - \chi_2(\omega(\chi_1)) - \omega([\chi_1, \chi_2]).$$

Direct computation shows that $d\omega$ is a skew symmetric operation bilinear over the action of smooth functions on each factor. This means that $d\omega$ is a 2-form.

Since any differential form can be written as a sum of terms of the form $fdg_1 \wedge dg_2 \wedge \cdots \wedge dg_k$ for smooth functions $f, g_1, \ldots, g_k$, there is a unique extension of d on functions and one-firms to an operation $\Omega^k(M) \to \Omega^{k+1}(M)$, for all k, satisfying linearity over $\mathbb{R}$ and the Leibnitz rule for homogeneous differential forms:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta,$$

where $|\alpha|$ is the degree of α.

Dual to the Jacobi identity, is the identity that $d^2(f) = 0$. This equation can be established by direct computation in local coordinates and again uses the equality of cross partial derivatives. Using the Leibintz rule one shows that for any differential form α we have $d^2(\alpha) = 0$. Again this can be established by direct computation in local coordinates.

All of this structure is summarized by saying that $\Omega^*(M)$ is a differential graded algebra: i.e., a graded algebra with a linear operator d raising degree by 1, satisfying $d^2 = 0$ and the Leibnitz rule.

If $f \colon M \to N$ is a smooth map, then there is an induced map of smooth vector bundles $Df \colon TM \to TN$. Suppose that ω is a differential 1-form on N, i.e., a section of the cotangent bundle T^*N. It defines a smooth bundle map $\omega \colon TN \to N \times \mathbb{R}$. The composition $Df \circ \omega$ is then a smooth bundle map $TM \to N \times \mathbb{R}$ covering the map $f \colon M \to N$. The fiber product of this map with the projection $TM \to M$ defines a smooth map $TM \to M \times_N (N \times \mathbb{R}) = M \times \mathbb{R}$. This is a differential form on M, usually denoted $f^*\omega$.

In local coordinate $(x^1, \ldots, x^n)$ on M and $y^1, \ldots y^k)$ on N, if $\omega = \sum_j \mu_j * y^1 \ldots, y^k)dy^i$, then

$$f^*\omega = \sum_j \left(\mu_j(y^1(\mathbf{x}), \ldots, y^k(\mathbf{x})) \sum_i \frac{\partial y^j}{\partial x^i}(f(\mathbf{x})dx^i \right).$$

More generally, there is a unique extension of this *pullback* of differential 1-forms, to a pullback of all differential forms compatible with wedge product. It turns out (see the problem set) that pullback of differential forms commutes with d.

2.9 Flows

A vector field on M generates, at least locally, a flow on M.

Definition 2.20. Let $\gamma\colon (a,b) \to M$ be a smooth map of an open interval to M. Then $D\gamma_{t_0}(\partial/\partial t)) \in T_{\gamma(t)_0}M$. These tangent vectors form a tangent field defined along the curve. Given a vector field χ on M we say that γ is an *integral curve* for χ if $D\gamma_{t_0}(\partial/\partial t) = \chi(\gamma(t_0))$ for all $t_0 \in (a,b)$.

Proposition 2.21. *Given a vector field χ on M and a point $x_0 \in M$ there is an $\epsilon > 0$ and an integral curve $\gamma\colon (-\epsilon, \epsilon) \to M$ for χ with $\gamma(0) = x_0$. Any two such agree on the intersection of their domains of definition. Furthermore, given a smooth map $\rho\colon X \to M$ there is an open neighborhood W of $X \times \{0\}$ in $X \times \mathbb{R}$ such that the intersection of W with each line $\{x\} \times \mathbb{R}$ is an open interval and such that for every $x \in X$ the integral curve with initial condition $\rho(x)$ exists on the interval $W \cap \{x\} \times \mathbb{R}$ and these integral curves define a smooth map $\hat{\rho}\colon W \to M$.*

Proof. In local coordinates χ is expressed as $\sum_i f_i(x^1, \ldots, x^n)(\partial/\partial x^i)$. The equation that a curve $\gamma\colon (-\epsilon, \epsilon) \to M$ is required to satisfy is $\dot{\gamma}^i(t) = f^i(\gamma(t))$. The usual theorem on local existence and uniqueness of solutions to vector-valued ODE's and smooth variation with parameters gives the result. $\square$

Now let us apply this to the family of initial conditions given by the identity map $M \to M$. We conclude that there is a neighborhood W of $M \times \{0\}$ in $M \times \mathbb{R}$ and a map $\Phi\colon W \to M$ such that (i) the restriction of Φ to $M \times \{0\} \subset W$ is the identity map of M to itself and (ii) the restriction of Φ to the intersection of W with $\{x\} \times \mathbb{R}$ is an integral curve for the vector field χ. Now let us suppose that M is compact. In this case there is $\epsilon > 0$ such that $M \times (-\epsilon, \epsilon) \subset W$. This allows us to assume that $W = M \times (-\epsilon, \epsilon)$. Thus, for each $t \in (-\epsilon, \epsilon)$ the map $\Phi_i = \Phi|_{M \times \{t\}}$ is a map $M \to M$ with $\Phi_0 = \mathrm{Id}_M$. Since Φ_t varies smoothly with t, possibly after making $\epsilon > 0$ smaller, we can assume that the differential of Φ_t is everywhere an isomorphism; i.e., Φ_t is a local diffeomorphism, for all $t \in (-\epsilon, \epsilon)$. On the other hand by the uniqueness of flow lines, it follows that Φ_t is one-to-one. That is to say for every $t \in (-\epsilon, \epsilon)$, the map Φ_t is a diffeomorphism and flow lines starting at any point of M extend in both directions for at least time ϵ.

This means that in fact the flow lines are defined for all time and they define a *flow* $\Phi\colon M \times \mathbb{R} \to M$, the flow generated by ch in the sense that

$$\frac{\partial \Phi}{\partial t}(x,t) = \chi(\Phi(x,t)).$$

One sees easily that the Φ_t define a group homomorphism from the additive group $\mathbb{R}$ to the group of diffeomorphisms of M with the C^∞-topology; i.e., $\Phi_s \circ \Phi_t = \Phi_{s+t}$ and the Φ_t vary smoothly with t.

This shows that a vector field on a compact manifold M can be integrated to a homomorphism of $\mathbb{R}$ to the group of diffeomorphisms with the C^∞-topology. Such homomorphisms are called *flows*.

3 A: Distributions and the Frobenius Theorem

Definition 3.1. A *k-dimensional distribution* on a smooth manifold M is a smoothly varying family of k-planes in TM.

Equivalently, we could require that for each point $x \in M$ the distribution is given by the span of k smooth sections of the tangent bundle, sections that are linearly independent at every point.

Definition 3.2. A *k-dimensional foliation* in a smooth manifold M is an atlas of coordinates, called *flow boxes*, of the form $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with coordinates $(\mathbf{x}, \mathbf{y})$ such that the overlap functions $h_{i,j}$ are of the form $(g_{i,j}(\mathbf{x}, \mathbf{y}), k_{i,j}(\mathbf{y}))$. The level sets of the $\mathbf{y}$-coordinate are called the *local leaves* of the flow box. As we pass from one flow box to another the local leaves match up.

Given a foliation on M we define an equivalence relation on M. It is generated by setting a and b equivalent if they are in the same flow box and lie on the same local leaf of that flow box. Each equivalence class is a *leaf* of the foliation. Each leaf is the image of a smooth one-to-one immersion of a smooth k-dimensional manifold to M. (The map is not necessarily an embedding since the image may not be a closed subset.

Notice that a k-dimensional foliation $\mathcal{F}$ in M determines a k-dimensional distribution, the tangent distribution to the leaves of the foliation. The theorem about integrating vector fields implies that every 1-dimensional distibution is the tangent distirbution to a foliation. (If the distribution is orientable, then choose a section producing a non-=where zero vector field which can then be integrated to give a 1-dimensional foliation. If the distribution is not orienatble, pass to the double cover where it is orientable. Then taking a non-where zero section and integrating gives a flow on the double covering that is tangent to induced distribution on the double cover. Taking the quotient by the covering transformation produces a foliation on the quotient as required.)

Consider the 2-dimensional distribution on $\mathbb{R}^3$ given by the kernel of the 1-form $dz - xdy$. At (x, y, x) the plane is spanned by $\{\partial_x, \partial_y + x\partial_z\}$. In particular, the two-planes in this distribution project isomorphically onto the (x, y)-plane. This distribution is not tangent to a foliation. If it were then every leaf of the foliation would project isomorphically onto the (x, y)-plane. But if we start and move along y-axis to $(0, 1, 0)$ the integral curve tangent to the distribution is the interval on the y-axis. Starting at $(0, 1, 0)$ and moving parallel to the x-axis for a unit distance the integral curve ends

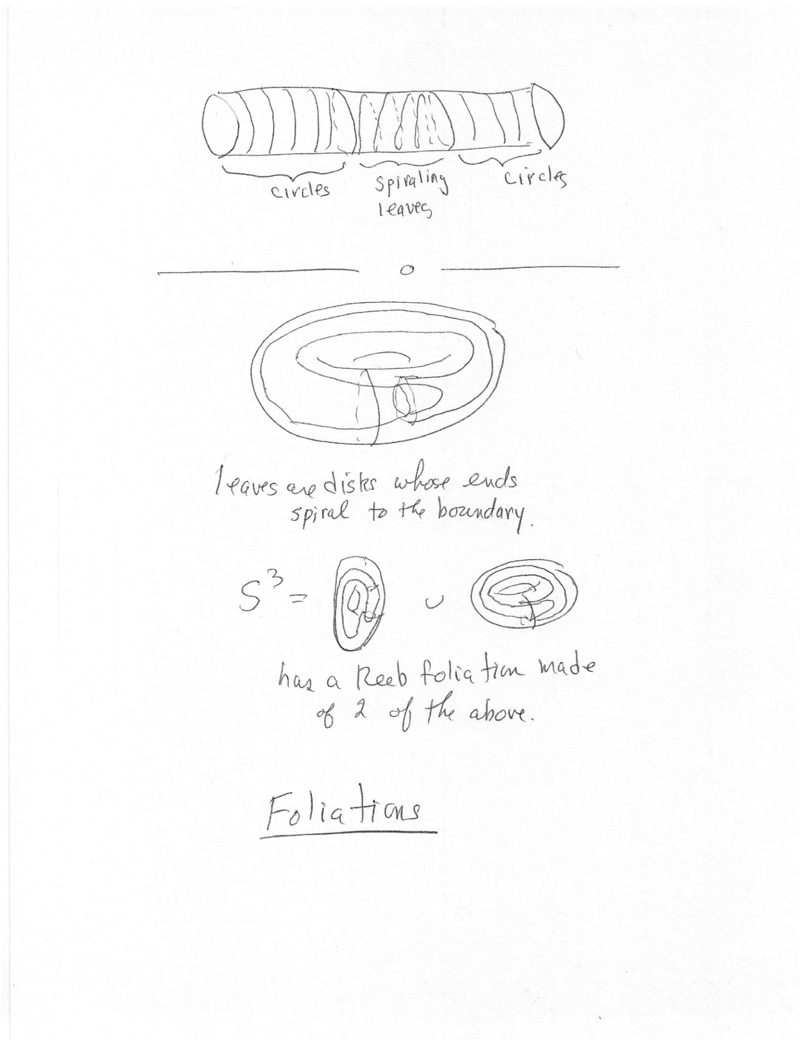

Figure 1: Foliations

at $(1,1,0)$. On the other hand, if we first move along an integral curve projecting to the x-axis a unit length we end at $(1,0,0)$. Then the integral curve projecting to an interval of length 1 parallel to the y-axis ends at $(1,1,1)$. Since these endpoints do not agree, there can be a foliation tangent to the distribution.

Here is the result that tells us when a distribution can be integrated to a foliation (i.e., is tangent to a foliation).

Theorem 3.3. *(Frobenius Theorem) Let $\mathcal{D}$ be a k-dimensional distribution on a smooth manifold M. The $\mathcal{D}$ is the tangent distribution to a foliation if and only if the space of vector fields whose values are contained in $\mathcal{D}$ is closed under Lie bracket.*

Definition 3.4. Distributions that are closed under Lie bracket in the sense given in the theorem are called *involutive.*

Proof. First let us show that the tangent distribution to a foliation is involutive. The result is local so we may as well work in a flow box $U^k \times V^{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with the leaves being given by $U^k \times \{\mathbf{y}\}$ as $\mathbf{y}$ varies in V. Then a vector field tangent to the foliation is of the form

$$\sum_{i=1}^{k} \mu_i(\mathbf{x}, \mathbf{y}) \partial_{x^i}.$$

Clearly, the Lie bracket of any two such vector fields is again a vector field in the space spanned over the smooth functions of the $\{\partial_{x^i}\}_{1 \le i \le k}$.

For the converse, I will prove the case when the dimension of the foliation is 2. This contains the essential idea and the higher dimensional proof requires only more intricate record keeping. Again we work locally. Given a 2-dimensional distribution $\mathcal{D}$ and a point $x \in M$ we can choose local coordinates $U^2 \times V^{n-2}$ near x that are open balls centered at the origin in $\mathbb{R}^2$ and $\mathbb{R}^{n-2}$, respectively, so the projection of the two-planes of $\mathcal{D}$ to the tangents planes to first factor are isomorphisms. We lift ∂_1, ∂_2 to vector fields χ_1, χ_2 in $\mathcal{D}$. These generate $\mathcal{D}$ throughout this region. Since $[\chi_1, \chi_2] \in \mathcal{D}$, it is a linear combination over the smooth functions of χ_1 and χ_2 and hence the bracket is zero if and only if its projection to the (x^1, x^2)-plane is zero. But since the projection of χ_i is ∂_i, the projection of $[\chi_1, \chi_2]$ to this plane is zero. Thus, $[\chi_1, \chi_2] = 0$. That is to say that, locally at least $\mathcal{D}$ is generated by two commuting vector fields.

Now we define coordinates on the subspace $\{x^2 = 0\}$ by integrating χ_1. That is to say we use the given coordinates $(x^3, \ldots, x^n)$ on $\{x^1 = x^2 = 0\}$

and then use the integral curves of χ_1 to extend these $n-2$ coordinates to the subspace $\{x^2 = 0\}$ by requiring these coordinates to be constant on the integral curves. These extension together with x^1 define new local coordinates on $\{x^2 = 0\}$. In these coordinates $\chi_1 = \partial_1$ along the subspace $\{x^2 = 0\}$. Now we do the same thing with χ_2 extending these $n-1$ coordinates to an entire neighborhood by requiring them to be constant along the integral curves of χ_2. These, together with the coordinate x^2 define a full coordinate system throughout a small neighborhood of x. In these coordinates $\chi_1 = \partial_1$ along $x^2 = 0$ and $\chi_2 = \partial_2$.

The condition on χ_1 means that in these coordinates it is written as $\chi_1 + \sum_i f^i(\mathbf{x})\partial_i$ with $f^i = 0$ along $\{x^2 = 0\}$. The condition $[\chi_1, chi_2] = 0$ now reads $-\sum_i \partial_2(f^i)\partial_i = 0$, meaning that $\partial_2(f^i) = 0$ for all i. Since $f^i = 0$ along $\{x^2 = 0\}$, it follows that $f^i = 0$ for all i; that is to say $\chi_1 = \partial_1$ in these coordinates. Thus, the foliation tangent to the distribution in this neighborhood is given by the family of surfaces $\{(x^3, \ldots, x^n) = \text{constant}\}$, which is the usual 2-dimensional foliation of $\mathbb{R}^2 \times \mathbb{R}^{n-2}$ restricted to this neighborhood. $\square$

3B: Integration of Differential Forms

We have seen where the term differential comes from, now let us explain where the term 'form' comes from. Originally, the expressions we now call differential forms were of the correct 'form' to be integrated (over oriented manifolds).

3.1 Integration

3.1.1 The case of smooth curves in a smooth manifold

Consider a differential 1-form ω on a smooth manifold M and a smooth map $\gamma\colon [0,1] \to M$. Then $\gamma^*\omega$ is a differential 1-form on $[0,1]$, and thus of the form $g(t)dt$. This is exactly the type of expression we can integrate: We define

$$\int_{\gamma([0,1])} \omega = \int_0^1 g(t)dt.$$

The key point here is that this definition is independent of the parameterization of the domain interval of γ (as long as the parameterization is to have the same initial and terminal points). For suppose that $\varphi\colon [a,b] \to [0,1]$ is an increasing reparameterization (meansing $\varphi(a) = 0, \varphi(b) = 1$, and $\varphi' > 0$

everywhere. Then the change of variables formula gives:

$$\int_a^b \varphi^*(g(t)dt) = \int_a^b g(\varphi(s))\varphi'(s)ds = \int_a^b (\gamma \circ \varphi)^*\omega.$$

Notice that if the repazrameterization reverses the direction, then the integral changes by a multiplicative factor of -1.

The upshot is that differential 1-forms on M can be integrated over compact, smooth directed curves in M, and the result multiplies by -1 when we reverse the direction on the smooth curve. The integral of a differential 1-form is additive under juxtaposition of smooth curves.

3.1.2 Orientations of smooth manifolds

A direction on a curve is an example of an orientation of a manifold. Consider a real vector space V of dimension finite dimension $n > 0$. Then $\Lambda^n V$ is a 1-dimensional real vector space. An *orientation* for V is the choice of a non-zero vector in $\Lambda^n V$ up to positive multiple; i.e., a choice of the 'positive' direction in $\Lambda^n V$. An ordered basis $\{e_1, \ldots, e_n\}$ for V determines an orientation for V given by the element $e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$. If we switch to another basis $\{f_1, \ldots, f_b\}$ then the effect on the orientation is given by the sign of the determinant of the matrix expressing the $\{f_i\}_i$ in terms of the $\{e_j\}_j$.

Now consider a vector bundle $\mathcal{E} \to M$ with fibers n-dimensional real vector spaces. Then $\Lambda^n \mathcal{E} \to M$ is a line bundle. An orientation for $\mathcal{E}$ is a trivialization of $\Lambda^n \mathcal{E} \to M$ up to a positive function on M; i.e., an orientation of each fiber of $\Lambda^n \mathcal{E}$ that are locally trivial.

An *orientation for a smooth manifold* M is an orientation for its tangent bundle. A manifold is orientable if there is an atlas for the smooth structure so that all overlap functions are orientation preserving maps of between open subsets of $\mathbb{R}^n$. Of course, as the Möbius band shows not every manifold is orientable, though of course every manifold is locally orientable in the sense that every point has an open neighborhood that is orientable. If the manifold has a boundary, then the orientation of the manifold induces one on the boundary with the property that a basis for $T_y\partial M$ gives the orientation of ∂X at y if and only if this basis preceded by an outward pointing normal gives the local orientation of X at y. This convention means that the boundary of the unit 2-disk with its orientation induced from the usual one on $\mathbb{R}^2$ is the circle with the orientation given by the counter-clockwise direction.

A connected zero manifold has two orientations and they are denoted $\pm$. The value of a 0-form (function) on an oriented 0-manifold is the product of the value of the function on the point times the sign of the orientation. With these conventions the boundary of a $[0,1]$ with its usual orientation is $\{1\} - \{0\}$. .

Associated to a manifold M is the orientation double covering $\widetilde{M} \to M$. The points of $\widetilde{M}$ consist of a point $p \in M$ and an orientation of T_pM. The manifold M is orientable if and only if $\widetilde{M}$ is isomorphic to the double cover $\{\pm\} \times M$, and a global section of this bundle is equivalent to an orientation of M.

3.1.3 Integration of compactly supported differential n-forms over open subsets of $\mathbb{R}^n$

Let $U \subset \mathbb{R}^n$ be an open subset and let ω be an n-form on U with compact support. Then

$$\omega = f(x^1, \ldots, x^n)dx^1 \wedge \cdots \wedge dx^n$$

for some smooth function f defined on U with compact support. We define

$$\int_U \omega = \int_U f dx^1 \cdots dx^n,$$

the usual Riemannian integral of the compactly supported function f on U. Once again the change of variables formula in integration tells us that if we have a diffeomorphism $\varphi\colon V \to U$ between open subsets of $\mathbb{R}^n$, with Cartesion coordinates $\{y^1, \ldots, y^n\}$ on V then

$$\int_V \varphi^*(f)|\det(D\varphi)|dy^1 \cdots dy^n = \int_U f dx^1 \cdots dx^n.$$

Of course,

$$\varphi^*(f dx^1 \wedge \cdots \wedge dx^n) = (f \circ \varphi)\det(D\varphi)dy^1 \wedge \cdots \wedge dy^n.$$

Thus, if the determinant of $D\varphi$ is everywhere positive, then we have

$$\begin{aligned}
\int_V \varphi^*(f dx^1 \wedge \cdots \wedge dx^n) &= \int_V \varphi^*(f)\det(D\varphi)dy^1 \wedge \cdots \wedge dy^n \\
&= \int_V \varphi^*(f)\det(D\varphi)dy^1 \cdots dy^n \\
&= \int_U f dx^1 \cdots dx^n \\
&= \int_U f dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Similarly, if $\det D\varphi$ is everywhere negative, then

$$\int_V \varphi^*(f dx^1 \wedge \cdots \wedge dx^n) = -\int_U f dx^1 \wedge \cdots \wedge dx^n.$$

In the above discussion we have assumed that we are working with an open subset U of $\mathbb{R}^n$ but the arguments easily adapt to the case when U is an open subset of the half-space $\{x^1 \geq 0\}$, and even to manifolds with corners given as open subsets of $\{x^1 \geq 0, \dots, x^k \geq 0\}$.

3.1.4 Integration over Compact Submanifolds

Suppose that M is a smooth n manifold and ω is a differential k-form on M. Let X be a compact,oriented k-manifold, possibly with boundary,. We can cover X by finitely many coordinate charts $U_1, \dots, U_N$ of M with the property that $U_i \cap X$ is the intersection of U_i with a coordinate k-dimensional subspace given by$\{x_j = 0\}$ for all $j > k$, or a half-space therein and the orientation of X agrees with the usual orientation of this k-dimensional subspace. Extend the $\{U_i\}_{1\leq i\leq n}$ to a covering of M by coordinate charts so that all the other charts are disjoint from X. Choose a partition of unity $\{\psi_i\}$ subordinate to this covering. Then

$$\omega|_X = \sum_{i=1}^{N} (\psi_i \omega)|_{U_i}.$$

The differential form $(\psi_i\omega)|_X$ is then supported in the coordinate chart $U_i \cap X$. It is a differential k-form on an open subset of a coordinate patch Hence, $\int_{U_i \cap X} (\psi_i\omega)|_X$ is defined as above

Then the integral $\int_X \omega$ is defined as the sum of these integrals. It is an easy exercise using the fact that the individual terms are independent of the choice of Euclidean coordinates to show that the result is independent of all choices of open covering and partition of unity.

This gives us an integration pairing between differential k-forms and compact smooth k-dimensional submanifolds, possibly with boundary (or even corners), of M.

3.2 Stokes' Theorem

Theorem 3.5. *Let M be a smooth n-manifold and let $X \subset M$ be a smoothly embedded k-dimensional oriented submanfiold with boundary. Then for any* *differential* $(k-1)$*-form* ω *on* M *we have*

$$\int_X d\omega = \int_{\partial X} \omega,$$

when we use the induced orientation on ∂X*.*

Proof. By restricting to X it suffices to assume that $M = X$. Next, let us consider the case when X is one dimensional and ω is a smooth function f. If X is an interval with $[a, b]$ with its induced orientation, Stokes' Theorem becomes

$$\int_a^b f'(t)dt = f(b) - f(a),$$

the fundamental theorem of calculus. If X is a circle, then we take a 'parameterization' by the interval $[0, 2\pi]$. Then $f(2\pi) = f(0)$ since 0 and 2π map to the same point of X. Thus,

$$\int_X df = \int_0^{2\pi} f'(t)dt = 0.$$

In this case $\partial X = \emptyset$, so that $\int_{\partial X} f = 0$. So once again the result comes down to the fundamental theorem of calculus.

Now we turn to the higher dimensional case. We cover X by coordinate patches. $\{U_1, \ldots, U_N\}$. Using a partition of unity subordinate to this cover we can write $\omega = \sum_i \omega_i$ where for each i the form ω_i has compact support contained in U_i. In fact, we can suppose that for each i the support of ω_i is supported in a box $B_i = \prod_{j=1}^k [a, b]$ in the Cartesian coordinates on U_i. Furthermore, we can assume that for each i either the box B_i is contained in the interior of X or it meets ∂X exactly along $(n-1)$-face $\{b\} \times B_i'$ where $B_i' = \prod_{j=2}^k [a, b]$. For boxes B_i in the interior of X, the differential form ω vanishes on ∂B_i. For boxes B_i that meet ∂X the differential form vanishes except possibly on the face $\{b\} \times B_i'$.

We write $\omega_i = \sum_j g_{i,j} dx^{I_j}$, where dx^{I_j} means the wedge product, in order, of the $dx^1, \ldots, dx^k$ omitting dx^j. Let us consider the germ $g_{i,1} dx^{I_1}$. Its differential is $\partial_1 g_{i,1} dx^1 \wedge \cdots \wedge dx^k$ and by Fubini's theorem we have

$$\int_{B_i} d(g_{i,1} dx^{I_1}) = \int_{B_i'} \left(\int_a^b (\partial_1 g_{i,1}) dx^1 \right) dx^2 dx^3 \cdots dx^k.$$

By the fundamental theorem of calculus the right-hand side is equal to

$$\begin{aligned}
&\int_{B_i'} \Big(g_{i,1}(b, x^2, \ldots, x^k) - g_{i,1}(a, x^2, \ldots, a_k) \Big)\, dx^2 \cdots dx^k = \\
&= \int_{B_i'} g_{i,1}(b, x^2, \ldots, x^k) dx^2 \cdots dx^k \\
&= \int_{\partial X} g_{i,1} dx^{I_1},
\end{aligned}$$

where the first equation comes from the fact that $g(a, x^2, \ldots, x^k)$ is identically zero.

A similar argument shows that for the cases $\ell > 1$ both $\int_{\partial X} g_{i,\ell} dx^{I_\ell}$ and $\int_X d(g_{i,\ell} dx^{I_\ell})$ are zero. $\square$

Definition 3.6. A differential form ω is said to be *closed* if $d\omega = 0$ and is said to be *exact* if there is a differential form η with $d\eta = \omega$.

Notice that since $d^2 = 0$ every exact form is closed.

Corollary 3.7. *1. Suppose that ω is a closed form and $X \subset M$ is a closed submanifold which is the boundary of an oriented submanfiold $Y \subset M$. Then*

$$\int_X \omega = 0.$$

2. Suppose that ω is an exact form and X is a closed oriented submanifold then

$$\int_X \omega = 0.$$

Proof. For 1

$$\int_X \omega = \int_{\partial Y} \omega = \int_Y d\omega = \int_Y 0 = 0.$$

For 2, write $\omega = d\eta$. Then

$$\int_X \omega = \int_X d\eta = \int_{\partial X} \eta = 0,$$

where the last equality is a consequence of the fact that $\partial X = \emptyset$. $\square$

Corollary 3.8. *Let ω be a closed differential k-form and X a closed, oriented smooth manifold. Suppose that $i_1 \colon X \to M$ and $i_2 \colon X \to M$ are smooth embeddings that are homotopic. Then*

$$\int_{i_1(X)} \omega = \int_{i_2(X)} \omega.$$

Proof. We can approximate the homotopy between i_1 and i_2 by a smooth homotopy between these embeddings. That is to say there is a smooth map $J: X \times I \to M$ with $J|_{\{j\}\times X} = i_j$ for $j = 0, 1$ Then $J^*\omega$ is a closed form on $I \times X$. Hence,

$$0 = \int_{I\times X} d\omega = \int_{i_1(X)} \omega - \int_{i_2(X)} \omega.$$

□

3.3 de Rham's Theorem: A brief introduction

Let $\Omega^*(M)$ denote the differential graded algebra of differential forms on a smooth manifold M. Recall that a form ω is *closed* if $d\omega = 0$ and it is exact if there is η with $d\eta = \omega$. The fact that $d^2 = 0$ implies that every exact form is closed. Clearly, the closed forms (being the kernel of a linear map) and the exact forms (being the image of a linear map) are real vector spaces, with the second bin a subspace of the first. the em k^{th}-de Rham cohomology of M is defined as the quotient of closed k-forms modulo exact k-forms

$$H^k_{dR}(M) = \frac{\mathrm{Ker}(d\colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}\ d\colon \Omega^{k-1}(M) \to \Omega^k(M)}.$$

[Since the kernel and image are infinite dimensional vector spaces with (various) natural topologies one might be concerned about for example whether the image of d is is a closed subspace. But these turn out not to be real issues.]

de Rham's Theorem compares this cohomology with algebraic topology's singular cohomology. For any toological space one defines the singular homology by forming a chain complex of abelian groups $\oplus_k \mathrm{Sing}_k(X)$. The group $\mathrm{Sing}_k(X)$ is the free abelian group generated by the continuous maps of the k-simplex to X. The boundary map $\partial\colon \mathrm{Sing}_k(X) \to \mathrm{Sing}_{k-1}(X)$ is determined setting the boundary of $\varphi\colon \Delta^k \to X$ equal to the linear combination

$$\sum_{i=0}^{k} (-1)^i \varphi|_{i^{th}-\text{face of } \Delta^k}.$$

This defines a chain complex (i.e., $\partial^2 = 0$). The singular homology of X is the homology of this chain complex: namely

$$H_k(X) = \frac{\mathrm{Ker}\partial\colon \mathrm{Sing}_k(X) \to \mathrm{Sing}_{k-1}(X)}{\mathrm{Im}\ \partial\colon \mathrm{Sing}_{k+1}(X) \to \mathrm{Sing}_k(X)}.$$

For any abelian group A one defines the singular chomology of X with coefficients in A by taking the cochain complex whose groups are $\mathrm{Sing}^k(X;A) = \mathrm{Hom}(\mathrm{Sing}_k(X), A)$ with differential

$$d\colon \mathrm{Sing}^k(X;A) \to \mathrm{Sing}^{k+1}(X;A)$$

to be the dual to the boundary map in the singular chain complex. Then $d^2 = 0$ and the singular cohomology

$$H^k(X;A) = \frac{\mathrm{Ker}\ d\colon \mathrm{Sing}^k(X;A) \to \mathrm{Sing}^{k+1}(X;A)}{\mathrm{Im}\ d\colon \mathrm{Sing}^{-1}(X;A) \to \mathrm{Sing}^k(X;A)}.$$

The basic idea for comparing de Rham cohomology with singular cohomology comes from the pairing of integrating differential forms over chains. There is a technical issue in that the objects in singular cohomology have no smooth requirements (since they are defined for arbitrary topological spaces not just smooth manifolds.

There are many ways to deal with this problem. One is to consider smooth chain complex of a smooth manifold. Here the chain groups are the free abelian groups generated by smooth maps of the simplicies into a smooth manifold M. The same boundary formula works in this case to show that we have an analogous smooth singular chain complex of a smooth manifold, which we denote by $\mathrm{SSing}_*(M)$. Now the integration pairing is defined:

$$\Omega^k(M) \otimes \mathrm{SSing}_k(M) \to \mathbb{R}$$

given by $\omega \otimes A \mapsto \int_A \omega$. We can view this as an $\mathbb{R}$-linear map

$$\Omega^k(M) \to \mathrm{SSing}^k(M;\mathbb{R}).$$

Stokes' Theorem is the statement that this map of cochain complexes is a cochain map (i.e., it computes with the differentials). As a result it defines a linear map

$$H^k_{dR}(M \to H^k_{\mathrm{SSing}}(M;\mathbb{R})$$

where the right-hand term is the cohomology of the smooth singular cochain complex with $\mathbb{R}$-coeffiients.

de Rham's Theorem consists of two isomorphisms;

Theorem 3.9. *1. The integration pairing above determines an isomorphism*

$$H^k_{dR}(M) \xrightarrow{\cong} H^k_{\mathrm{SSing}}(M;\mathbb{R})$$

2. The inclusion of the chain complex $\mathrm{SSing}_*(X) \to \mathrm{Sing}_*(X)$ *induces an isomorphism on homology.*

I shall not give a proof of de Rham's Theorem in this course.

It follows from the second result that the inclusion map $\mathrm{SSing}_*(M) \to \mathrm{Sing}_*(M)$ induces isomorphisms on the associated cohomology with any coefficients, in particu;lar with $\mathbb{R}$-coefficients.

Corollary 3.10. *The de Rham cohomology of a smooth manifold is identified with its singular cohomology with $\mathbb{R}$-coefficients. This identification is natural for smooth maps between smooth manifolds.*

Even more is true: Wedge product of diferential forms induces a multiplication de Rham cohomology making it a graded commutative algebra over $\mathbb{R}$. There is for singular cohomology a cup product formula due to Whitney. This makes singular cohomology with integer coefficients a graded commutative ring and hence makes singular cohomology with $\mathbb{R}$-coeficients a graded commutative algebra over $\mathbb{R}$. The isomorphism of de Rhan's theorem is a ring isomorphism.

4 Complex plane curves and holomorphic one-forms

4.1 The Complex Curves

In this lecture we shall study curves in $\mathbb{C}^2$ given by a single complex polynomial equation $C = \{P(x, y) = 0\}$. We will only consider those curves that are smooth, i.e., curves given by polynomials $P(x, y)$ with the property that $dP \neq 0$ at every point of C. The equations we shall study are of the form $P(x, y) = y^2 - p(x)$. Equivalently, we are considering curves given by the equation $y^2 = p(x)$. In this case the condition that the curve C defined by this equation is smooth is that $p(x)$ has only simple roots, meaning that there is no $a \in \mathbb{C}$ such that $p(x)$ factors as $(x - a)^2 q(x)$. We make this assumption from now on in this lecture.

The projection $\mathbb{C}^2 \to \mathbb{C}$ sending (x, y) to x induces a holomorphic map $\pi \colon C \to \mathbb{C}$ that is 2-to-1 everywhere except at the roots of p where there is only one point in the pre-image. Furthermore, for any $(x, y) \in C$ with $y \neq 0$ (or equivalently x not a root of p), the differential of π is an isomorphism so that x is a local coodrinate locally at any point of $C \setminus C \cap \{y = 0\}$. At the remaining points y is a local coordinate. The reason is that $\partial_x(y^2 - p(x)) = -p'(x)$ does not vanish at $(0, x) \in C$ since the roots of $p(x)$ are simple.

Near $(0, x) \in C$ the projection to $C \to \{y = 0\}$ looks like $z \mapsto z^2 + a$ in appropriate local coordinates on C. The point is that we can write $y^2 = (x-a)q(x)$ where $q(a) \neq 0$. Thus, locally we can take an holomorphic square root of $q(x)$, denoted $\sqrt{q(x)}$ and

$$\left(\frac{y}{\sqrt{q(x)}}\right)^2 = (x - a).$$

By the implicit function theorem we know that $q(x)$ is a non-zero holomor[phic function of y near $(0, a)$, hence so is its square root. Hence we can use

$$z = \frac{y}{\sqrt{q(x)}}$$

as a local holomorphic coordinate on C near $(0, a)$ and $(x - a) = z^2$ so that $\pi(z) = z^2 + a$. In particular the pre-image in C of a small disk centered at a is a disk centered at the origin in the y_1 coordinate on C. These points are *branch points* for the map to x-axis. Centered at any other point $a \in \mathbb{C}$ distinct from the roots of $p(x) = 0$ there is a small disk in the x-axis whose preimage under π is two disjoint disks in C, each mapping down by a holomorphic isomorphism. (This is the statement that x is a local coordinate at all such points of C that do not project to a root of $p(x)$.)

Definition 4.1. The algebraic curve C is called *the Riemann surface* of the holomorphic function $\sqrt{p(x)}$. Indeed, C is displayed over the x-axis and has on it a holomorphic function y. Then values of y on the points above a in the x-axis all the possible values of $\sqrt{p(a)}$. Usually there are two of them, but for a a root of p there is only one, $y = 0$.

4.1.1 Nature of the curve near infinity

Lemma 4.2. *Let γ be a smooth simple closed curve in the x-axis that does not pass through any root of p. The pre-image $\widetilde{\gamma} = \pi^{-1}(\gamma)$ maps to γ two-to-one by a local diffeomorphism. There are two possibilities: $\widetilde{\gamma}$ is itself a loop and under π it wraps twice around γ or $\widetilde{\gamma}$ is a disjoint union of two loops each of maps diffeomorphically onto γ. If the disk in the x-axis bounded by γ contains an odd number of points the first case holds, otherwise the second case holds.*

Proof. The simple closed curve is homotopic in the complement of the roots of $p(x)$ to the simple closed curve as drawn in the upper part of Figure 1. Since homotopic loops are in the same case, it suffices to consider the smaller loops near the arcs connecting pairs of roots as in this figure. Since traversing the boundary of a disk centered at a root interchanges the sheets of of C above the x-axis, the result follows easily. $\square$

Corollary 4.3. *If p is of even degree, then the pre-image of any sufficiently large loop in the x-axis is two loops in C. If p is of odd degree, then the pre-image of any sufficiently large circle in the x-axis is a single loop.*

Proof. The disk bounded by a sufficiently large loop contains all the roots of p and the number of roots of p is its degree. $\square$

Corollary 4.4. *Let γ be a sufficiently large circle in the x-axis. Then the complement of the disk bounded by γ is an annulus A and its pre-image in C is either two annuli when p has even degree or a single annulus double covering A. Thus, a neighborhood of infinity in C is either two punctured disks each mapping isomorphically onto A when p has even degree, or a single punctured disk mapping by $z \mapsto z^2$ onto A (in appropriate local coordinates).*

In the case when the degree of p is even we compactify the curve C by adding the centers of the two punctured disks. In the case when the degree of p is odd we compactify the curve C by adding the center of the punctured disk neighborhood of infinity. The resulting curves maps two-to-one to $\mathbb{C}P^1$

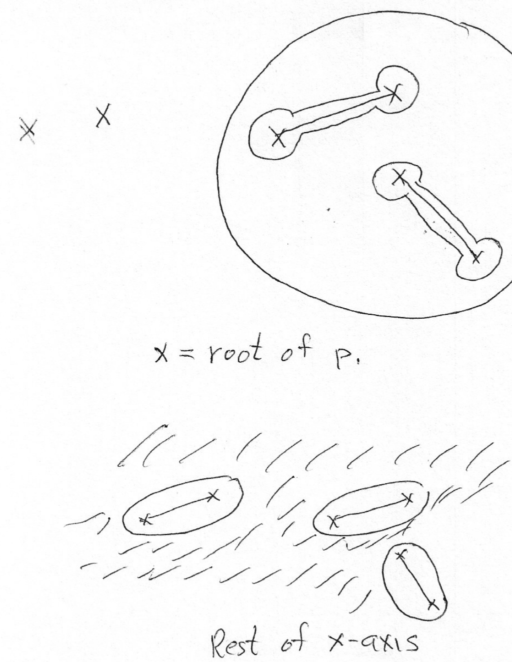

Figure 2: Simple Closed curves in the x-axis

which is a local diffeomorphism at each point in the preimage of ∞ when the degree of $p(x)$ is even and is locally of the form $z \mapsto z^2$ when the degree of p is odd. In the latter case $\infty \in \mathbb{C}P^1$ is a branch point.

Claim 4.5. *When p has even degree, the coordinate $\zeta = x^{-1}$ is a local coordinate near each of the completion points. When p has odd degree $\sqrt{x^{-1}}$ is a local coordinate at the completion point. Thus, the compactified curves are complex curves with a generically two-to -one holomorphic map to $\mathbb{C}P^1$, the completion of the x-axis. If the degree of p is $2d$ or $2d-1$ this projection has $2d$ branch points.*

Proof. Since each point at infinity has a neighborhood in C that maps by holomoprhic isomorphism onto the complement of a large disk centered at the origin in the plane, the first statement is clear. In the second case $\psi = \sqrt{\zeta}$ is a local coordinate on a neighborhood of infinity in C and the missing point is the origin of the ψ disk. $\square$

From now on C refers to the completed curve as defined above.

4.1.2 The Topology of the Curve C

We are considering the curve $C = \{y^2 = p(x)\}$ in $\mathbb{C}^2$ and its compactification over the compactification $\mathbb{C}P^1$ of the x-axis. First we consider the case when $p(x)$ has even degree, say $2d$. Pair up the roots on $p(x)$ and draw disjoint arcs between each pair. We get a picture as in the lower half of Figure 1. The pre-image of a small neighborhood of an arc connecting two of the roots of p is an annulus and the pre-image of the complement of these d neighborhoods is two copies of a 2-sphere with d small disks removed. Each of the d annuli connects the two differ copies. The result is pictured in Figure 2. It is a surface with $(d-1)$ holes, called the surface of *genus* $d-1$.

In the case when p is of odd degree $2d-1$, the point ∞ in $\mathbb{C}P^1$ is also a branch point but the analysis above still holds. Pair up the branch points and draw disjoint arcs between them on $\mathbb{C}P^1$. As before the preimages of neighborhoods of the arcs are annuli, and the pre-image of the rest of $\mathbb{C}p^1$ is two copies of S^2 with d disks removed. Thus, once again the surface is a surface of genus $d-1$.

4.2 Homolomorphic differential 1-forms

Given a smooth curve $C \subset \mathbb{C}^2$, with linear coordinates (x, y) on $\mathbb{C}^2$, given by an equation $y^2 = p(x)$ (with no repeated roots for p), we define a holo-

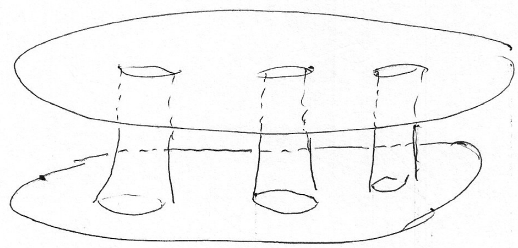

Figure 3: Surface when $d = 3$

morphic differential

$$\frac{dx}{y}.$$

This expression makes good sense near any $(a, b) \in C$ with $b \neq 0$, for at such points x is a local coordinate and $y \neq 0$. We have to examine what happen near points of the form $(a, 0) \in C$. Of course, on the open subset of C where both x and y are local coordinates we have the equation

$$2ydy = p'(x)dx.$$

Since $p'(a) \neq 0$ (this is the no repeated root condition), in a small neighborhood of $(a, 0) \in C$, y is a local coordinate and x is a local coordinate except that $(a, 0)$. Thus, in the punctured neighborhood of $(a, 0)$ we have

$$\frac{dx}{y} = \frac{2dy}{p'(x)}.$$

On the other hand, since $p'(a) \neq 0$, the expression

$$\frac{dy}{2p'(x)}$$

is holomorphic through this neighborhood. This shows that dx/y has a unique holomorphic extension to all of C. By a slight abuse of notation we denote this global form on C by dx/y.

Let us see what happens at infinity .

Case (i): The degree of p is even, say $2d$. We introduce the local coordinate $\zeta = x^{-1}$ which is a local coordinate near every point of C mapping to a small neighborhood of ∞ in $\mathbb{C}P^1$. Then differential one-form dx/y near infinity is $-\xi\zeta^{-2}d\zeta$ where $\xi = y^{-1}$.

The function $y^2 = \zeta^{-2d}r(\zeta)$ where r is a polynomial with $r(0) \neq 0$, and hence $y = \zeta^{-d}h(\zeta)$ where $h(\zeta)$ is holomorphic with $h(0) \neq 0$. There are two choice for the square root and the two functions correspond to the two sheets of C over this disk in ζ. Each of the resulting functions ξ then is holomorphic in this region with a zero of order $d = \deg(p)/2$ at $\{\zeta = 0\}$. This implies that $(dx/y) = -\xi\zeta^{-2}d\zeta$ has a zero of order $d - 2$ at infinity.

In each sheet of the pre-image of a small neighborhood $\infty \in \mathbb{C}P^1$, the differential one-form dx/y has a zero of order $d - 2$. This means that when the degree of p is 2, the form has a simple pole at both pre-images of ∞ and is meromorphic rather than holomorphic. When the degree of p is four, dx/y is everywhere holomorphic and non-zero. When the degree of p is $2d > 4$, the form dx/y is everywhere holomorphic, has a zero of order $d-2$ at each of the pre-images of $\infty \in \mathbb{C}P^1$ and is otherwise non-zero.

For $2d > 4$ we can create similar holomorphic differential one-forms by taking

$$\frac{q(x)dx}{y}$$

where q is a polynomial of degree $\leq d-2$. This produces a vector space of dimension $d-1$ of global holomorphic differentials. Direct computation (see the Problem Set) shows that this is all the global holomorphic differential one-forms. on C.

Case (ii): The degree of p is odd, say $2d-1$. Now a local coordinate near the unique point at infinity is z projecting to $\zeta^2 = x^{-2}$. In this coordinate the form dx is $-2z^{-3}dz$ and $p(x) = \zeta^{-(2d-1)}r(\zeta) = z^{-(4d-2)}r(z^2)$, where r is a polynomial with $r(0) \neq 0$. Thus,

$$y = z^{-(2d-1)}h(z)$$

where h is a holomorphic function with $h(0) \neq 0$. It follows that in this local coordinate

$$\frac{dx}{y} = \frac{-2z^{-3}dz}{z^{-(2d-1)}h(z)} = -2z^{2d-4}h^{-1}(z)dz.$$

This means that $\frac{dx}{y}$ has a zero of order $2d-4$ at infinity. Hence the form

$$\frac{q(x)dx}{y}$$

with q a polynomial of degree at most $d-2$ is holomorphic on all of C. (If $q(x)$ is a polynomial of degree k, then it creates a meromorphic function on the z-disk with a pole of order $2k$.)

Thus, in either case degree of p is $2d$ or $2d-1$ all differential forms of the type $\frac{q(x)dx}{y}$ with $q(x)$ a polynomial of degree at most $d-2$ are global holomorphic differentials on the compact curve C. This gives is a complex vector space of dimension $d-1$ of holomorphic 1-forms.

Claim 4.6. *Holomorphic 1-forms on C are closed. No non-zero holomorphic 1-form on C can be exact.*

Proof. Let ω be a non-zero holomorphic differential form. Since ω is locally of the form $\psi(z)dz$ with ψ homomorphic, it is clear that $\partial_{\overline{z}}\psi(z) = 0$ and hence $d\overline{z} \wedge \partial_{\overline{z}}\omega = 0$. Since the complex dimension of C is one, it follows that $dz \wedge \partial_z\omega = 0$. Hence, $d\omega = 0$.

Proof 1. If f is a complex-valued smooth function with $df = \omega$, where ω is a holomorphic 1-form, then in local holomorphic coordinate on C, we have $\partial_{\overline{z}} f = 0$, meaning that f is a holomorphic function. Since C is compact, f is contant and hence $df = 0$.

Proof 2. The differential form $\overline{\omega}$ is a closed diffierential form and $\omega \wedge \overline{\omega}$ is a two-form which in local holomorphic coordinates is of the form $|f|^2 dz \wedge d\overline{z}$ and $dz \wedge d\overline{z} = (-2i) dx \wedge dy$. This means that $i\omega \wedge \overline{\omega}$ is a two-form and

$$\int_C i\omega \wedge \overline{\omega} > 0.$$

If ω were exact then by the Leibnitz rule $i\omega \wedge \overline{\omega}$ is also exact and would have integral 0 over C. $\square$

Since the wedge product of two holomorphic 1-forms is zero, it follows that there is no sum $\omega_1 + \overline{\omega_2}$ is exact, in particular the direct sum of the space of holomorphic differentials and the space of their conjugates (called *anti-holomorphic differentials*) is a space of closed 1-forms that injects into the de Rham cohomology $H^1_{dR}(C; \mathbb{C})$. The image of the holomorphic differentials is denoted

$$H^{1,0}(C) \subset H^1_{dR}(C; \mathbb{C})$$

and the image of the space of anti-holomorphic differentials is denoted

$$H^{0,1}(C) \subset H^1_{dR}(C; \mathbb{C}).$$

We have just shown that when p has degree $2d$ or $2d-1$ the dimension of $H^{1,0}$ and $H^{0,1}$ are $d-1$ and that there is a linear embedding

$$H^{1,0}(C) \oplus H^{0,1}(C) \subset H^1_{dR}(C; \mathbb{C}).$$

On the other hand, when p has $2d$ or $2d-1$, C is a $(d-1)$-holed surface. Thus, its singular homology of C is a free abelian group of rank $2(d-1)$. We have also seen that integrating over smooth simple closed curves gives an injection

$$H^1_{dR}(C; \mathbb{C}) \to \mathrm{Hom}(H_1(C), \mathbb{C}).$$

Since $H^{1,0}(C) \oplus H^{0,1}(C)$ and $\mathrm{Hom}(\mathbb{H}_1(C), \mathbb{C})$ are complex vector spaces of dimension $2d-2$ and the map between them is an injection, it follows that

$$H^{1,0}(C) \oplus H^{0,1}(C) \xrightarrow{\cong} H^1_{dR}(C; \mathbb{C}) \xrightarrow{\cong} \mathrm{Hom}(H_1(C), \mathbb{C}).$$

In particular, every closed 1-form on C with complex coefficients differs by an exact form from the sum of a holomorphic and anti-holomorphic 1-form. Since complex conjugation on $H^1_{dR}(C;\mathbb{C})$ interchanges $H^{1,0}(C)$ and $H^{0,1}(C)$, it follows that $H^{1,0}(C)$ and $H^{0,1}(C)$ meet $H^1_{dR}(M:\mathbb{R})$ only at $\{0\}$.

There is a real linear isomorphism $H_1(M;\mathbb{R}) \to \mathrm{Hom} H^1_{dR}(C;\mathbb{R}),\mathbb{R})$ and $\mathrm{Hom}(H^1_{dR}(C;\mathbb{R}),\mathbb{R})$ embeds in $\mathrm{Hom}(H^1_{dR}(C;\mathbb{C}),\mathbb{C})$ as those homomorphisms φ that satisfy $\varphi(\overline{\omega}) = \overline{\varphi(\omega)}$. Since $\overline{H^{0,1}(C)} = H^{1,0}(C)$, it follows that any non-zero $\varphi \in \mathrm{Hom}(H^1_{dR}(C;\mathbb{R}),\mathbb{R})$ is non-trivial when restricted to $H^{1,0}(C)$. Counting dimensions, we see that the map

$$\mathrm{Hom}(H^1_{dR}(C;\mathbb{R}),\mathbb{R}) \to \mathrm{Hom}(H^{1,0}(C),\mathbb{C})$$

is a real linear isomorphism. This means that

$$\mathrm{Hom}(H^1(C;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H^{1,0}(C),\mathbb{C})$$

is an embedding onto a lattice of full rank $(2(d-1))$ in $\mathrm{Hom}(H^{1,0}(C),\mathbb{C})$. Of course, $\mathrm{Hom}(H^1(C;\mathbb{Z}),\mathbb{Z}) = \mathbb{H}_1(C)$. Thus, we have an embedding $H_1(C;\mathbb{Z})$ as a lattice of full rank in $\mathrm{Hom}(H^{1,0}(C),\mathbb{C})$. Tracing through this map it is given as follows: the homology class of curve γ maps to the homomorphism that sends a holomorphic differential ω to

$$\int_\gamma \omega.$$

Choosing an integral basis $\gamma_1,\ldots,\gamma_{2(d-1)}$ for $H_1(C)$ and a complex basis $\omega_1,\ldots,\omega_{d-1}$ for $H^{1,0}(C)$ we have a $(d-1)\times 2(d-1)$ matrix

$$\int_{\gamma_i} \omega_j.$$

The $2(d-1)$ columns of this matrix generate a lattice of rank $2(d-1)$ in the $\mathbb{C}^{d-1}$; i.e., these $2(d-1)$ vectors are linearly independent over $\mathbb{R}$.

The quotient of $\mathbb{C}^{d-1}$ by this lattice is a complex torus, called the *Jacobian* of the curve and denoted $J(C)$ call the *Abel-Jacobi* map. Integarting the holomorphic differentials along paths from the base point $x_0 \in C$ defines a holomorphic map $C \to J(C)$. The Jacobian $J(C)$ and the Abel-Jacobi map $C \to J(C)$ exist for all compact smooth complex curves and are an essential ingredient in the study of complex structures on Riemann surfaces.