1 Riemannian Metrics and Covariant Derivatives

1.1 Quadratic Forms

Let V be a finite dimensional real vector space. A *quadratic form* on V is a symmetric 2-tensor on V^*; i.e., an element $B \in V^* \otimes V^*$ invariant under the interchange of factors. Such a tensor defines a bilnear function, also called B, on V, namely $v_1 \otimes v_2 \mapsto A(v_1, v_2)$. Since B is required to be symmetric, this bilinear function is symmetric in v_1 and v_2. We define the associated quadratic form $Q = Q_B$ on V by $Q(v) = B(v, v)$. This form is homogeneous of degree 2, meaning that $Q(\lambda v) = \lambda^2 Q(v)$ for any $\lambda \in \mathbb{R}$ and any $v \in V$. The quadratic form Q and the bilinear form B are related by following two equations

$$Q(v_1 + v_2) = Q(v_1) + Q(v_2) + 2B(v_1, v_2)$$

$$Q(v) = B(v, v).$$

As long as we are working in a context in which 2 is invertible (which is the case here) the bilinear form and the quadratic form determine each other.

Choosing a basis $\{e_1, \ldots, e_n\}$ for V define an $n \times n$ symmetric matrix M defined by $M_{i,j} = B(e_i, e_j)$. This is a symmetric matrix that represents both the bilinear form and the quadratic form in the sense that

$$B(\sum_i a^i e_i, b^j, e_j) = \sum_{i,j} a^i b^j M_{i,j}$$

$$Q(\sum_i a^i e_i) = \sum_{i,j} a^i a^j M_{i,j}.$$

So, in this basis, Q is given by a homogeneous quadratic polynomial in n variables $(a^1, \ldots, a^n)$. If we change bases by a linear automorphism L then the matrix for Q changes from M to $L^{tr} M L$. The reason is that the quadratic form is given by matri multiplication

$$Q(v) = v^{tr} M v,$$

and as we change basis v is replaced by Lv.

A quadratic form Q is said to be *positive definite* if $Q(v) > 0$ for all $v \neq 0$ in V. Analogously, Q is said to be *negative definite* if $Q(v) < 0$ for all $v \neq 0$ in V. The null space of a quadratic form is the largest subspace $W \subset V$ with the property that $B|_{W \otimes V} = 0$.

The following is easily established.

Lemma 1.1. *If Q is a quadratic form on a finite dimensional real vector space V, then there is a basis $\{e^1, \ldots, e^n\}$ for V in which the matrix representative for Q is diagonal, and in fact has all diagonal entries in $\{+1, -1, 0\}$. The number of positive, resp. negative, diagonal entries is the dimension of any maximal subspace on which Q is positive, resp., negative, definite. The space spanned by the basis vectors with zero diagonal entry is the null space of the quadratic form.*

A positive definite quadratic form on V determines a length $|v|$ by $|v|^2 = Q(v)$ and angles between vectors by setting the angle between v and w equal to

$$\mathrm{Arccos}\left(\frac{\langle v, w\rangle}{|v| \cdot |w|}\right),$$

where $\langle v, w\rangle = B(v, w)$ with B being the bilinear form associated to the quadratic form.

1.2 The Definition of a Riemannian Metric

Definition 1.2. A Riemannian metric for. a smooth manifold M is a symmetric section under the switch of factors of the tensor square of the cotangent bundle, $T^*M \otimes T^*M$, whose restriction to every tangent space determines a positive definite quadratic form. The symmetric tensor square of T^*M is denoted $\mathrm{Symm}^2(T^*M)$.

In local coordinates $(x^1, \ldots, x^n)$, a Riemanian metric takes the form $g_{i,j} dx^i \otimes dx^j$ where $g_{i,j}$ is a symmetric matrix of smooth functions with $g_{i,j}(x^1, \ldots, x^n j)$ a positive definite (real matrix) for every $(x^1, \ldots, x^n)$ in the coordinate patch.. The transformation formula as we pass from one local coordinate system to another is given by the tensor square of the transformation formula for differential 1-forms. That is to say if $(y^1, \ldots, y^n)$ is another set of local coordinates then

$$\sum_{i,j} g_{i,j}(\mathbf{x}) dx^i \otimes dx^i = \sum_{i,j} g_{i,j}(\mathbf{x}(\mathbf{y})) \sum_r \left(\frac{\partial x^i}{\partial y^r} dy^r\right) \sum_s \left(\frac{\partial x^j}{\partial y^s} dy^s\right)$$

so that

$$g_{r,s}(\mathbf{y}) = \sum_{i,j} g_{i,j}(\mathbf{x}(\mathbf{y}) \frac{\partial x^i}{\partial y^r} \frac{\partial x^j}{\partial y^s}.$$

As above Riemannian metric determines a positive definite pairing, denoted $\langle \cdot, \cdot \rangle$, on each tangent space resulting in notion of length for each tangent vector $|v| = \sqrt{\langle v, v\rangle}$ and determines an angle between two tangent vectors at the same point by the Arcccos-formula given above. These notions vary smoothly as we move tangent vectors smoothly. in the tangent bundle.

If $\gamma\colon [0,1] \to M$ is a smooth curve, we define the *length* of γ to be

$$\int_0^1 |\gamma'(t)|dt.$$

For a connected Riemannian manifold M we define the distance $d(x,y)$ for points $x, y \in M$ to be the infimum over all smooth curves from x to y of the length of the curve.

Theorem 1.3. *For a connected, Riemanian manifold M the distance function $d\colon M \times M \to \mathbb{R}$ as defined above is a metric.*

Proof. Clearly, $d\colon M \times M \to \mathbb{R}$ is a non-negative, symmetric, contiunous function that satisfies the triangle ineqaulity. To see that it is a metric we need only see that $d(x,y) = 0$ implies that $x = y$. Suppose that $x \neq y$. Then there is a closed ball B in a coordinate patch centered at x that is disjoint from y. By compactness there is $\epsilon > 0$ such that for any non-zero tangent vector v at a point of B the ratio of the length of v measured in the Riemannian metric divided by the Euclidean length of v computed using the given local coordinates is at least ϵ. That implies that for any $y \in B$ any curve in B from x to y has length in the RIemannian metric at least ϵ times its Euclidean length. Thus, the infimum over all paths in B from x to y of the length of the path is at least the ϵ times the Euclidean distance from x to y. In particular, there is a uniform positive lower bound to the length of any smooth curve from x to any point in ∂B. It now follows that for $y \notin B$ there is a uniform positive lower bound, independent of y, to the length of any smooth curve from x to y, and for any point $y \in B$ any smooth curve from x to y has length at least ϵ times the Euclidean distance from x to y. □

Definition 1.4. A connected Riemannian M is said to be *complete* if the metric $d\colon M \times M \to \mathbb{R}$ defined by the Riemannian metric is complete metric in the usual sense, meaning every Cauchy sequence has a limit point.

1.3 Existence of Riemannian Metrics

Theorem 1.5. *Every smooth manifold has a Riemannian metric.*

Proof. Let $V \subset \mathbb{R}^n$ be an open subset. Then we have a trivialization of TV as $V \times \mathbb{R}^n$. We give V the Riemannian metric which is the standard Euclidean metric on each tangent space in this trivialization. That is to say, $\{\partial_1, \ldots, \partial_n\}$ form an orthonormal basis of the tangent space at each point of V. We cover M by a union of coordinate patches U_α, Each U_α is diffeomorphic to an open subset in $V_\alpha \subset \mathbb{R}^n$. We transport the Riemannian metric on V_α by this diffeomorphism to give a Riemannian metric ω_α on U_α.

Since M is paracompact, there is a smooth partition of unity $\{\lambda_\alpha\}_\alpha$ subordinate to this covering in the sense that the support of λ_α is contained in U_α and for each $x \in M$ there is an open subset U_x of x meeting the support of only finitely many of the λ_α. Consider the sum $\sum_\alpha \lambda_\alpha \omega_\alpha$. This is a locally finite sum and on each fiber it is a finite affine linear combination of positive definite quadratic forms. But the space of quadratic forms is a vector space, so this sum makes sense. Also, an affine linear combination of positive definite forms is positive definite. Hence the result is a Riemannian metric on M. □

1.4 Examples

Eucldean space $\mathbb{R}^n$ has a natural metric: Its tangent space at every point is identified with the vector space $\mathbb{R}^n$ which has its usual metric $\langle x, x\rangle = |x|^2 = \sum_i (x^i)^2$. Viewing the metric as a symmetric 2-tensor in the cotangent bundle it is given by $\sum_i dx^i \otimes dx^i$.

Let N be a smooth submanifold of a Riemannian manifold M with Riemannian metric g. The metric g gives a smoothly varying positive definite quadratic form on each tangent space of M. Hence, it restricts to give a smoothly varying positive definite quadratic form on the tangent spaces of N, which are subspaces of the tangent spaces of M. That is to say $g|_N$ is a Riemannian metric on N. In particular any manifold embedded as a submanifold of $\mathbb{R}^N$ has an induced Riemannian metric. In the case of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ the induced metric is invariant under the action of the orthogonal group $O(n)$ on the unit sphere. A Riemannian manifold whose group of isometries acts transitively on the manifold is said to be *homogeneous.* Thus, $\mathbb{R}^n$ and S^{n-1} are our first examples of a homogeneous Riemannian manifolds.

Consider the upper half-plane $\{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. We writ $z = x + iy$. The metric is

$$g_{\text{hyp}} = \frac{ds^{\otimes 2}}{y^2} = \frac{dx \otimes dx + dy \otimes dy}{y^2},$$

where $ds^{\otimes 2}$ is the restriction of the usual Euclidean metric to the upper half-plane. Thus, if v is a tangent vector at $x+iy$, then

$$g_{\mathrm{hyp}}(v,v) = \frac{\langle v, v\rangle}{y^2} = \frac{|v|^2}{y^2},$$

where $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product. If v is written as a complex vector then $|v|^2 = v\cdot\overline{v}$.

Let the group $SL(2,\mathbb{R})$ act on the upper half-plane as follows: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $SL(2,\mathbb{R})$, i.e., $a,b,c,d\in\mathbb{R}$ and $ad-bc=1$. Then

$$A\cdot z = \frac{az+b}{cz+d}.$$

It is a direct computation to show:

$$\mathrm{Im}(A\cdot z) = \frac{\mathrm{Im} z}{|cz+d|^2},$$

and the action on the complex derivative is given by

$$(A(z))' = \frac{z'}{(cz+d)^2}.$$

It follows immediately that the action preserves the Riemannian metric g_{hyp}. It is easy to see that the action of $SL(2,\mathbb{R})$ on the upper half-plane is transitive, so this gives another example of a homogeneous Riemannian manifold.

There is a higher dimensional generalization of this metric. Consider the quadratic form

$$Q(x^0,\ldots,x^n) = -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$$

and the hypersurface

$$\mathbf{H}^n = \{Q=-1\}\cap\{x^0>0\}.$$

It is an easy exercise to show that the restriction of the form Q to any tangent space $T_x\mathbf{H}^n$ is positive definite. Hence, it determines a Riemannian metric on $\mathbf{H}^n$. This is *n-dimensional hyperbolic space.* Notice that the isometries group of Q preserves $\mathbf{H}^n\coprod -\mathbf{H}^n$ and the subgroup stablizing $\mathbf{H}^n$ is a subgroup of index 2. This subgroup acts as a group of isometries of hyperbolic n-space, so hyperbolic n-space is also a homogeneous Riemannian manifold.

1.5 General Relativity

In Newtonian mechanics we have flat Euclidean 3-space with the usual Euclidean metric and time as an independent linear coordinate. Newtonian space-time is simply their product – a flat 4-dimensional space with Riemannian metric

$$dt^{\otimes 2} + dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2}$$

together with the foliation by slices in the spatial directions at constant time and the orthogonal time lines.

Using the fact that Maxwell's equations produce a constant – the speed of light, denoted c, – that is a physical constant independent of the inertial frame, Einstein realized that this was incompatible with the Newton product picture separating flat space and flat time. The Euclidean 4-dimensional metric is not what is invariant, but rather the "metric"

$$c^2 dt^{\otimes 2} - (dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2}).$$

While this is a non-degenerate quadratic form, it is not positive definite so it is not a Riemannian metric. By a slight abuse of terminology we call it the *Lorentz metric*. There are analogous notions for Lorentzian geometry to those we encounter in Riemannian geometry.

First, let us understand what plays the role of the usual orthogonal group for this metric. We are considering a vector space with linear coordinates t, x, y, z and constant 'metric' $c^2 dt^{\otimes 2} - (dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2})$ on each tangent space (identified in the usual way with the underlying vector space). We want to find the group of all linear transformations that preserve this metric. A simpler case is when 'space' is 1-dimensional so that we have 'metric' $c^2 dt^{\otimes 2} - dx^{\otimes 2}$. We have an invariant 'light cone' given by the subspaces $x = \pm ct$. The linear transformations that preserve the form are

$$\begin{pmatrix} cosh(t) & c \cdot sinh(t) \\ c^{-1} \cdot sinh(t) & cosh(t) \end{pmatrix}$$

where

$$cosh(t) = \frac{e^t + e^{-t}}{2}; sinh(t) = \frac{e^t - e^{-t}}{2}$$

so that

$$cosh'(t) = sinh(t); sinh'(t) = cosh(t)$$

and

$$cosh^2(t) - sinh^2(t) = 1.$$

We can also multiply by the group of diagonal matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

that interchange the two parts of the light cone and also reverse orienations of the hyberbolic sheets. This is the Lorentz group in one spacial dimension. It is the orthogonal group that stabilizes the quadratic form

$$\begin{pmatrix} c^2 & 0 \\ 0 & -1 \end{pmatrix}.$$

In 3-spatial dimensions we have the group generated by the orthogonal group of the spatial 3-dimensional space and the Lorentz group in (t, x) described above. [The latter are often called 'boosts' in the physics literature.] We are describing the group of linear transformations of 4-space that is stabilizing the diagonal quadratic form with diagonal entries $(c^2, -1, -1, -1)$. This form divides space along the light cone, which is the space of vectors of length 0 in this 'metric' from the light-like vectors that have positive length under the form and the space-like vectors that have negative length under the form.

Einstein formulation of general relativity is as a smooth 4-manifold with a Lorentzian metric, namely a smoothly varying quadratic form on the tangent spaces of the manifold with the form on each tangent space being isomorphic to the Lorentz metric on 4-space. The flat example of such Lorentzian geometry is *Minkowski space.* It is $\mathbb{R}^4$ with Lorentzian metric

$$c^2 dt^{\otimes 2} - (dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2})$$

on each tangent space. One encoutners non-flat Lorentzian manifolds in general relativity. Indeed Einsteins equations of state for a space-time relate the Lorentzian metric tensor of space-time, a curvature tensor for this metric called the *Ricci curvature,* and the so-called *stress-energy tensor* which is a measure of the matter and energy contained in space-time.

1.6 Covariant Differentiation for Riemannian manifolds

There is not in general a natural notion of the derivative of a vector field in a direction. The reason is that as we move in the manifold the vector space where the vector field takes values changes, so that the derivative of its value makes no sense. Nevertheless, on a Riemannian manifold there is a natural way to do this leading to the notion of covariant differentiation.

In this section we denote by $VF(M)$ the real vector space space of vector fields on M. It is a Lie algebra under bracket and it is a module over the smooth functions on M.

Definition 1.6. In general a covariant derivative of vector fields on a smooth manifold M is an $\mathbb{R}$-linear homomorphism

$$\nabla\colon VF(M)\otimes VF(M)\to VF(M),\ \text{denoted}\ \ X\otimes Y\mapsto \nabla_X(Y).$$

It satisfies the following axioms:

1. ∇ is linear over the smooth functions in the first variable, i.e., $\nabla_{fX}(Y) = f\nabla_X(Y)$.
2. $\nabla_X(fY) = X(f)Y + f\nabla_X(Y)$ (this is a type of Leibnitz rule).

Lemma 1.7. *If one of X or Y is identically zero near p then $\nabla_X(Y)$ is identically zero near p. If $X = X'$ and $Y = Y'$ near p then $\nabla_X Y = \nabla_{X'}Y'$*

Proof. Let U be a neighborhood of p on which one of X and Y vanishes. Let f be a smooth function identically equal to 0 near p and identically 1 outside U. Suppose that X vanishes on U. Then $fX = X$. Hence $\nabla_X Y = \nabla_{fX}Y = f\nabla_X Y$ vanishes near p. If instead Y vanishes on U, then $fY = Y$ and

$$\nabla_X Y = \nabla_X(fY) = X(f)Y + f\nabla_X Y.$$

Of course $f\nabla_X Y$ vanishes near p. Also, $X(f)$ vanishes near p since f is constant near p. This establishes the first statement.

Applying this, we have $\nabla_{X-X'}Y$ vanishes near p so that $\nabla_{X'}Y = \nabla_X Y$ near p. Applying the first statement again, we see $\nabla_{X'}(Y - Y')$ vanishes near p, so that $\nabla_{X'}Y = \nabla_{X'}Y'$ near p. □

Let M be a smooth manifold X and Y vector fields on M. Let $U\subset M$ be a coordinate patch with local coordinates $(x^1,\dots,x^n)$. Fix $p\in U$ and fix a smooth function f identically 1 near p and with support contained in U. Let χ_i be the vector field $f\partial_i$ on U extended by 0 outside. Then by Lemma 1.7 the covariant derivative $\nabla_{\chi_i}\chi_j$ is supported in U so that we can write

$$\nabla_{\chi_i}\chi_j = \sum_k \Gamma^k_{f,i,j}\partial_k,$$

with the $\Gamma^k_{f,i,j}$ smooth functions supported in U.

Claim 1.8. *If g is some other function supported in U identially 1 near p then $\Gamma^k_{g,i,j} = \Gamma^k_{f,i,j}$ on some neighborhood of p.*

Proof. Since $f\chi_i = g\chi_i$ for all i in some neighborhood of p, this follows immediately from Lemma 1.7. □

Corollary 1.9. *There are smooth functions $\Gamma^k_{i,j}$ defined throughout U such that if $\{\chi_i\}$ of the form $f\partial_i$ with f compactly supported in U then on any open subset of U on which f is identically 1 we have*

$$\nabla_{\chi_i}\chi_j = \sum_k \Gamma^k_{i,j}\partial_k.$$

These functions $\Gamma^k_{i,j}$ on U are called the *Christoffel symbols* for the covariant derivative written in the local coordinates on U.

Proposition 1.10. *Suppose A and B are vector fields on M whose restriction to U are $\sum_i a_i\partial_i$ and $\sum_i b_i\partial_i$ for arbitrary smooth functions a_i and b_i on U. Then*

$$(\nabla_A B)|_U = \sum_{i,j} a_i b_j \Gamma^k_{i,j}\partial_k + a_i\partial_i(b_j)\partial_j.$$

Proof. It suffices to prove this equality in a neighborhood of each point $p \in U$. By Lemma 1.7 it suffices to replace A and B by fA and fB for some function f supported in U and identically 1 near p. The result then follows from Corollary 1.9. □

Remark 1.11. If the Christoffel symbols are all zero on the coordinate patch, the covariant derivative is the usual derivative using the trivialization of the bundle with basis $\{\partial_1, \ldots, \partial_n\}$ on every fiber.

Definition 1.12. Suppose that we have a Riemannian metric on a smooth manifold. We say that a connection ∇ for vector fields on this manifold. We say that ∇ is the *Riemannian covariant derivative* (or the *Levi-Cività covariant derivcative*) if the following two conditions hold:

(i) $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$.

(ii) $X(\langle Y, Z\rangle) = \langle \nabla_X(Y), Z\rangle + \langle Y, \nabla_X(Z)\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product on tangent vectors determined by the Riemannian metric

The second condition says that ∇ preserves the metric. The first goes under the rubric 'the covariant derivative is *torsion-free*'.

Proposition 1.13. *Every Riemannian manifold (M, g) has a unique Riemannian covariant derivative.*

Proof. It suffices to work in local coordinates where the connection is given by Christoffel symbols $\Gamma^k_{i,j}$. Define $\Gamma_{i,j,k} = \sum_\ell g_{k,\ell}\Gamma^\ell_{i,j}$. Since $[\partial_i, \partial_j] = 0$, the torsion-free condition is equivalent to $\Gamma^k_{i,j} = \Gamma^k_{j,i}$ which implies $\Gamma_{i,j,k} = \Gamma_{j,i,k}$.

Since $g_{i,j} = \langle\partial_i, \partial_j\rangle$, the second condition is equivalent to

$$\begin{aligned}
\partial_k(g_{i,j}) &= \langle\sum_\ell \Gamma^\ell_{k,i}\partial_\ell, \partial_j\rangle + \langle\partial_i, \sum_\ell \Gamma^\ell_{k,j}\partial_\ell\rangle \\
&= \sum_\ell g_{\ell,j}\Gamma^\ell_{k,i} + g_{\ell,i}\Gamma^\ell_{k,j} \\
&= \Gamma_{k,i,j} + \Gamma_{k,j,i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_i(g_{j,\ell}) + \partial_j(g_{i,\ell}) - \partial_\ell(g_{i,j}) &= \Gamma_{i,j,\ell} + \Gamma_{i,\ell,j} + \Gamma_{j,i,\ell} + \Gamma_{j,\ell,i} - (\Gamma_{\ell,i,j} + \Gamma_{\ell,j,i}) \\
&= 2\Gamma_{i,j,\ell}
\end{aligned}$$

(To get the last last we use repeatedly the symmetry $\Gamma_{i,j,\ell} = \Gamma_{j,i,\ell}$.))

Let $g^{\ell,k}$ denote the inverse matrix to $g_{i,k}$. The we have

$$\begin{aligned}
\sum_\ell g^{\ell,k}\left(\partial_i g_{j,\ell}) + \partial_j(g_{i,\ell}) - \partial_\ell(g_{i,j})\right) &= 2\sum_\ell g^{\ell,k}\Gamma_{i,j,\ell} \\
&= 2\sum_\ell g^{k,\ell}\sum_t g_{\ell,t}\Gamma^t_{i,j} \\
&= 2\sum_t \delta^k_t\Gamma^t_{i,j} = 2\Gamma^k_{i,j}.
\end{aligned}$$

(Here, δ^k_t is the delta function which is nonzero only when $k = t$ when it is 1.)

This shows that $\Gamma^k_{i,j}$ is determined (in any local coordinate system) by the metric tensor in that coordinate system. Namely,

$$\Gamma^k_{i,j} = \frac{1}{2}\left(\sum_\ell g^{\ell,k}\left(\partial_i(g_{j,\ell}) + \partial_j(g_{i,\ell}) - \partial_\ell(g_{i,j})\right)\right).$$

□

Remark 1.14. Notice that the derivation above only requires that the quadratic form defining the metric on the tangent spaces be invertible, not necessarily positive definite. Thus, Lorentzian manifolds also have a unique torsion-free connection preserving the Lorentz metric. The formulas above for the Christoffel symbols in local coordinates are valid in tis context as well.

1.7 More General Covariant Derivatives and the Connection Description

Suppose that $\mathcal{E}$ is a smooth vector bundle over a smooth manifold M. By a covariant derivative on $\mathcal{E}$ we mean an $\mathbb{R}$-linear map

$$\nabla\colon VF(M)\otimes\Gamma^\infty(\mathcal{E})\to\Gamma^\infty(\mathcal{E}),$$

where $\Gamma^\infty(\mathcal{E})$ is the space of smooth sections on V, that is required to satisfy:

- $\nabla_{fX}(\sigma)=f\nabla_X(\sigma)$ (linearity over the functions in the first variable).
- $\nabla_X(f\sigma)=X(f)\sigma+f\nabla_X(\sigma)$ (a Leibnitz rule in the second variable).

Over any open subset $U\subset M$ with local coordinates $(x^1,\ldots,x^n)$ over which there is a smooth trivialization. of $V|_U=U\times\mathbb{R}^k$ the connection is given by Christoffel symbols $\Gamma^\beta_{i,\alpha}$. These are smooth functions, where i ranges over the indices of the local coordinates for U and α,β range over the indices of the basis for $\mathbb{R}^k$. The definition is

$$\nabla_{\partial_i}(e_\alpha)=\sum_\beta\Gamma^\beta_{i,\alpha}e_\beta.$$

(As in the case on the tangent bundle one must localize by multiplying by a bump function supported in U and 1 near the point where we are making the computation.). Then, the same argument as in the case of the tangent bundle shows that for any section $\sigma\in\Gamma^\infty(\mathcal{E})$ if $\sigma|_U=\sum_\alpha\sigma^\alpha e_\alpha$ and $A|_U=\sum_i f^i\partial_i$ then

$$(\nabla_A\sigma)|_U=\sum_i f^i\left(\partial_i(\sigma^\alpha)e_\alpha+\sum_\beta\sigma^\alpha\Gamma^\beta_{i,\alpha}e_\beta\right).$$

Notice that there are no symmetry conditions on these symbols.

Suppose that $f\colon N\to M$ is a smooth map and $\mathcal{E}\to M$ is a smooth vector bundle. Let ∇ b a covariant derivative on $\mathcal{E}$. We have defined the pull back bundle $f^*\mathcal{E}=N\times_M\mathcal{E}$. We claim that this pull back bundle carries a pulled back connection $f^*\nabla$. Recall that the fiber of $f^*\mathcal{E}$ over x is identified with

$\mathcal{E}_{f(x)}$. For any tangent vector $A \in T_xN$, we have $Df_x(A) \in T_{f(x)}(M)$. Then $\nabla_{f(x)}$ is an endomorphism of $\mathcal{E}_{f(x)}$ or equivalently of $f^*\mathcal{E}|_x$. This is the definition of $(f^*\nabla)_A$ on $f^*\mathcal{E}$. Given local coordiantes $(x^1, \ldots, x^k$ of N near x and $(y^1, \ldots, y^n)$ of M near $f(x)$ and a trivialization $U \times V$ for $\mathcal{E}$ near $f(x)$ then ∇ is given by

$$\nabla = \sum_j \Gamma^{\beta}_{j,\alpha} \partial_{y^i}.$$

There is an induced trivialization of $f^*(\mathcal{E})$ near x and $f^*\nabla$ is given by

$$f^*\nabla = \sum_i \sum_j \frac{\partial y^j}{\partial x^i} \Gamma^{\beta}_{j,\alpha} \partial_{x^i}.$$

1.8 Relations to a Connection

Let us consider the case of a covariant derivative ∇_A on a smooth vector bundle $\mathcal{E} \to M$ we work in local coordinates $(x^1, \ldots, x^n)$ for M on an open set $U \subset M$ with a trivialization $\mathcal{E}|_U = U \times \mathbb{R}^k$ given by a basis of sections $\{e_\alpha\}$. Any connection ∇_A is given in U and with respect to the trivialization by Christofel symbols $\Gamma^{\beta}_{i,\alpha}$ in the sense that if $\sigma = \sum_\alpha \sigma^\alpha e_\alpha$ is a section of $\mathcal{E}$ then

$$\nabla_{\partial_i}(\sigma)|_U = \sum_\alpha \left(\partial_i \sigma^\alpha e_\alpha + \sum_\beta \sigma^\alpha \Gamma^{\beta}_{i,\alpha} e_\beta \right).$$

Claim 1.15. *There is an n-dimensional distribution $\mathcal{D}$ in the total space $\mathcal{E}$ of the bundle transverse to the tangent spaces to the fibers of the vector bundle; i.e., the differential of the projection $\mathcal{E} \to M$ maps each plane of the distribution $\mathcal{D}$ isomorphcally onto a tangent plane of M, with the following property: For a section σ and any element $\in T_X M$, the covariant derivative $\nabla_A(\sigma) = 0$ if and only if $D\sigma_x(A) \in \mathcal{D}_{\sigma(x)}$.*

Proof. It suffices to work locally. Fix an open set $U \subset M$ with local coordinates $(x^1, \ldots, x^n)$ and with a trivialization of $\mathcal{E}|_U = U \times V$ given by a sections $\{e_\alpha\}_\alpha$ that form a basis at each point.

For each i and each $x \in U$ we have the endomorphism $\Gamma_i(x)$ defined by

$$\Gamma_i(x)(e_\alpha) = \sum_\beta \Gamma^{\beta}_{i,\alpha}(x) e_\beta.$$

That is to say the matrix $\Gamma^{\beta}_{i,\alpha}(x)$ with entries indexed by α and β is an endomorphism of the vector space V spanned by the $\{e_\alpha(x)\}$. At each

$(x, \sigma) \in U \times V$ consider the tangent vector

$$(\partial_i|_x, -\Gamma_i(x)(\sigma)).$$

(In this expression the second coordinate is a point in V, which is identified in the usual way with the tangent space of V.) Thus $(\partial_i|_x, -\Gamma_i(x)(\sigma))$ is a tangent vector to $U \times V$.

These vectors vary smoothly with $x \in U$ and $\sigma \in V$. Furthermore, under the projection $\mathcal{E}|_U \to U$, the vector maps to $\partial_i|_x$, so that the plane that the vectors indexed by all $1 \leq i \leq n$ span maps isomorphically onto T_xU. The planes spanned by these vectors is a distribution $\mathcal{D}(U)$ on $\mathcal{E}|_U$ whose planes map isomorphically onto tangent planes to U under the projection.

It follows easily from the definition that if σ is a local section defined near $x \in U$ then $\nabla_A(\sigma) = 0$ for all $A \in T_xU$ if an only if $D\sigma_x \colon T_xM \to T_{\sigma(x)}\mathcal{E}$ has image $\mathcal{D}(\sigma(x))$. This proves that we change the trivialization of $\mathcal{E}|_U$ the distribution $\mathcal{D}(U)$ is invariant and as we pass from a coordinate patch U to a coordinate patch U' the distributions $\mathcal{D}(U)$ and $\mathcal{D}(U')$ agree on the overlap. That is to say the $\mathcal{D}(U)$ fit together to form a global distribution on $\mathcal{E}$, with each plane in the distribution projecting isomorphically onto a tangent space of M. □

The distribution $\mathcal{D}$ determined by the covariant derivative ∇ is called the *connection* associated with the covariant derivative. In fact, the connection is equivalent to the covariant derivative. To see this, for any $e \in \mathcal{E}$, define $P_e \colon T_{\pi(e)}M \to \mathcal{D}(e)$ to be the inverse isomorphism to the projection $\mathcal{D}(e) \to T_{\pi(e)}M$. We shall show that

$$\nabla_A(\sigma)(x) = D\sigma_x(A) - P^{-1}_{\sigma(x)}(A). \tag{1.1}$$

Since the two vectors on the right-hand side each project to $A \in T_{\pi(e)}M$, the right-hand side of this equation is a tangent vector to $\mathcal{E}$ that projects to zero in M; that is to say, the right-hand side is a tangent vector to the fiber $\mathcal{E}|_{\{x\}}$, and hence a vector in $\mathcal{E}|_{\{x\}}$.

In a local coordinate system and a local trivialization of $\mathcal{E}$ as before we have the following three equations:

$$\nabla_{\partial_i}(\sigma)(x) = \sum_\alpha \left(\partial_i(\sigma^\alpha)e_\alpha + \sum_\beta \Gamma^\beta_{i,\alpha}(x)\sigma^\alpha e_\beta \right),$$

$$P^{-1}_{\sigma(x)}(\partial_i) = \left(\partial_i, -\sum_{\alpha,\beta}(\Gamma^\beta_{i,\alpha}(x)\sigma^\alpha e_\beta) \right),$$

$$D\sigma_x(\partial_i) = (\partial_i, \sum_\alpha (\partial_i(\sigma^\alpha)e_\alpha).)$$

From these Equation 1.1 follows for $A = \partial_i$. From this and the linearity of covariant differentiation over the smooth functions in the vector field variable, it is immediate that Equation 1.1 holds for general tangent vectors.

Remark 1.16. In the special case of the Riemannian covariant derivative on the tangent bundle of a Riemannian manifold, the associated connection is called the *Riemannian* connection. The above discussion carries over *mutatis-mutandis* to the case of Lorentzian manifolds so that the Lorentzian covariant derivative also has a connection description, the *Lorentzian connection.*

1.9 Parallel Translation

Definition 1.17. Let $\pi\colon \mathcal{E} \to M$ be a smooth vector bundle with a connection. Let $\gamma\colon (a,b) \to M$ and lifting $\widetilde{\gamma}\colon (a,b) \to \mathcal{E}$ is said to be a *parallel lifting* or a *horizontal lifting* of γ if $\nabla_{\gamma'(t)}(\widetilde{\gamma}(t)) = \widetilde{\gamma}'(t)$ for all $t \in (a,b)$. equivalently, the lifting is parallel if $\pi \circ \widetilde{\gamma} = \gamma$ and if $\widetilde{\gamma}(t)$ is tangent to the distribution in $\mathcal{E}$ associated to ∇.

Lemma 1.18. *Given a covariant derivative on a smooth vector bundle $\pi\colon \mathcal{E} \to M$, given a smooth curve $\gamma\colon [a,b) \to M$, and a point $e \in \mathcal{E}|_{\gamma(a)}$ there is a unique horizontal lifting $\widetilde{\gamma}$ of γ with $\widetilde{\gamma}(a) = e$.*

Proof. We pull the vector bundle $\mathcal{E} \to M$ back by γ^* to a smooth vector bundle over $[a,b)$ and we pull the connection ∇ back to a connection on this bundle. Then the problem becomes to show that for each initial value in the fiber over $\{a\}$ there is a parallel lifting for $\gamma^*\nabla$ with this value as initial condition. But the horizontal distribution for the pulled back covariant derivative is a line field, and there is a unique vector in each line that projects to the vector field $\partial/\partial t$ on $[a,b)$. These vectors form a smooth vector field on $\gamma^*\mathcal{E}$. The equation that a parallel lift has to satisfy is that it is an integral curve for this vector field. Thus, the existence and unqiueness of solutions for the initial value problem for ODE's applies to show that there is a unique horizontal lift with the given initial value. Once we have the horizontal lift $\widetilde{\varphi}(t)$ in $\gamma^*(\mathcal{E})$ of the identity map, we push it forward to give the horizontal lift $\widetilde{\gamma}\colon [a,b) \to \mathcal{E}$ of $\gamma\colon [a,b) \to M$. □

For any $t \in (a,b)$, the point $\widetilde{\gamma}(t) \in \mathcal{E}$ is called the *result parallel translation along $\gamma|_{[a,t]}$ applied to e* or *the result of parallel translating e along γ from $\gamma(a)$ to $\gamma(t)$.*

Claim 1.19. *Parallel translation along γ from $\gamma(a)$ to $\gamma(t)$ defines a linear isomorphism $\mathcal{E}|_{\gamma(a)} \to \mathcal{E}|_{\gamma(t)}$.*

Proof. It suffices to work locally in coordinates $(x^1, \dots, x^n)$ on an open set $U \subset M$ over which we have a trivialization $U \times \mathbb{R}^k$ with basis $\{e_\alpha\}$ for $\mathcal{E}$. We have $\gamma\colon [a, b) \to U$. Then the equation that the horizontal lift $\widetilde{\sigma}$ of γ with initial condition $v = \sum_\alpha v^\alpha e_\alpha$ is

$$(\sigma_v^\beta)'(t) = -\sum_\alpha \sigma_v^\alpha(t)\Gamma^\beta_{\gamma(t),\alpha}$$

with initial condition $\sigma_v^\beta(a) = v^\beta$ for all β. This is a vector-valued linear ODE, so that the solution varies linearly with the initial condition v. It follows that the value of the solution $\sigma_v(t)$ is a linear function of v. Thus, parallel translation along γ from $\gamma(a)$ to $\gamma(t)$ is a linear map $\mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(t)}$. It is a linear isomorphism since parallel translation along γ in the opposite direction is its inverse. □

1.10 Parallel translation on the tangent bundle of a submanifold of $\mathbb{R}^N$

Claim 1.20. *Let (P, g_P) be a Riemannn manifold with induced Riemannian covariant derivative ∇^P Let $M \subset P$ be a smooth submanifold and g_M the induced Riemannian metric on M and denote by ∇^M its covariant derivative. Then for $A \in T_xM$ we have*

$$\nabla^M_A = \pi_{T_xM} \circ \nabla^P_A,$$

where π_{T_xM} is the orthogonal projection $T_xP\colon T_xP \to T_xM$ defined using the inner product $g_P|_{T_xP}$.

Proof. First note that, computing in the inner product defined by g_P we have

$$\langle \pi_{T_xM}(v), w\rangle = \langle v, w\rangle$$

provided that $w \in T_xM$. Thus, given a curve $\gamma(t)$ in M and families of tangent vectors $v(t), w(t) \in T_{\gamma(t)}M$ and setting $A = \gamma'(0)$, we have

$$\begin{aligned}\langle \nabla^M_A(v(t)), w(0)\rangle + \langle v(0), \nabla^M_A(w(t))\rangle &= \langle \pi_{T_xM}(\nabla^P_A(v(t)), w(0))\rangle + \langle v(0), \pi_{T_xM}(\nabla^P_A(w(t))\rangle \\ &= \langle \nabla^P_A(v(t)), w(0)\rangle + \langle v(0), \nabla^P_A(w(t))\rangle \\ &= \frac{d}{dt}\langle v(t), w(t)\rangle|_{t=0}.\end{aligned}$$

(The last equation uses the fact that ∇^P preserves the metric g_P.) This proves that ∇^M preserves the induced metric g_M.

Now let v and w be vector fields along $M \subset P$. We have

$$\nabla_v^P(w) - \nabla_w^P(v) = [v, w],$$

and

$$\nabla_v^M(w) - \nabla_w^M(v) = \nabla_v^P(w) - \nabla_w^P(v) + L$$

where L is orthgonal to TM. Thus,

$$\nabla_v^M(w) - \nabla_w^M(v) = [v, w] + L.$$

But $[v, w] \in TM$ and hence $L = 0$. This proves that ∇^M is torsion-free.

Thus, the connection ∇^M satisfies the two properties that make it the Riemannian connection of $g_M = g_P|_M$.

□

As a special case let us consider a submanifold M of $\mathbb{R}^N$. Let g_M be the metric induced by the Euclidean metric on $\mathbb{R}^N$. Parallel translation along a curve in $\mathbb{R}^N$ for the Riemannian connection of the Euclidean metric is simply ordinary parallel translation. Thus, for a curve $\gamma(t)$ in M, a family of tangent vectors $v(t) \in T_{\gamma(t)}M$ is parallel if and only if for every t, $v'(t)$ is orthogonal (in Euclidean space) to $T_{\gamma(t)}M$.

2 Curvature of Connections

Let M be a smooth manifold, let $E \to M$ be a smooth vector bundle. We denote $\Gamma(E)$ the space of smooth sections of E. Let $\nabla\colon VF(M) \otimes \Gamma(E)$ be a connection, with the notation $\nabla_X(\sigma) + \nabla(X \otimes \sigma)$..

Lemma 2.1. *For vector fields X, Y the expression*

$$R(X,Y)(\cdot) = \nabla_X \circ \nabla_Y(\cdot) - \nabla_Y \circ \nabla_X(\cdot) - \nabla_{[X,Y]}(\cdot)$$

is an endomorphism of $\Gamma^\infty(E)$ that is linear over $C^\infty(M)$, meaning that $R(X,Y)(f\sigma) = fR(X,Y)(\sigma)$ for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$.

Proof. This is a direct computation.

$$\nabla_X(\nabla_Y(f\sigma)) = \nabla_X(Y(f)\sigma + f\nabla_Y(\sigma)) = X(Y(f))\sigma + Y(f)\nabla_X(\sigma) + X(f)\nabla_Y(\sigma) + f\nabla_X \circ \nabla_Y(\sigma).$$

This plus the symmetric expression tells us that

$$(\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X)(f\sigma)) - f(\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X)(\sigma) = (X(Y(f)) - Y(X(f)))\sigma.$$

On the other hand,

$$\nabla_{[X,Y]}(f\sigma) - f\nabla_{[X,Y]}(\sigma) = [X,Y](f).$$

The lemma follows by subtracting the second equality from the first. □

Corollary 2.2. *There is a section $R(X,Y)$ of the endomorphism bundle of E, i.e., of the vector bundle* $\mathrm{Hom}(E,E)$*, such that for all sections σ of E*

$$\nabla_X \circ \nabla_Y(\sigma) - \nabla_Y \sigma \nabla_X(\sigma) - \nabla_{[X,Y]}(\sigma) = R(X,Y)(\sigma).$$

Proof. This is a special case of the statement an endomorphism of $\Gamma^\infty(E)$ is induced by an endomorphism of E if and only if it is linear over the action of the smooth functions on $\Gamma^\infty(E)$. □

There is in fact much more linearity over the functions as the next result shows.

Lemma 2.3. *For all smooth functions f, g on M we have*

$$R(fX, gY) = fgR(X,Y).$$

Proof. By symmetry it suffices to prove that $R(fX,Y) = fR(X,Y)$. We compute:

$$\nabla_{fX} \circ \nabla_Y - \nabla_Y \circ \nabla_{fX} = f(\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X) - Y(f)\nabla_X.$$

On the other hand $[fX,Y] = f[X,Y] - Y(f)X$. The result follows immediately. □

Corollary 2.4. *There is a section* $\mathbf{R}$ *of* $\mathrm{Hom}(T_*M \otimes T_*M, \mathrm{End}(E))$ *such that for all vector fields* X, Y *the endomorphism of* E *given by* $R(Y,Y)$ *as above is the value of the tensor* $\mathbf{R}$ *on* (X,Y). *By duality we can view* $\mathbf{R}$ *as a section of*

$$T^*(M) \otimes T^*M \otimes \mathrm{End}(E).$$

Since from the definition we see that $R(X,Y)$ *is skew symmetric in* X *and* Y, *in fact* $\mathbf{R}$ *is a section of*

$$\Lambda^2 T^*M \otimes \mathrm{End}(E).$$

Definition 2.5. $\mathbf{R}$ is *the curvature tensor* of the connection ∇. Notice that there are four variables in $\mathbf{R}$: there are the two vector fields on which one evaluates $\mathbf{R}$ and the result is an endomorphism of a vector bundle, i.e., an element of $E^* \otimes E$. Thus, in local coordinates $(x^1, \ldots, x^n)$ on M and a local basis $\{e_\alpha\}_\alpha$ for E the expression for $\mathbf{R}$ is a matrix of two forms on M. Thus, its coefficients are $R^{\beta}_{i,j,^\alpha}$, meaning that

$$\mathbf{R}(\partial_i, \partial_j)(e_\alpha) = \sum_\beta R^\beta_{i,j,\alpha} e_\beta.$$

Said another way, given local coordinates $(x^1, \ldots, x^n)$ for M and a local trivialization of E with bases $\{e_\alpha\}_\alpha$, if $X = \sum_i x^i \partial_i$, $Y = \sum_j y^j \partial_j$ and $\sigma = \sum_\alpha s^\alpha e_\alpha$, then

$$\mathbf{R}(X,Y)(\sigma) = \sum_{i,j,\alpha,\beta} R^\beta_{i,j,\alpha} x^i y^j s^\alpha e_\beta.$$

Let us rewrite $R(\partial_i, \partial_j)$ in terms of the Christoffel sysbols.

Lemma 2.6. *With respect to local coordinates we have*

$$R(\partial_i, \partial_j) = \partial_i \Gamma_j - \partial_j \Gamma_i - [\Gamma_i, \Gamma_j].$$

Here we are viewing Γ_i *as the endomorphism of* E *given by*

$$\Gamma_i(\sum_\alpha \sigma^\alpha e_\alpha) = \sum_{\alpha,\beta} \sigma^\alpha \Gamma^\beta_{i,\alpha} e_\beta.$$

Proof. Since $R(\partial_i, \partial_j) = \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_j$ and $\nabla_i = \partial_i + \Gamma_i$, we see that

$$R(\partial_i, \partial_j) = [\partial_i, \partial_j] + [\partial_i, \Gamma_j] + [\Gamma, \Gamma_j].$$

The first term is zero. The second term is

$$\partial_i(\Gamma_j(\cdot)) - \Gamma_j(\partial_i(\cdot) = (\partial_i \Gamma_j)(\cdot),$$

and by symmetry the third term is $\partial_j(\Gamma_i)$. Putting all this together establishes the lemma. □

This has a direct and interesting corollary in terms of the connection.

Theorem 2.7. *The curvature of a covariant derivative on a bundle E if and only if the connection, thought of as a distribution in the total space of E, is integrable. In particular, the curvature is trivial if an only if near every point x of the base there is a neighbohood U_x and a tirivialization of $E|_{U_x}$ in which ∇_i is equal to ∂_i for all i.*

Proof. First notice that the condition is necessary for there to be a trivialization in which the ∇_i are equal to ∂_i, for then $[\nabla_i, \nabla_j] = 0$ for all I, j, and the curvature is zero..

We consider the converse. Vector fields that span the distribution are $\chi_i = \partial_i + \Gamma_i$ and we have just computed the bracket $[\chi_i \chi_j] = R(\partial_i, \partial_j)$ where both sides are tangent vectors to the fibers of the bundle.

If the curvature vanishes, then by the Frobenius Theorem, the distribution given by the connection is integrable; i.e., there is a foliation tangent to the distribution. The leaves of the foliation map by local diffeomorphisms to the base and parallel translation along the leaves over a curve in the base determines a linear automorphism of the fiber over the initial point to the fiber over the final point. Fix a basis $\{e_1, \ldots, e_k\}$ for the fiber E_{x_0}. For each i, the leaf $\mathcal{L}_i$ through e_i has the property that for a small ball B_i in the base centered at x_0 the intersection of $\mathcal{L}_i$ with the preimage of B_i contains an open subset U_i containing e_i mapping isomorphically to B_i. By restriction to the intersection B of the B_i, for each i we find a subset $U_i \subset \mathcal{L}_i$ containing e_i and mapping isomorphically onto the ball B in M. Parallel translation along these leaves then determines elements $e_i(y)$ for all $y \in B$. Parallel translation along any curve in B from x_0 to y sends e_i to $e_i(y)$. This means that parallel translation along any curve in the ball from x_0 to y is the unique linear map sending e_i to $e_i(y)$ for all i. That is to say, parallel translation along the leaves of the foliation determines a trivialization of $E|_B$ with the property that the horizontal spaces $\{e\} \times B$ of this trivialization are the leaves of the foliation. Thus, in this trivialization $\Gamma_i = 0$ for all i, meaning that $\nabla_i = \partial_i$. □

Definition 2.8. A connection with zero curvature is said to be a *flat connection.*

Proposition 2.9. *Let $E \to M$ b a vector bundle over a compact manifold and let ∇ be a flat connection. Then the leaves of the foliation tangent to the distribution giving the connection, when they are given the 'leaf' topology rather than the subspace topology, are manifolds and are covering spaces of. M. Furthermore, parallel translation determines a linear representation of*

$$\pi_1(M, x_0) \to \mathrm{Hom}(E_{x_0}, E_{x_0}).$$

Proof. We cover M by finitely many connected open subset $\{U_\alpha\}_\alpha$ with trivializations $E|_{U_\alpha} = U_\alpha \times \mathbb{R}^N$ such that the connection is given by $\nabla_i = \partial_i$ in each local trivialization. Thus, each leaf $\mathcal{L}$ has the property that $\mathcal{L} \cap \pi^{-1}(U_\alpha)$ is a disjoint union of components mapping isomorphically onto U_α. The components of the intersection are indexed by the intersection with the fiber over any point of U_α. If the intersection with a fiber is not a discrete subset of the fiber then the union of the components is not homeomorphic to the product of a discrete set with B and the map from the subspace topology on the leaf to M is not a covering map. But the leaf topology is defined so that its basic open subsets are exactly the components of the intersections of the leaf with a flow box. With this topology the union of the components of the intersection of a leaf with $\pi^{-1}(U_\alpha)$ is homeomorphic to a product of a discrete set with U_α, and hence the projection of a leaf to M is a covering space when we give the leaf its 'leaf topology'. In particular, the holonomy around a loop based at x_0 does not change as we deform the loop keeping the base point fixed at x_0. Thus, the holonomy factors to give a function $\pi_1(M, x_0) \to \mathrm{Hom}(E_{x_0}, E_{x_0})$. We have already seen that this function preserves compositions. It also sends the trivial loop to the identity homomorphism. Thus, this function is a representation of $\pi_1(M, x_0)$ as linear automorphisms of the vector spacxe E_{x_0} □

Remark 2.10. For a general connection as we deform a loop based at x_0 keeping the base point fixed, the holonomy will vary by the integral of the Riemannian curvature over the 2-dimensional track of the homotopy.

2.1 Case of the Tangent Bundle

Everything we have done in general holds for the Riemannian connection on tangent bundle bundle of a Riemannian manifold M. But in this case there are extra restrictions in the curvature tensor. the first to remark is that in this case all four of the coordinates involved in the curvature tensor are tangent or cotangent coordinates. The two form $R(X,Y)$ takes values in the endomorphism bundle of the tangent bundle. Thus, for X,Y,Z vector fields the expression $R(X,Y)(Z)$ is a vector field The best presentation of this curvature is to use the metric to make all the variables in the cotangent bundle. In other words we define a 4-tensor

$$\mathbf{RR}(X,Y,Z,W) = \langle R(X,Y)(W), Z\rangle.$$

Here, X,Y,Z,W are vector fields. Indeed, it follows immediately from what we have already established that $\mathbf{RR}$ is linear over the functions in all four variables. Hence, it is a section of $\Lambda^2(T^*(M)) \otimes T^*(M) \otimes T^*(M)$.

Proposition 2.11. *1. The tensor* $\mathbf{RR}$ *is skew-symmetric in the last two variables; i.e.,*

$$\mathbf{RR}(X,Y,Z,W) = -\mathbf{RR}(X,Y,W,Z).$$

2. The tensor $\mathbf{RR}$ *is symmetric under the interchange of variables* $1,2$ *with variables* $3,4$*; i.e.,*

$$\mathbf{RR}(X,Y,Z,W) = \mathbf{RR}(Z,W,X,Y).$$

Proof. As before, it suffices to consider the case when all four vector fields commute. Let us establish the first item. We compute

$$\begin{aligned}
\mathbf{RR}(X,Y,Z,W) =& \langle R(X,Y)(W), Z\rangle = \langle \nabla_X\nabla_Y(W) - \nabla_Y\nabla_X(W), Z\rangle \\
=& -\langle \nabla_Y(W), \nabla_X(Z)\rangle + X(\nabla_Y(W), Z\rangle) + \langle \nabla_X(W), \nabla_Y(Z)\rangle \\
& - Y(\langle \nabla_X(W), Z\rangle) \\
=& \langle W, \nabla_Y\nabla_X(Z)\rangle - Y(\langle W, \nabla_X(Z)\rangle) + X(\nabla_Y(W), Z\rangle) \\
& - \langle W, \nabla_X\nabla_Y(Z)\rangle + X(W, \nabla_Y(Z)\rangle) - Y(\langle Z, \nabla_X(W)\rangle \\
=& -\langle W, R(X,Y)(Z)\rangle - Y(\langle W, \nabla_X Z\rangle + \langle Z, \nabla_X(W)\rangle) \\
& + X(\langle \nabla_Y(W), Z\rangle + \langle W, \nabla_Y(Z)\rangle) \\
=& -\langle W, R(X,Y)(Z)\rangle - Y(X(\langle W, Z\rangle)) + X(Y(\langle W, Z\rangle)) \\
=& -\langle W, R(X,Y)(Z)\rangle = -\mathbf{RR}(X,Y,W,Z)
\end{aligned}$$

For the second statement we begin by showing what is called the *Bianchi identity*:

$$R(X,Y)(Z) + R(Z,X)Y + R(Y,Z)(X) = 0.$$

Since we know that this expression is linear over the functions in all three variables, it suffices to consider the case when X, Y, Z are three mutually commuting vector fields. (In fact, we could restrict to the case of coordinate direction partial derivatives in a local coordinate system.) Since all the brackets vanish we have

$$
\begin{aligned}
&R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = \\
&= \nabla_X\nabla_Y(Z) - \nabla_Y\nabla_X(Z) + \nabla_Z\nabla_X(Y) - \nabla_X\nabla_Z(Y) + \nabla_Y\nabla_Z(X) - \nabla_Z\nabla_Y(X).
\end{aligned}
$$

Again using the fact that the brackets vanish we interchange the second and third variables in each of the terms with negative signs. These terms are replaced respectively by

$$-\nabla_Y\nabla_Z(X), \quad -\nabla_X\nabla_Y(Z), \quad -\nabla_Z\nabla_X(Y),$$

which cancel the other three terms.

Now using the Bianchi identity and Part 1 of this proposition and the skew symmetry we show that $R(X,Y) = -R(Y,X)$. We compute:

$$
\begin{aligned}
\mathbf{RR}(X,Y,Z,W) =& \langle R(X,Y)W,Z\rangle = -\langle R(X,Y)Z,W\rangle \\
=& \langle R(Z,X)Y,W\rangle + \langle R(Y,Z)X,W\rangle \\
=& -\langle R(Z,X)W,Y\rangle - \langle R(Y,Z)W,X\rangle \\
=& \langle R(W,Z)X,Y\rangle + \langle R(X,W)Z,Y\rangle + \langle R(W,Y)Z,X\rangle + \langle R(Z,W)Y,X\rangle
\end{aligned}
$$

In the last expression the first and the fourth terms are equal because of the skew symmetries. Each of these terms is equal to $\langle R(Z,W)Y,X\rangle$ and hence they add to give $2\mathbf{RR}(Z,W,X,Y)$. The sum of the second and third terms is

$$
\begin{aligned}
&-\langle R(X,W)Y,Z\rangle - \langle R(W,Y)X,Z\rangle \\
&\quad = \langle R(Y,X)W,Z\rangle = -\langle R(X,Y)W,Z\rangle = -\mathbf{RR}(X,Y,Z,W).
\end{aligned}
$$

We have shown that

$$\mathbf{RR}(X,Y,Z,W) = 2\mathbf{RR}(X,Y,Z,W) - \mathbf{RR}(Z,W,X,Y),$$

and the result follows. □

The consequence of all these symmetries is that **RR** is a quadratic form (or symmetric bilinear pairing) on $\Lambda^2 T_x M$ varying smoothly with x. Equivalentlly **RR** can be viewed as a smooth section of $\mathrm{Symm}^2(\Lambda^2 T^M)$.

Definition 2.12. Given a 2-plane $P \subset T_xM$ chose an orthonormal basis $\{e_1, e_2\}$ for it. Then $\mathbf{RR}(e_1 \wedge e_2, e_1 \wedge e_2) = \mathbf{RR}(e_1, e_2, e_1, e_2)$ is called the *sectional curvature* of the Riemannian manifold M in the P-direction.

It is an easy exercise to show that any finite dimensional quadratic form over the reals has a basis in which the form is diagonal. If we diagonalize $\mathbf{RR}$ at a point $x \in M$, then diagonal entries are the *principal curvatures* at x. It is important to realize that these do not have to be sectional curvatures: i.e., the principal directions in $\Lambda^2 T_xM$ do not have to be given by elementary elements of the form $a \wedge b$ in $\Lambda^2 T_xM$.

2.2 Examples

Example 1. Surfaces in 3-space In the case of a surface Σ the bundle $\Lambda^2 T\Sigma$ is a real line bundle, the orientation bundle. The metric on M determines a metric on this bundle. Consequently, the bundle of quadratic forms on $\Lambda^2 T\Sigma$ which is the tensor square of the dual to the orientation bundle has a metric and an orientation. Hence, the bundle of quadratic forms on $\Lambda^2 T_x\Sigma$ has a preferred identification with the trivial line bundle $\Sigma \times \mathbb{R}$ over Σ, which given by evaluation of the tensor square of the unit section of the orientation bundle. Thus, the Riemannian curvature of Σ is a section of this trivial bundle and hence a function on the surface. The value of his function at a point $p \in \Sigma$ is the sectional curvature in the unique 2-plane direction at this point. That is to say it is

$$\mathbf{RR}(e_1, e_2, e_1, e_2) = \langle R(e_1, e_2)(e_1), e_2 \rangle,$$

where $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$. We shall show that in this case the Riemannian curvature agrees with the Gauss curvature of $\Sigma \subset \mathbb{R}^3$.

We start by recalling the defnitinion of the Gauss curvature of an oriented surface Σ in 3-space and let (u, v) be local coordinates near a point p. We view these coordinates as a smooth math φ from an open subset of the (u, v)-plane to $\mathbb{R}^3$ with image parameterizing a neighborhood of p in Σ. We define $N\colon \Sigma \to S^2$ to be the smooth function that associates to each point of Σ the positive unit normal vector to Σ at that point. Clearly $T_{N(p)}S^2$ and $T_p\Sigma$ are the same plane in $\mathbb{R}^3$; that is to say, since $\langle N, N \rangle = 1$, both $\partial_u N$ and $\partial_v N$ lie in $N(p)^\perp = T_p\Sigma$. Thus,

$$dN_p \colon T_p\Sigma \to T_{N(p)}S^2 = T_p\Sigma.$$

Claim 2.13. *The map $dN\colon T_p\Sigma \to T_p\Sigma$ is self-adjoint with respect to the metric on $T_p\Sigma$ induced by the restriction of the Euclidean metric.*

Proof. Using the local coordinates φ from an open subset of (u, v) space to a neighborhood of p in Σ, $\partial_u\varphi$ and $\partial_v\varphi$ form a basis for $T\Sigma$ and

$$N(u,v) = \frac{\partial_u\varphi \times \partial_v\varphi}{|\partial_u\varphi \times \partial_v\varphi|}.$$

Adjointness is the statement that

$$\langle dN_p(\alpha), \beta\rangle = \langle dN(\beta), \alpha\rangle$$

for all tangent vectors α, β in $T_p\Sigma$. It suffices to check that

$$\langle dN_p(\partial_u\varphi), \partial_v\varphi\rangle = \langle dN_p(\partial_v\varphi), \partial_u\varphi\rangle.$$

Of course $\partial_u N_p = dN_p(\partial_u)$, and analogously for v. The adjointness then becomes

$$\langle \partial_u N, \partial_v\varphi\rangle = \langle \partial_v N, \partial_v\varphi\rangle.$$

Since N is orthogonal to the image of $D\varphi_*$

$$\partial_v\langle N, \partial_u\varphi\rangle = \partial_u\langle N, \partial_u\varphi\rangle = 0.$$

Thus,

$$\langle \partial_u N, \partial_v\varphi\rangle = -\langle N, \partial_u\partial_v\varphi\rangle$$

and

$$\langle \partial_v N, \partial_u\varphi\rangle = -\langle N, \partial_v\partial_v\varphi\rangle.$$

The result follows from the equality of cross partials of φ. □

We define the *second fundamental form* to be $-dN\colon T_p\Sigma \to T_p\Sigma$. [The change of sign is to make the curvatures line up with the usual geometric notions.] Thus, there is an orthonormal basis $\{e_1.e_2\}$ for $T_p\Sigma$ in which the bilinear form $a \otimes b \mapsto \langle -dN(a), b\rangle$ is diagonal; i.e. $\langle -dN(e_1), e_2\rangle = 0$. Then e_i are the*principal directions* and $\lambda_i = \langle -dN(e_i), e_i\rangle$ is the prinicipal curvature in the e_i direction. The *Gauss curvature* of Σ is defined to be $\lambda_1\lambda_2$. Notice that reversing the orientation does not change the λ_i and hence leaves the Gauss curvature unchanged, so in fact we do not need an orientation on Σ. [Also, notice that multiplying the form by -1 has not changed the Gauss curvature, only the principal curvatures.

One of Gauss's fundamental results is that this Gauss curvature (which we have defined for surfaces embedded in 3-space) is given by a purely intrinsic computation using only the restriction of the metric to the surface, not the way the surface is isometrically embedded in $\mathbb{R}^3$.. (This is not true of the principal curvatures and the mean curvature; they depend on the way the surface sits in $\mathbb{R}^3$ not just on its intrinsic metric.)

Theorem 2.14. *Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface. Then the Gauss curvature of Σ at p is given as*

$$\lim_{r\mapsto 0}\frac{\pi r^2 - A(B(p,r))}{12\pi r^4},$$

where $B(r,p)$ is the ball of radius r in Σ centered at p; i.e., the open set or all points in Σ that are connected to p by a curve of length less than r.

I shall not give a proof of this result. But using it we can define the Gauss curvature of any smooth surface using this formula as the definition of the Gauss curvature.

Definition 2.15. Let Σ be a smooth surface and $p \in \Sigma$ a point. The Gauss curvature of Σ at p, denoted $K_p(\Sigma)$ is given by

$$K_p(\Sigma) = \lim_{r\mapsto 0}\frac{\pi r^2 - A(B(p,r))}{12\pi r^4}.$$

Now let us compute the Riemannian curvature of $\Sigma \subset \mathbb{R}^3$ at the point p. We choose ambient Euclidean coordinates so that $p \in \Sigma$ is the origin, $T_p\Sigma$ is the (x, y)-plane and that x and y are the principal directions with principal curvatures being λ_1 and λ_2, respectively, and the map $\varphi(u, v) = (u, v, f(u, v)$ where f is a function vanishing at $(0, 0)$ and with $df(0, 0) = 0$.

Claim 2.16. *The second fundamental form at $p = (0, 0, 0)$ is given by the negaitve of the Hessian matrix of f*

$$\begin{pmatrix} \partial_x^2 f(0,0) & \partial_x\partial_y f(0,0) \\ \partial_y\partial_x f(0,0) & \partial_y^2 f(0,0) \end{pmatrix}.$$

The Gauss curvature at p is $(\partial_x^2 f(0.0))(\partial_y^2 f(0,0))^2$.

Proof. This is a direct exercise. □

We rename the coordinate $x_1 = x$ and $x_2 = y$ and denote ∇_{∂_i} by ∇_i Our hypothesis on the coordinates and the principal curvatures implies that

$$\partial_i^2 f = \lambda_i; \quad \partial_1\partial_2 f(0,0) = 0.$$

Thus, the second corder Taylor expansion of f at $(0, 0)$ is $f(0, 0)+(\lambda_1/2)x_1^2+ (\lambda_2/2)x_2^2$.

Since $\nabla_1(\partial_2) = \sum_k \Gamma_{1,2}^k \partial_k$ and similarly for the other indices and the curvature at p is given by

$$\langle(\nabla_1 \circ \nabla_2 - \nabla_2 \circ \nabla_1)(e_2), e_1\rangle = \langle(\partial_1\Gamma_2 + \Gamma_1 \circ \Gamma_2 - \partial_2\Gamma_1 - \Gamma_2 \circ \Gamma_1)(e_2), e_1\rangle,$$

where we write Γ_i for the endomorphism of $T_p\Sigma$ given by the Christoffel symbols $\Gamma_{i,\alpha}^{\beta}$ as α, β range over the indices $1, 2$. W see that the curvature only depends on the Christoffel symbols at the point and their first order partial derivatives, and hence the curvature only depends on the metric at the point and its first and second partial derivatives.

We compute second order Taylor expansion of the metric at $(0, 0)$. Our coordinates are $(u, v) \mapsto (u, v, f(u, v))$ It is given by the matrix

$$\begin{pmatrix} 1 + (\partial_1 f)^2 & \partial_1 f \partial_2 f \\ \partial_2 f \partial_1 f & 1 + (\partial_2 f)^2 \end{pmatrix}.$$

Hence, the metric to second order at p is

$$\begin{pmatrix} 1 + \lambda_1^2 x_1^2 & \lambda_1\lambda_2 x_1 x_2 \\ \lambda_1\lambda_2 x_2 x_1 & 1 + \lambda_2^2 x_2^2 \end{pmatrix}.$$

Since the first order Taylor expansion of the metric is constant and equal to the identity, all the Christoffel symbols vanish at the origin and hence the curvature at p is

$$\langle \partial_1\Gamma_2 - \partial_2\Gamma_1)(e_2), e_1\rangle = \partial_1\Gamma_{2,2}^1 - \partial_2\Gamma_{1,2}^1.$$

Computation gives of the first-order Taylor expansion of Γ at p gives

$$\Gamma_{2,2}^1 = \frac{1}{2}(\partial_2 g_{1,2} + \partial_2 g_{1,2} - \partial_1 g_{2,2}) = \partial_2 g_{1,2} = \lambda_1\lambda_2 x_1$$

$$\Gamma_{1,2}^1 = \frac{1}{2}(\partial_1 g_{1,2} + \partial_2 g_{1,1} - \partial_1 g_{1,2}) = 0.$$

Hence, the Riemannian curvature at p is $\lambda_1\lambda_2$, agreeing with the Gauss curvature at this point.

Example 2. Manifolds in $\mathbb{R}^N$

Suppose that M is a smooth k-manifold in $\mathbb{R}^N$. Let $P \subset T_pM$ be a 2-plane. Consider the surface Σ that is the intersection of M with the affine space of dimension $N - k + 2$ through p tspanned by $\{p\} + P$ and $\{p\} + (T_pM)^\perp$.

Claim 2.17. *The sectional curvature of M at p in the 2-plane direction P is equal to the Riemannian curvarture of Σ at p.*

Proof. The argument above works in this case as well with minor modifications. Now the surface Σ near p is defined by a function $U \to \mathbb{R}^{N-2}$ where

U is a neighborhood of the origin in $\mathbb{R}^2$ where f vanishes to second order at the origin. Then the metric is given by

$$\begin{pmatrix} (1 + \partial_x f \cdot \partial_x f & \partial_x f \cdot \partial_y f \\ \partial_y f \cdot \partial_x f & 1 + \partial_y f \cdot \partial_y f \end{pmatrix}.$$

By choosing the orthonormal coordinates correctly, we can arrange that this is a diagonal matrix. The computations in the case of a surface in 3-space, then carry over to show that the curvature of Σ at p is the product of the diagonal entries of this matrix. This agrees with the Gauss curvature of the surface. □

3 Geodesics

Let M be a Riemannian manifold. A parameterized smooth curve $\gamma\colon [a,b) \to M$ is said to be a *geodesic* if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for all $t \in [a,b)$, meaning the one-sided derivative at a.

In local coordinates if $\gamma(t) = (x^1(t), \ldots, x^n(t))$, then using the basis $\{\partial_1, \ldots, \partial_n\}$ for the tangent space at any point of the local coordinate systme, we have

$$\gamma'(t) = \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right),$$

and the geodesic equation reads:

$$\frac{d^2x^k}{dt} + \sum_{i,j} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma^k_{i,j}(x^1(t), \ldots, x^n(t)) = 0.$$

This is a second order ODE and locally has a unique solution given arbitrary initial condition $\gamma(a)$ and an initial velocity $\gamma'(a)$.

There is another way to think about this equation using the calculus of variations. Consider a Lagrangian for smooth curves given by

$$\mathcal{L}(\gamma) = E(\gamma) = \int_0^1 |\gamma'(t)|^2 dt,$$

which is the *energy* of the curve.

Proposition 3.1. *The critical points of this energy functional restricted to the space of curves $\gamma\colon [0,1] \to M$ subject to the condition that $\gamma(0) = \overline{x}_0$ and $\gamma(1) = \overline{x}_1$ for chosen points $\overline{x}_0$ and $\overline{x}_1$ are exactly the geodesics with these endpoints.*

Proof. Consider a variation of $\gamma(t)$ to a two-parameter family $\widetilde{\gamma}(t,s)$ defined for s near 0, with $\widetilde{\gamma}(t,0) = \gamma(t)$. Since we are working in the space of curves with fixed endpoints, $\widetilde{\gamma}(1,s) = \overline{x}_1$ and $\widetilde{\gamma}(0,s) = \overline{x}_0$ for all s. Let $\dot{\gamma}(t,s)$ denote the derivative in the t-direction. Now we compute the first order variation

$$\begin{aligned}\delta_s|_{s=0}\left(\mathcal{L}(\widetilde{\gamma}(t,s))\right) &= \delta_s|_{s=0}\int_0^1 |\dot{\gamma}(t,s)|^2\, dt\\ &= 2\int_0^1 \langle\dot{\gamma}(t), \nabla_s\dot{\gamma}(t,s)|_{s=0}\rangle dt\end{aligned}$$

The fact that the connection is torsion-free, implies that

$$\nabla_s \dot{\gamma}(t,s) = \nabla_{\dot{\gamma}(t)} \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}.$$

Thus,

$$\delta_s|_{s=0} \left(\mathcal{L}(\widetilde{\gamma}(t,s))\right) = 2\int_0^1 \langle \dot{\gamma}(t), \nabla_{\dot{\gamma}} \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}|_{s=0}\rangle dt.$$

Integration by parts gives

$$\delta_s|_{s=0} \left(\mathcal{L}(\widetilde{\gamma}(t,s))\right) = -2\int_0^1 \langle \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}|_{s=0}\rangle dt.$$

[Integration by parts is derived by using the fact that the covariant derivative preserves the metric so that

$$\frac{d}{dt}\langle \dot{\gamma}(t), \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}|_{s=0}\rangle = \langle \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}|_{s=0}\rangle + \langle \dot{\gamma}(t), \nabla_{\dot{\gamma}} \frac{\partial \widetilde{\gamma}(t,s)}{\partial s}|_{s=0}\rangle dt$$

and the integral over $[0,1]$ of the left-hand side vanishes since $\frac{\partial \widetilde{\gamma}(t,s)}{\partial s}$ vanishes at $t=0$ and $t=1$.]

Now γ is a critical point of $\mathcal{L}$ if and only if the right-hand side of the last equation vanishes for all variations $\widetilde{\gamma}(t,s)$. But this is equivalent to saying that $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, which is the geodesic equation. [If $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)$ is non-zero at a point, then in local coordinates about that point we see that this vector is smooth. Hence we can choose a smooth variation in the local coordinates supported in a small neighborhood of this value of t with positiveinner product for the term on the right-hand side of the last equation.] □

Lemma 3.2. *If M is a compact manifold, then given any $x \in M$ and $v \in T_xM$ there is a unique geodesic $\gamma(t)$ defined for all $t \in \mathbb{R}$ with $\gamma(0) = x$ and $\gamma'(0) = v$.*

Proof. The theory of solutions to ODE's tells us that for each (x,v) with $|v| = 1$ there is an $\epsilon > 0$ and a neighborhood of (x,v) in the unit tangent bundle of M so that for every (y,w) in this neighborhood there ia a geodesic $\gamma_{y,w}(t)$ defined for $t \in (-\epsilon,\epsilon)$ with initial conditions given by (y,w). By compactness of the unit tangent bundle of M, there is $\epsilon > 0$ so that for every (x,v) in the unit tangent bundle there is a geodesic defined on the interval $(-\epsilon,\epsilon)$ with these initial conditions. Now piecing these ϵ intervals together and using uniquensee, we see that any geodesic in M extends to one defined for all $t \in \mathbb{R}$. This proves the result for all initial conditions of unit speed. Rescaling by any non-negative constant gives the result for all initial velocities. $\square$

Remark 3.3. One of your homework problems is to show this result does not hold for non-compact manifolds.

Remark 3.4. The same argument shows that a Lorentzian manifold has local geodesics unqieuly determined by the initial position and initial velocity and that these are the critical points for the 'energy' functional of a path (which no longer has to be positive). For example in Minkowski space the path $\gamma(s) = (s, s, 0, 0)$ has $|\gamma'(s)|^2 = 0$ for all s whereas the path $\gamma(s) = (0, s, 0, 0)$ has $\langle \gamma'(s), \gamma'(s) \rangle < 0$ for all s. Neverthelesss, the argument is valid as long as the quadratic form is non-degenerate, which means given a non-zero tangent vector at a point there is another tangent vector at the point that pairs non-trivially with it under the bilinear form determined by the quadratic form on the tangent space.

3.1 Examples of Geodescis

Let S^n be the unit sphere is $\mathbb{R}^{n+1}$. The intersection of the sphere with any two-dimensional linear subspace L^2 is called *a great circle* on the sphere. k When poarameterized at constant speed, these are geodesics, as is easy to see from the description of parallel translation for manifolds in $\mathbb{R}^N$. A similiar argument shows that the intersection of hyperbolic space H with a linear 2-dimensional space is a curve, which when parameterized a constant speed produces a minimal geodesic..

The geodesics in the upper half-pane $\{z|\mathrm{Im}(z) > 0\}$ with the metric $(1/y^2)ds^{\otimes 2}$ are vertical lines and semi-circles perpendicular to the x-axis, when they are parameterized at unit speed.

Let $T = \mathbb{R}^n/L$ where L is a full-rank lattice be a torus with the flat metric induced from the Euclidean metric on $\mathbb{R}^n$ that is invariant under translation by the lattice. The geodesic are the images of straight lines in $\mathbb{R}^n$. Straight lines in rational directions with respect to the lattice (i.e., lines that pass through a point of $\mathbb{Q} \otimes L$ and hence a point of L) become periodic (closed) geodesics and those in irrational directions map to geodesics whose images are dense copies of $\mathbb{R}$ in T.

3.2 Relation of Energy to Length

Suppose that $\gamma\colon [a,b] \to M$ is a smooth curve. Then the length of γ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)|dt.$$

The Cauchy-Schwartz inequality tells is that $E(\gamma) \geq L(\gamma)^2/(b-a)$ with equality if and only if γ has constant speed, i.e., $|\gamma'(t)|$ is constant. Reparametrizing at a constant speed $L(\gamma)/(b-a)$ show us that there is a parameterized curve with the same image as γ whose energy is $L(\gamma)^2/(b-a)$ and that this minimizes the energy among all curves with this image. If follows that if γ minimizes the length of all smooth curves from $\gamma(a)$ to $\gamma(b)$ then reparametrizing γ at constant speed produces a minimizer of the energy for all smooth curves with these endpoints. This then is a minimal geodesic connecting $\gamma(a)$ and $\gamma(b)$.

3.3 The exponential map

Let M be a Riemannian manifold. For every $x \in M$ consider the map $T_xM \to M$ that assigns to $v \in T_xM$ the value at 1 of the unique geodesic with initial point x_0 and initial velocity v. Such a geodesic may not of course exist, but by the uniqueness and smooth variation with parameters, there is a small ball B_x about the origin of 0 in T_xM such that for all $v \in B_x$ there is such a geodesic γ_v. The map $B_x \to M$ defined by $v \mapsto \gamma_v(1)$ is a smooth map $B_x \to M$, called the *Gauss map* or the *exponential map*.

Notice that if M is compact then the exponential map is defined for all $v \in T_x(M)$ for all $x \in M$ and defines the smooth map $TM \to M$. If M is non-compact then the exponential map is only defined in a neighborhood B of the 0-section.

Lemma 3.5. *The exponential map* $\exp_x\colon B_x \to M$ *is a smooth map. Its differential at the origin is the identity map from* $T_xM \to T_xM$. *Thus, possibly after replacing* B_x *with a small ball centered at* $0 \in T_xM$ *the map* $\exp_x\colon B_x \to M$ *is a difeomorphism onto an open subset of* M.

Proof. The exponential map is smooth by the general theory of ODEs. Since $\gamma'_(0) = v$, we see that $d\exp_x(0) = \mathrm{Id}$. The last statement follows by the inverse function theorem. □

Corollary 3.6. *For* M *a Riemannian manifold and* $x \in M$ *a point, for* $\epsilon > 0$ *sufficiently small, every point* y *in the image* $\exp_x(B(0,\epsilon))$ *is connected to* *x by a unique shortest smooth curve that being the image of a ray in T_xM under the exponential mapping. The length of that geodesic is the norm of the preimage under* $\exp_x$ *of y in $B_x \subset T_xM$, the norm calculated using the quadratic form determined by the restriction of the Riemannian metric to* T_xM

Proof. Choose $\epsilon > 0$ so that $\exp_x$ is a diffeomorphism on $B_x = B_0(\epsilon) \subset T_xM$. Then geodesics of length less than ϵ emanating from x are exactly those of the form $\gamma_v : [0,1] \to M$ for some $v \in B(0,\epsilon)$. No two of these geodesics have the same final point. Thus, every point in $\exp_x(B(0,\epsilon))$ is connected to x by a unqiue geodesic of length less than ϵ.

Using the exponential map to transfer the polar coordinates on T_xM to M in a neighborhood of x. Since each γ_v is parameterized by arc length $\partial(\gamma_v(t))/\partial t = 1$, meaning that in the radial direction the metric agrees with $dr^{\otimes 2}$. If $\gamma(t)$ is any smooth curve starting at x in this neighborhood and parameterized by arc length, then $1 = |\gamma'(t)| \geq \langle \partial_r, \gamma'((t)\rangle$. Integrating gives $r(\gamma(t)) \leq t$ for all t. This shows that any smooth curve parameterized by arc length from x to y has length at least the length of the radial geodesic from x to y, and it has greater length unless it is that geodesic. □

Remark 3.7. Notice that this same result holds for curves that are uniform limits of smooth curves parameterized by arc length starting at x.

3.3.1 Rectifiable Curves

Definition 3.8. Let $\gamma : [a,b] \to M$ be a smooth curve. Given $a = t_0 < t_1, \cdots < t_n = b$ the Riemann sum approximation for this division of $[a,b]$ to the length of γ is $\sum_{i=1}^N d(\gamma(t_i), \gamma(t_{i-1})$. The curve is said to be *rectifiable* if the Riemann sum approximations for divisions of $[a.b]$ converge to a finite value L as the mesh size of the division tends to zero. (The mesh size is the maximum of $t_i - t_{i-1}$.) This limit L is the *length* of the curve. If γ is rectifiable, then or any $a \leq a' < b' \leq b$ the restriction $\gamma|_{[a',b']}$ is rectifiable. A rectifiable curve $\gamma : [0,L] \to M$ is said to be *parameterized by arc length* if for every $0 \leq a \leq L$ the length of $\gamma|_{[0,a]} = a$.

Lemma 3.9. *Let M be a compact Riemannian manifold. Let $\gamma_i : [0, L_i] \to M$ be a sequence of smooth curves starting at a given point $x \in M$ and parameterized by arc length. Suppose that $L_i = L(\gamma_i)$ converges to $L > 0$ as $i \mapsto \infty$. Then after passing to a subsequence, the γ_i converge uniformly to a rectifiable curve $\gamma_\infty : [0, L] \to M$ starting at x and parameterized by arc length.*

Proof. By passing to a subsequence we can assume that for any fixed $t \in [0, L)$ the points $\gamma_i(t)$ converge as $i \mapsto \infty$ to a limit point $\gamma_\infty(t)$. The standard diagonalization subsequence argument then shows that after passing to a subsequence we can assume that $\gamma_i(t)$ converges to a limit $\gamma_\infty(t)$ for every $t \in \mathbb{Q} \cap [0, L)$ and that this convergence is uniform in t in the sense that for every $\epsilon > 0$ there is $N < \infty$ such that for all $t \in \mathbb{Q} \cap [0, L)$ and all $i > N$ we have $d(\gamma_i(t), \gamma_\infty(t)) < \epsilon$.

Since $|\gamma_i'(t)| = 1$ for all i and all $t \in [0.L_i)$, the γ_i are uniformly lipschitz in the sense that $d(\gamma_i(t), \gamma_i(t')) \le |t-t'|$ for all i and t, t'. This means that in fact for any $t \in [0, L)$ the points $\gamma_i(t)$ converge to a limit $\gamma_\infty \colon [0, L) \to M$. The same argument shows that this curve extends is a rectifiable curve defined on all of $[0, L]$. This extended limit curves starts at x is easily seen to be continuous, rectifiable, and parameterized by arc length. □

Remark 3.10. This lemma holds for rectifiable curves that are uniform limits of smooth curves parameterized by arc length

Theorem 3.11. *Let M be a compact, connected Riemannian manifold and x and y distinct points of M. Then there is a smooth curve*

$$\gamma \colon [0, L] \to M$$

pazrameterized by arc length that minimizes the length of smooth curves from x to y. This curve is a geodesic and L is the distance from x to y in M.

Proof. Let $x \neq y$ be points of M. Choose a sequence of smooth curves γ_i from x to y parameterized by arc length such that $\lim_{i \mapsto \infty} L(\gamma_i) = d(x, y)$. Since the function $d(x, y)$ given by the infimum of the lengths of smooth curves from x to y is positive, there is a positive lower bound $L = d(x, y)$ to the lengths of these curves. By Lemma 3.9 a subsequence of these curves converges uniformly on $[0, L]$ to a rectifiable curve γ parameterized by arc length. This curve has length less than or equal to that of every smooth curve from x to y. Therefore for by Lemma 3.9 and the remark after it, given any t there is an open interval J about t such that the restriction of the curve to an interval of the form $[t, t')$ or $(t', t]$ in J is a geodesic. This implies that γ is a geodesic. □

Remark 3.12. This result and argument hold for complete Riemannian manifolds not just compact ones. There is a similar result for non-homotopically trivial loops in a compact manifold. Let M be a compact Riemannian manifold with non-trivial fundamental group. In any non-trivial free homotopy class of maps $[0, 2\pi] \to S^1 \to M$ there is one that minimizes the length

$$\int_{S^1} |\gamma'(t)| dt.$$

Parameterizing this curve at constant speed then gives a periodic geodesic in M minimizing length in the free homotopy class. Notice that the sphere has no length minimizing geodesics since there are deformations a great circle decreases the length (though not to first order). One of the homework problems is to show that this result does not hold in general for complete manifolds.