Modern Geometry: Exercises for week of Sept. 13, 2021

September 13, 2021

1. (a) Consider a smooth atlas consisting of $U_1 = \mathbb{R}^1$ and $U_2 = \mathbb{R}^1$ with $W_{1,2} \subset U_1$ the complement $\{0\}$ and $W_{2,1} \subset U_2$ also the complement of $\{0\}$ and with overlap function $h_{1,2}\colon W_{1,2} \to W_{2,1}$ given by $x \mapsto x^{-1}$. Show that the resulting smooth manifold is diffeomophic to the circle.

(b) With the subsets as in (a) suppose that $h_{1,2}\colon W_{1,2} \to W_{2,1}$ is the identity map. Show that the result of gluing U_1 to U_2 using $h_{1,2}$ is a non-Hausdorff space.

2.(a) Let $Gr_{\mathbb{R}}(k,n)$ is the set of linear k-dimensional linear subspaces in $\mathbb{R}^n$. Define the 'natural' topology on this set. For each $P \in Gr_{\mathbb{R}}(k,n)$ define an embdding $\varphi_P\colon \mathrm{Hom}(P, P^\perp) \to Gr_{\mathbb{R}}(k,n)$ by assigning to a homomorphism $P \to P^\perp$ its graph. Let U_P denote the image of φ_P. Show that there are linear isomorphisms $L_P\colon \mathrm{Hom}(P, P^\perp) \to \mathbb{R}^{k(n-k)}$. Show that as P varies over the points of $Gr_{\mathbb{R}}(k,n)$ the maps $L_P \circ \varphi_P^{-1}\colon U_P \xrightarrow{\cong} \mathbb{R}^{k(n-k)}$ define a smooth atlas for $Gr_{\mathbb{R}}(k,n)$.

(b) Show that for $k = 1$ this is a definition of $\mathbb{R}P^{n-1}$.

(c) Give the analogues of (a) and (b) for $\mathbb{C}$ replacing $\mathbb{R}$.

3. (a) Let $f(x_0, \ldots x_n)$ be a homogeneous polynomial of degree $k > 0$. Show that $X = \{f = 0\}$ is a union of one-dimensional subspaces in $\mathbb{R}^n$. Denote by $\overline{X} \subset \mathbb{R}P^n$ be the corresponding subset. Show that $\overline{X}$ is a smooth $(n-1)$-manifold if and only if both $\nabla f = 0$ and $f = 0$ only at the origin. [Hint: Use the Euler relation $\sum_i x_i \frac{\partial f}{\partial x_i} = kf$.]

(b) Same question with $\mathbb{C}$ replacing $\mathbb{R}$.

4. Show that the space of ordered pairs $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $|v_1|^2 = |v_2|^2 = 1$ and $\langle v_1, v_2 \rangle = 0$ is a smooth manifold. What is its dimension?

5. Consider the complex curve given by $\{z_1 z_2 = 1\}$ in the complex 2-dimensional linear space. Describe its topology.

6. Define products in the categories of smooth manifolds, complex manifolds, and lie groups.

7. (a) A smooth group action $G \times M \to M$ of a Lie group on a smooth manifold has *local slices* if for every $x \in M$ there is an embedding of $U \subset M$ so that the map $G \times U \to M$ given by the group action is a diffeomorphism onto an open subset of M. Show that in this case the quotient space $G \backslash M$ is naturally a smooth manifold (including the Hausdorff condition) with the property that a local function on an open subset of $G \backslash M$ is smooth if and only if its pull back is a smooth function on an open subset of M.

(b) Show that a smooth free action of a finite group G on a smooth manifold M always has local slices, and hence that the quotient $G \backslash M$ inherits a smooth structure. Show further that the map $M \to G \backslash M$ is a local diffeomorphism..

8. Consider the subset of $S^1 \times \mathbb{R}^2$ given by the equation

$$\cos(\theta/2)x_1 + \sin(\theta/2)x_2 = 0$$

is a smooth manifold. Describe its topology.

9. Show that $\mathbb{R}P^2$ is homeomorphic to the union of a closed 2-disk and a Möbius band glued together by a homeomorphism of their boundaries.

10. Suppose that $M \subset GL(n, \mathbb{R})$ is a smooth submanifold closed under multiplication. Show that the induced multiplication makes M a Lie group.