Modern Geometry: Problems for week of Sept 20, 2021

September 13, 2021

1. Suppose that a vector bundle $\mathcal{E} \to M$ is trivialized over each subset U_i of an open cover of M with transition functions $h_{ij} \colon U_i \cap U_j \to GL(n, \mathbb{R})$. Show that the dual bundle $\mathcal{E}^* \to M$ has a 'dual' trivialization over the same open subsets of M. Determine the transition functions for this dual trivializatin.

2. Establish the transistion formula given in the lecture for differential 1-forms.

3. Show that if $f \colon M \to N$ is a smooth map, then it induces smooth vector bundle maps $Df \colon TM \to TN$ and $Df^* \colon T^*N \to T^*M$. Applying the second map to sections gives us a'pull back' map from differential 1-forms on N to differential 1-forms on M, and by multiplicative extension to differential k-forms on M to differential k-forms on N

4. Let M be a smooth manifold. Show that there is a category whose objects are smooth vector bundles over M and whose morphisms are vector bundle maps covering the identity on M. Likewise show that there is a category whose objects are smooth vector bundles over smooth manifolds and whose morphsims are smooth maps of total spaces covering smooth maps of the bases.

5. Given a smooth vector bundle $\mathcal{E} \to M$ and a smooth map $f \colon N \to M$, define the pullback vector bundle $f^*\mathcal{E} \to N$ and show that there is a bundle map $f^*\mathcal{E}) \to \mathcal{E}$ covering the map f on the bases and inducing a linear isomorphism on each fiber of $f^*\mathcal{E}$.

6. Show that an $\mathbb{R}$-linear function L from smooth functions to smooth functions is the evaluation of a vector field if and only if it is a derivation in the sense that $L(fg) = L(f)g + fL(g)$. Show that a skew-symmetric

operation on k-tuples of vector fields is a vector field if it is linear over the smooth functions in each variable.

7. Show that for a 1-form ω the formula $d\omega(\chi_1, \chi_2$ given in the lecture notes is skew-symmetric and bilinear over the smooth functions.

8. Let $\omega = f(x^1, \ldots, x^n) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Compute directly $d\omega$ and show that $d^2\omega = 0$.

9. Show that any differential k-form can be written as a locally finite sum of forms of the type $f dg_1 \wedge \cdots \wedge dg_k$. [Hint: first prove this result for open subset of $\mathbb{R}^n$ where the g_i are coordinate functions. Then use a partition of unity to extend to all of M.]

10. Prove that the bracket of vector fields satisfies the Jacobi identity.

11. Show that vector fields with their Lie bracket on M form a sheaf of Lie algebras on M and that the differential forms on M form a sheaf of differential graded algebras on M.