Modern Geometry:
Problems for week of Sept 27, 2021

September 25, 2021

1. Suppose that $\mu\colon G \times M \to M$ is a smooth action. Show that there is an induced map of Lie algebras $L\mu\colon \mathfrak{g} \to Vect(M)$, where $Vect(M)$ is the Lie algebra of vector fields on M. Show the composition of $L\mu$ with evaluation of vector fields at $x \in M$ agrees with the differential the restriction of μ to $G \times \{x\}$ at (e, x).

2. Show that the exponential map

$$\exp\colon M(n \times n) \to GL(n, \mathbb{R})$$

defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{A^n}{n}$$

is well-defined. Compute its differential at $0 \in M(n \times n)$. Fix $A \in M(n \times n)$ and define a curve $\gamma(t) = e^{tA}$. Show that this is an abelian subgroup of $GL(n, Ar)$, which is an immersion of $\mathbb{R} \to GL(n, \mathbb{R})$. (N.B. It might not be a Lie subgroup since it may not be closed.) Show that γ is the integral curve to the left invariant vector field whose value at the identity is A.

3. Show that if $G \subset GL(n, \mathbb{R})$ is a Lie subgroup and if $A \in M(n \times n)$ is in tangent to G, then $\gamma(t) = e^{tA}$ is contained in G for all $t \in \mathbb{R}$.

4. For any Lie group G and any $A \in \mathfrak{g}$ let $\gamma_A(t)$ to be the integral curve to the left-invariant vector field whose value at the identity is A. Show that $\gamma(t)$ is defined for all $t \in \mathbb{R}$ and gives a group homomorphism $\mathbb{R} \to G$ whose differential at the identity sends ∂_t to A. Show that if $D\mu_{(e,x)}(A, 0) = 0$, i.e., if A is in the kernel of $D(\mu|_{G\times\{x\}]}$, the $\gamma(t) \cdot x = x$ for all t.

5. More generally, with notation as above, show that for any $x \in M$ the kernel of $D(\mu|_{G\times\{x\}})$ at (e, x) is a sub Lie algebra of $\mathfrak{g}$.

6. Show that there is a unique extension of pullback of differential 1-forms to all differential forms and that this is a map of differential graded algebras. That is to say, if $f\colon M \to N$ is a smooth map then $f^*\colon \Omega^*(M) \to \Omega^*(N)$ is a map of differential graded algebras. Show that this is a contravariant functor from the category of smooth manifolds and smooth maps to the category of differential graded algebras and morphisms between them. This means showing that the association sends the identity map of M to the identity map of $\Omega^*(M)$ (obvious) and preserves compositions (reversing the order of the composition since the functor is contravaiant).

7. Suppose that ω is a closed 1-form on a smooth manifold M and that the period of ω on every smooth closed loop in M is zero, i.e. for any smooth map $f\colon S^1 \to M$, we have

$$\int_{S^1} f^*\omega = 0.$$

Show that ω is exact; i.e., there is a function φ with $d\varphi = \omega$.

8. [More advanced problem for those who know about the fundamentgroup.] With ω as in the previous problem, show that integration of ω over smooth loops defines a homomoprhism $\pi_1(M, x_0) \to \mathbb{R}$. Show that any such homomorphism arises in this way.

9. Suppose that ω is 1-form with every period of ω over any smooth simple closed curve being of the form $2n\pi$ for some integer n. Show that there is a map $f_\omega\colon M \to S^1$ so that $\omega = (f_\omega)^* d\theta$. Now suppose that ω is nowhere zero, so that f_ω is a submersion [everywhere surjective differential] and if M is compact, then f_ω determines a locally smooth fibration of M over S^1.

10. Show that an n-manifold M has a nowhere vanishing n-form if and only if it is orientable, and any such form determines an orientation of M.

11. Show that a complex structure on a manifold determines an orientation for that manifold.

12. Let M be the solid 2-holed torus with loops A and B as marked below. Show that there are closed 1-forms ω_A and ω_B with

$$\int_A \omega_A = 1; \quad \int_B \omega_B = 1; \quad \int_A \omega_B = 0; \quad \int_B \omega_A = 0.$$

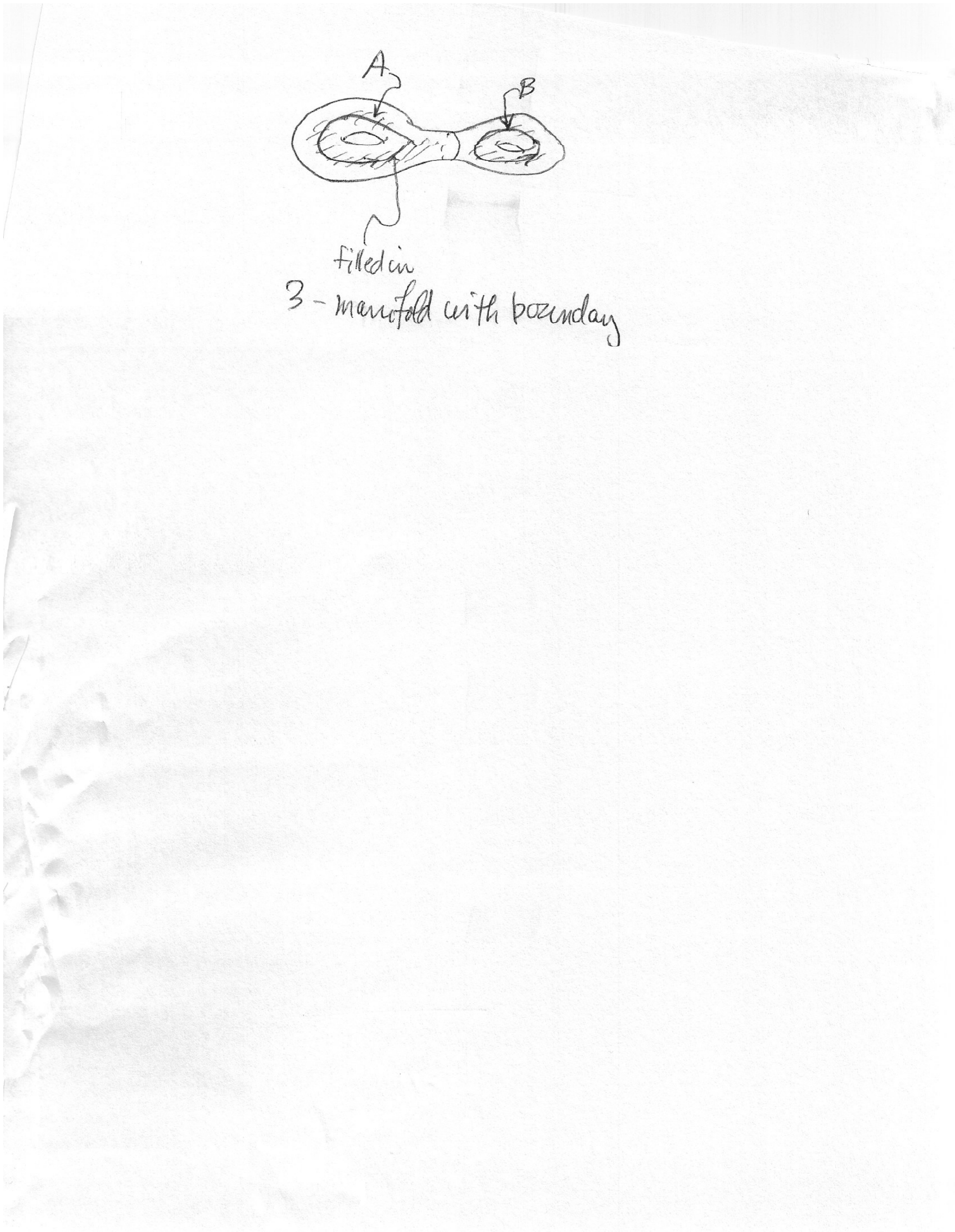

Figure 1: A and B curves