Modern Geometry:
Problems for week of Oct. 4, 2021

September 21, 2021

1. Let $(z^1, \ldots, z^n)$ be the usual complex coordinates on an open subset $U \subset \mathbb{C}^n$. We write $z^j = x^j + iy^j$, identifying $\mathbb{C}^n$ with $(\mathbb{R}^2)^n$. We define

$$\{\partial_{x^1}, \partial_{y^1}, \partial_{x^2}, \partial_{y^2} \ldots, \partial_{x^n}, \partial_{y^n}\}$$

as the induced basis for TU at every point, giving U an orientation determined by the $\{z^j\}$. Show that if we take any other holomorphic coordinates $\{w^1, \ldots, w^n\}$ on U this determines the same orientation. Conclude that a complex structure on a manifold determines an orientation for that manifold.

2. Let $(z^1, \ldots, z^n)$ be the usual complex coordinates on an open subset $U \subset \mathbb{C}^n$. Then we have the complex-valued differential forms

$$dz^j = dx^j + idy^j;\ 1 \leq j \leq n$$

on U. Define

$$\partial_{\overline{z}^j} = \partial/\partial\overline{z}^j = \frac{1}{2}(\partial_{x^j} + i\partial_{y^j})$$

$$\partial_{z^j} = \partial/\partial z^j = \frac{1}{2}(\partial_{x^j} - i\partial_{y^j}).$$

Show that exterior d is given by

$$d = \sum_j (dx^j \partial_{x^j} + dy^j \partial_{yj}) = \sum_j (dz^j \partial_{z^j} + d\overline{z}^j \partial_{\overline{z}^j})$$

and that

$$\langle dz^j, \partial_{z^k} \rangle = \delta_{j,k}; \quad \langle d\overline{z}^j, \partial_{\overline{z}^k} \rangle = \delta_{j,k};$$

whereas for all j, k we have

$$\langle dz^j, \partial_{\overline{z}^k} \rangle = 0; \quad \langle d\overline{z}^j, \partial_{z^k} \rangle = 0.$$

3. With notation as above, show that a complex-valued function f on U is holomorphic (i.e., has a complex linear differential) if and only if $\partial_{\overline{z}^j}(f) = 0$ for all j. In particular $\partial(z^j)/\partial\overline{z}^k = 0$ for all j, k. Show $\partial(z^j)/\partial z_j = 1$.

4. Show that if $f\colon T \to S^1$ is a smooth map that is a local diffeomorphism and is a two-to-one mapping then there is either:
(i) a diffeoorphism $g\colon S^1 \to T$ so that the composition $f \circ g\colon S^1 \to S^1$ is given by $\theta \mapsto 2\theta$, or
(ii) there is a diffeomorphism $g\colon S^1 \coprod S^1 \to T$ such that the composition $f \circ g$ is the identity on each component of $S^1 \coprod S^1$.

Describe in a similar way all two-to-one local holomorphic isomorphisms of a complex curve onto the punctured unit disk in $\mathbb{C}$.

5. Let $D \subset C$ be a convex open set. Suppose that $T \to D$ is a holomorphic mapping that is generically two-to-one and has two branch points. That is to say that there are two points of D whose pre-image is a single point of T and in appropriate holomorphic local coordinates for T and D near each of these points the map is given by $z \mapsto z^2$. Show that T is diffeomorphic to an annulus $S^1 \times (0, 1)$.

6. Suppose that $U \subset \mathbb{C}$ is a closed submanifold with boundary whose interior is a non-empt open subset of U. Suppose that $f\colon T \to U$ is a local holomorphic isomorphism that is a two-to-one map and suppose the pre-image in T of each boundary component has two components. Show that there is a holomorphic isomorphism g from two copies of U to T such that $f \circ g$ is the identity on each copy of U.

7. Let C be the compactification curve given by $y^2 = p(x)$ where p is a polynomial of degree greater than 2 without repeated roots and fix $x_0 \in C$. Fix a basis $\omega_1, \ldots, \omega_g$ for $H^{1,0}(C)$ and a smooth loops $\gamma_1, \ldots, \gamma_{2g}$ whose homology classes form a basis for $H_1(C)$.. For any point $y \in C$ chose a smooth path ν_y from x_0 to y. an consider

$$AJ((\nu_y) = \left(\int_{\nu_y} \omega_1, \ldots, \int_{\nu_y} \omega_g\right) \in \mathbb{C}^g.$$

Show that if we choose any other path μ_y from x_0 to y then for

$$AJ(\mu_y) = \left(\int_{\mu} y\omega_1, \ldots, \int_{\mu_y} \omega_g\right) \in \mathbb{C}^g.$$

the difference $(AJ(\nu_y) - AJ(\mu_y)) \in \mathbb{C}^g$ is an integral linear combination of the vectors $\left\{\left(\int_{\gamma_i} \omega_1, \ldots, \int_{\gamma_i} \omega_g\right)\right\}_{1 \le i \le 2g} \in \mathbb{C}^g$. Denote by $L \subset \mathbb{C}^g$ the integral lattice of rank $2g$ spanned by these vectors. Show that the image of $AJ(\nu_y)$ in the Jacobian $J(C) = \mathbb{C}^g/L$ depends only on the choice of x_0 and not the path ν_y from x_0 to y.

Show that this construction results in a holomorphic map $C \to J(C)$. This is the *Abel-Jacobi* map for the curve C.