Modern Geometry: Fall, 2021: First Problem Set for Part 2

November 1, 2021

1. Let Q be a quadratic from on a finite dimensional vector space V. Fixing a basis $\{e_1, \ldots, e_n\}$ for V, let $M(Q)$ be the symmetric matrix $N(Q)_{i,j} = B(e_i, e_j)$ where B is the bilinear form determined by Q. Show that changing the basis changes $M(Q)$ to a matrix $A^{tr}M(Q)A$ where A is the invertible change of basis matrix.

2. With assumptions and notation as in Problem 1 suppose that V has a positive definitive inner product Show that there is an orthonormal basis in which $M(Q)$ is diagonal and that in fact that all orthonormal bases in which $M(Q)$ is diagonal produce the same diagonal matrix representing Q.

3. Compute the curvature of the upper half plane with the hyperbolic metric.

4. Show that if $M^n \subset \mathbb{R}^N$ is a smooth submanifold with the Riemannian metric induced from the usual metric on $\mathbb{R}^N$ and if $P \subset T_p\Sigma$ is a tangent plane, then the sectional curvature of M in the P direction is equal to the sectional curvature of $A^{N-n+2} \cap M$ at p where A^{N+n-2} is the $(N-n+2)$-dimensional affine space through p whose underlying vector space is the direct sum of P and the orthogonal complement to T_pM in $\mathbb{R}^N$.

5. Let $\Sigma \subset \mathbb{R}^N$ is a surface with the Riemannian metric induced from the usual metric on $\mathbb{R}^N$. Suppose that there is an open subset U of the origin in $\mathbb{R}^2$ such that Σ is the graph of a function $f\colon \mathbb{R}^2 \to \mathbb{R}^{N-2}$ with $df(0,0) = 0$. Compute the Riemannian metric of Σ. Compute that the sectional curvature of Σ at p in terms of f.

6. Show that Gauss's formula for the curvature of a surface at a point in terms of the area defect formula holds for the unit sphere; for the sphere of radius r.

7. Let $\mathcal{E} \to M$ be a smooth vector bundle with with a connection. Let $\gamma\colon [0,1] \to M$ be a loop starting at $x \in M$. Show that parallel translation

around γ defines a linear isomorphism of $\mathcal{E}_x$. this is the *holonomy* of the connection around γ. If we set γ equal to the composition of a loop γ_1 starting at x followed by γ_2 another loop starting at x show that the holomony of the connection around γ is the composition of the holomony of the connection around γ_1 followed by the holonomy of the connection around γ_2.